

Termination Analysis of Programs with Periodic Orbit on the Boundary

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Abstract. The termination problem of a class of simple while program: While (constraints) do {updates} end is proven to be decidable by computing periodic orbit of nonlinear updating function over the reals. The termination problem of such a program with open constraint domain which have periodic orbit on the boundary is also discussed and the corresponding algorithms are given.

Introduction

Plenty of computer software has been applied to people's work and entertainment in modern life. Because some bugs in software may cause catastrophic consequences, many software companies release bug repairing programs for their products frequently. The problem of program verification has been widely concerned by computer scientists (see [1-3]). Termination analysis as the essential problem in program verification is of great significance and extremely difficult.

In general, the classical approach for checking the termination of program is the synthesis of so-called ranking function which maps each program state to a value in a well-founded domain. The progress is that, by demonstrating that each step in the execution of program reduces the measure assigned by the ranking function, we can make sure such given program terminates. By constructing a ranking function of a given program, we make each process of program execution correspond to a chain of elements of the well-founded domain. Therefore, we conclude that the given program terminates. Namely, the existence of a ranking function of a given loop implies that such loop must terminate. Several methods about synthesizing ranking functions have been studied in [4,5,6,7,8,9]. In contrast to ranking function discovering, recently some algebraic approaches have been applied to program verification.

Let \mathbb{R} denotes reals and $\mathbb{R}^n = \overbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}^n$. A simple while Program over \mathbb{R}^n can be described specifically as follows:

while $(X \in \Omega)$ do $\{X := F(X)\}$ end. (1)

Where $X \in \mathbb{R}^n$, $\Omega \subset \mathbb{R}^n$ and $F: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a continuous mapping. A. Tiwari[10] proved the decidability of a linear case of (1) as while $(BX > b)$ do $\{X := AX + c\}$ end. by real eigenvectors belong to positive eigenvalue of A , where A is an $n \times n$ matrix, B is an $m \times n$ matrix, x , b , and c are vectors. M. Braverman[11] discussed the termination of such a program over integers. To avoid errors caused by floating-point computation, Yang. L et al. [12, 13] further proposed a method to the termination of these programs by calculating symbolic conditions.

A nonlinear loop over reals can be described as P_1 : while $(x \in \Omega)$ do $\{x := f(x)\}$ end. Where $\Omega \subset \mathbb{R}$, $f: \mathbb{R} \rightarrow \mathbb{R}$ is continuous. Some cases of P_1 were discussed by Yao [14] and an interesting result was given as follows:

Theorem ^[14, Theorem 1] Loop program

$$P_2 : \text{while } (x \in (a, b)) \quad \text{do } \{x := f(x)\} \quad \text{end},$$

Where a, b are reals, and $f(x)$ is a real continuous function. If program P_2 is non-terminating, then f has fixed point in the closed interval $[a, b]$.

In more complicated cases, i.e., Ω is union of disjoint intervals, [14] gave following example:

Example 1

$$P : \text{while } (x \in [0, \frac{11}{20}] \cup [\frac{19}{20}, 1]) \quad \text{do } \{x := f(x)\} \quad \text{end, where } f(x) = \begin{cases} x + \frac{1}{2}, & x \in [-\infty, \frac{1}{2}) \\ 2(1-x), & x \in [\frac{1}{2}, -\infty) \end{cases}$$

The updating function f has no fixed point in Ω . But this loop does not terminate because $f(0) = \frac{1}{2}$, $f(\frac{1}{2}) = 1, f(1) = 0$.

Example 1 shows that the non-terminating of nonlinear loops over intervals may be caused by periodic orbits instead of fixed point of f . And termination of a special case of nonlinear loops over intervals is decided in [15] by fixed point either. Reference [16, 17] presented methods to discuss such programs by calculating periodic orbit of f . In this paper we discuss the termination of P_1 on open constraint domain Ω with periodic orbit on the boundary of Ω . As an application, corresponding algorithms and examples are given.

Preliminaries

Basic notions.

- $\Omega = \bigcup_{i=1}^s I_i = (a_1, b_1) \cup (a_2, b_2) \cup \dots \cup (a_s, b_s)$, where $s \in \mathbb{N}$ are integers, $a_i, b_i \in \mathbb{R}$, $I_i = (a_i, b_i)$, and $a_i < b_i$, $i=1, \dots, s$.
- The n -th iteration of a continuous $f : \mathbb{R} \rightarrow \mathbb{R}$ defined as $f^n(x) = f(f^{n-1}(x))$, $n \in \mathbb{N}$, and $f^0(x) = x$.
- An orbit of f starting from a point $x \in \Omega$ denoted by $Orb_f\{x\} = \{x, f(x), f^2(x), \dots, f^n(x), \dots\}$.
- A real x_0 is called a k -periodic point of f , if $f^k(x_0) = x_0$, and $f^r(x_0) \neq x_0$ for all $1 \leq r < k$, $k \in \mathbb{N}$. Specially, 1-periodic point is called as fixed point.
- A real x_0 is called a locally monotonic point of f , if x_0 has a left neighborhood $(x_0 - \varepsilon, x_0)$ and a right neighborhood $(x_0, x_0 + \varepsilon)$, and f is monotone in both of the neighborhoods.

Termination of Nonlinear Loop over Intervals.

In following sections the loop condition of P_1 is described as open set Ω and every point $x \in \Omega$ is assumed to be locally monotonic. P_1 is terminating iff for all $x \in \mathbb{R}$, P_1 is terminating. Applying the definition of iteration, termination of nonlinear loop P_1 can be stated as following forms:

- P_1 is terminating $\Leftrightarrow \forall x_0 \in \mathbb{R}$, there exists a positive integer i such that $f^i(x_0) \notin \Omega$.
- P_1 is non-terminating $\Leftrightarrow \exists x_0 \in \Omega$, $f^i(x_0) \in \Omega, i=1, 2, \dots$ i.e., $Orb_f\{x_0\} \subset \Omega$.

If $Orb_f\{x_0\} \subset \Omega$, x_0 is called a non-terminating point. Otherwise we call it a terminating point. If every x_0 in Ω is a terminating point, P_1 is terminating.

Termination of P_1 with periodic orbit on the boundary

We first discuss the termination of P_2 with fixed point on the boundary. [14] gave a complete algorithm for P_2 but following properties are useful to the case of periodic orbit.

Lemma 1. Assume that a (resp. b) is the unique fixed point of f in P_2 , f has a monotonic neighborhood (a, c) of a (resp. (c, b) of b). If P_2 is non-terminating, then

1). The non-terminating orbit $Orb_f\{x_0\}$ is an infinite orbit.

2). For every non-terminating orbit $Orb_f\{x_0\} \subset [a, b]$, it has a sub-sequence $\{f^{n_k}(x_0)\}_{k=1}^{\infty}$ such that

$$\lim_{k \rightarrow \infty} f^{n_k}(x_0) = a \text{ (resp. } \lim_{k \rightarrow \infty} f^{n_k}(x_0) = b).$$

Proof: For an indirect proof, we assume that the non-terminating orbit $Orb_f\{x_0\}$ is finite. Then there exists a small positive real number ε such that $Orb_f\{x_0\} \subset (a + \varepsilon, b - \varepsilon)$. Applying Theorem 1 in [14] to loop while $(x \in (a + \varepsilon, b - \varepsilon))$ do $\{x := f(x)\}$ end, we get a contradiction that this loop is non-terminating but f has no fixed point in $[a + \varepsilon, b - \varepsilon]$.

If P_2 is non-terminating and the non-terminating orbit $Orb_f\{x_0\}$ is an infinite orbit. For arbitrary given small positive $\delta_k \rightarrow 0$, there exists $n_k \in \mathbb{N}$ such that $f^{n_k}(x_0) \in (a, a + \delta_k)$. Else we have a positive small ε such that $Orb_f\{x_0\} \subset (a + \varepsilon, b - \varepsilon)$. Similarly it is a contradiction to the hypothesis that a is the unique fixed point of f . This is the end of the proof. \square

According to function f which has a monotonic neighborhood (a, c) of a (resp. (c, b) of b), following lemmas are very simple:

Lemma 2. Assume that $x = a$ is the fixed point of f in P_2 , f is monotonic in a neighborhood (a, c) of a .

If f has no fixed point in (a, b) , then one of the following four cases happens:

- i). $f(x) \equiv a$ in (a, c) .
- ii). f is decreasing and $f(x) < x$ in (a, c) . (see Fig. 1)
- iii). f is increasing and $f(x) < x$ in (a, c) . (see Fig. 2)
- iv). f is increasing and $f(x) > x$ in (a, c) . (see Fig. 3)

Lemma 3. Assume that $x = b$ is the fixed point of f in P_2 , f is monotonic in a neighborhood (c, b) of b .

If f has no fixed point in (a, b) , then one of the following four cases happens:

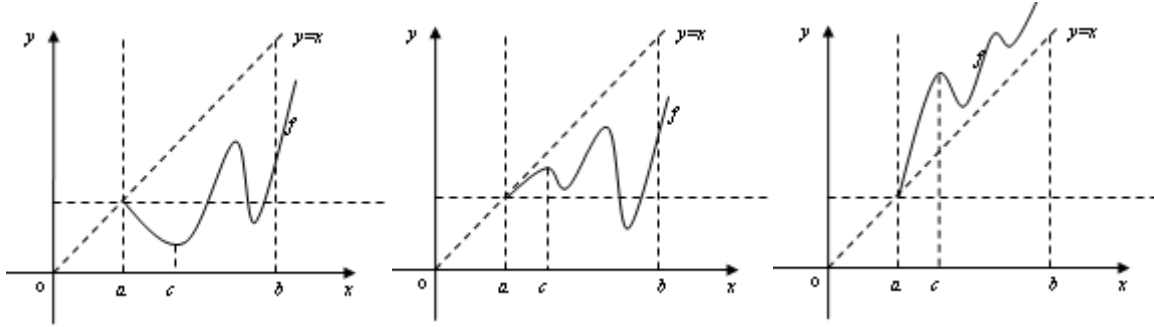
- v). $f(x) \equiv b$ in (c, b) .
- vi). f is decreasing and $f(x) > x$ in (c, b) .
- vii). f is increasing and $f(x) > x$ in (c, b) .
- viii). f is increasing and $f(x) < x$ in (c, b) .

Theorem 1. Assume a (resp. b) is a fixed point of f in P_2 , f is monotonic in a neighborhood (a, c) of a (resp. (c, b) of b). If f has no fixed point in (a, b) , then P_2 is non-terminating iff the cases iii) or vii) happens.

Proof: Assume that $x = a$ (resp. b) is the unique fixed point of f . If cases i) happens, then every point x in (a, c) is obviously terminated because $f(x) = a$. For point x in $[c, b]$, x is non-terminating means $Orb_f\{x\} \subset (a, b)$. By Lemma 1 we have a sub-sequence $\{f^{n_k}(x)\}_{k=1}^{\infty}$, $\lim_{k \rightarrow \infty} f^{n_k}(x) = a$, which is a contradiction with the fact that every point x in (a, c) is terminating. Termination of P_2 is clear for the case ii) or vi). If case iii) happens, then every point in (a, c) is non-terminating because $x > f(x) > \dots > f^{n-1}(x) > f^n(x) > \dots$ by monotonicity of f . Therefore $\lim_{n \rightarrow \infty} f^n(x) = a$ and $Orb_f\{x\} \subset (a, b)$. Case iv) implies that $x < f(x) < f^2(x) < \dots$ for x in (a, c) , which means that $Orb_f\{x\}$ could not have a subsequence $\{f^{n_k}(x_0)\}_{k=1}^{\infty}$, $\lim_{k \rightarrow \infty} f^{n_k}(x) = a$. The other cases can be proved similarly.

If the endpoint a and b are all fixed points of f . By Lemma 2 and 3, we only have six cases happened in a and b simultaneously: (1) i) and $viii$); (2) ii) and $viii$); (3) iii) and $viii$); (4) iv) and v); (5) iv) and vi); (6) iv) and vii).

It is clear that P_2 is non-terminated when case (3) or (6) happens. This is the end of the proof.



The case that f is differential in a neighborhood (a, c) of a (resp. (c, b) of b) is much simple. A fixed point x_0 of f is said to be a stable fixed point if $|f'(x_0)| < 1$ (resp. unstable fixed point if $|f'(x_0)| > 1$).

Every point in the neighborhood of x_0 will be attracted (resp. rejected) by x_0 under the iteration of f (see [8] P26). Case i) to $viii$) can be described as follows:

i) $f'(x) \equiv 0$ in (a, c) ; ii) $f'(a+0) < 0$; iii) $0 < f'(a+0) < 1$; iv) $f'(a+0) > 1$; v) $f'(x) \equiv 0$ in (c, b) ; vi) $f'(b-0) < 0$; vii) $0 < f'(b-0) < 1$; $viii$) $f'(b-0) > 1$.

The case that $f'(a+0) = 1$ and $f'(b-0) = 1$ need to be decided further. From Lemma 1-3 and Theorem1 we establish an algorithm TPI1(Termination of Program over an finite Interval) for Loop P_2 :

TPI1. Input: continuous and locally monotonic function f ,

Output: T (termination), NT (non-termination).

Step1: Calculate the fixed point of f . Whether f has a fixed point which belongs to (a, b) ? Yes, Output: NT, Quit; No, go to the next step.

Step2: a (or b) is a fixed point? Yes, Calculate the point c such that f is monotone in (a, c) (resp. (c, b)), go to step 3; No, Output T, Quit.

Step3: Whether f satisfies case iii) in (a, c) or case vii) in (c, b) ? Yes, Output: NT, Quit; No, Output: T, Quit.

Note that we can use $0 < f'(a+0) < 1$ or $0 < f'(b-0) < 1$ for judgment in Step2.

Next we discuss P_1 with an open domain Ω . Let r be a nonnegative integer. Define $N_i := \{r \mid \text{there are exactly } r \text{ intervals } I_{n_1}, I_{n_2}, \dots, I_{n_r} \text{ in } I_1, I_2, \dots, I_s, \text{ such that } f(I_i) \cap I_{n_j} \neq \emptyset, j=1, \dots, r\}$. We discuss termination of P_1 under following hypotheses:

$$N_i \leq 1, i=1, \dots, s. \quad (2)$$

$$f \text{ is monotonic in every interval } I_i, i=1, \dots, s. \quad (3)$$

The case that $N_i \leq 1, i=1, \dots, s$ and $N_i > 1$ were discussed over $\overline{\Omega}$ in [16] and [17] respectively, where $\overline{\Omega}$ is the closure of Ω . Under hypotheses (2) and (3) the termination of P_1 can be determined by k -periodic orbit of f with $k \leq s$. If the k -periodic orbit is in Ω , P_1 is non-terminating. But the case that the k -periodic orbit is on the boundary of Ω is still undecided. Without loss of generality, we assume:

f has a s -periodic orbit $\gamma = \{x_1, x_2, \dots, x_s\}$, $f(x_i) = x_{i+1}$, $i=1, \dots, s-1$ and $f(x_s) = x_1$. There exist integer k and a set $\{x_{i_1}^-, \dots, x_{i_{k_1}}^-, x_{i_1}^+, \dots, x_{i_{k_2}}^+\} \subset \{x_1, x_2, \dots, x_s\} \cap \{a_1, \dots, a_s, b_1, \dots, b_s\}$, $1 \leq k \leq s$, where $x_{i_1}^-, x_{i_1}^+$ are left and right endpoint respectively of some intervals in Ω . (4)

A neighborhood of the periodic orbit can be described as $L_\gamma := (x_1 - \varepsilon, x_1 + \varepsilon) \times (x_2 - \varepsilon, x_2 + \varepsilon) \times \dots \times (x_s - \varepsilon, x_s + \varepsilon)$.

Lemma 4. Loop P_1 satisfies (2), (4), and f has not other s -periodic orbit in $\overline{\Omega}$. If P_3 is non-terminated, then

1). The non-terminating orbit $Orb_f\{x_0\}$ is an infinite orbit.

2). For every non-terminating orbit $Orb_f\{x_0\} \subset \Omega$, it has s sub-sequences converge to x_1, x_2, \dots, x_s respectively.

Proof: By hypotheses (2) and (4), we have $N_i = 1$ for $i=1, \dots, s$. Without loss of generality, assume that $x_i \in \overline{I_{t_i}}$, $i=1, \dots, s$. Therefore $f(I_{t_1}) \cap I_{t_2} \neq \emptyset$, $f(I_{t_2}) \cap I_{t_3} \neq \emptyset$, ..., $f(I_{t_{s-1}}) \cap I_{t_s} \neq \emptyset$, $f(I_{t_s}) \cap I_{t_1} \neq \emptyset$ by (2) and the continuity of f . If f has a non-terminating orbit $Orb_f\{x_0\} = \{x_0, f(x_0), f^2(x_0), \dots, f^n(x_0), \dots\} \subset \bigcup_{i=1}^s I_{t_i}$, by the order of f iterating in Ω , we have $A_1 := \{f^{ns}(x_0)\}_{n=0}^\infty \subset I_{t_1}$, $A_2 := \{f^{ns+1}(x_0)\}_{n=0}^\infty \subset I_{t_2}, \dots$, $A_s := \{f^{(n+1)s-1}(x_0)\}_{n=0}^\infty \subset I_{t_s}$. Let $G := f^s$, G is continuous and satisfies $A_1 = \{G^n(x_0)\}_{n=0}^\infty \subset I_{t_1}$, $A_2 = \{G^n(f(x_0))\}_{n=0}^\infty \subset I_{t_2}, \dots$, $A_s = \{G^n(f^{s-1}(x_0))\}_{n=0}^\infty \subset I_{t_s}$. Discuss function G by Lemma 1 we have 1) and 2) hold. This is the end of the proof. \square

From lemma 4, if f has a period orbit $\gamma = \{x_1, x_2, \dots, x_s\}$, $x_i, i=1, \dots, s$ are fixed points of function $G = f^s$. Therefore, a periodic orbit γ is said to be attracting (see [18]) if there is a neighborhood L_γ of γ ,

$$f(L_\gamma) \subset L_\gamma, \bigcap_{j>0} f^j(L_\gamma) = \gamma. \quad (5)$$

If f is differential, (5) can be replaced by $\mu_\gamma := |f'(x_1)f'(x_2)\cdots f'(x_s)| < 1$. Similarly, if $\mu_\gamma > 1$, γ is a repelling orbit.

Because the updating function f of loop P_1 has a s -periodic orbit $\gamma = \{x_1, x_2, \dots, x_s\}$ with $\{x_{i1}, x_{i2}, \dots, x_{ik}\}$ on the boundary of Ω , the neighborhood of γ can be described as

$$(x_1 - \varepsilon, x_1 + \varepsilon) \times \cdots (x_{i1}^-, x_{i1}^- + \varepsilon) \times \cdots (x_{ir}^+ - \varepsilon, x_{ir}^+) \times (x_s - \varepsilon, x_s + \varepsilon) \quad (6)$$

where x_{it}^- s are left endpoints and x_{ir}^+ s are right endpoints of some intervals in Ω . The termination of loop P_1 can be decided by algorithm TPI1 for G infor $i=1, \dots, s$. Sometimes computing $G = f^s$ may be difficult (e.g. piecewise polynomial-functions), we can discuss the termination of P_1 by following algorithm:

TPI2. Input: continuous and locally differential function f , the unique s -periodic orbit $\gamma = \{x_1, x_2, \dots, x_s\}$ of f , and $\{x_{i1}, x_{i2}, \dots, x_{ik}\} \subset \gamma$ are endpoints of $\Omega, 1 \leq k \leq s$.

Output: T (termination), NT (non-termination). ND (non-determined)

Step1: Calculate μ_γ for f . If x_i is the left (resp. right) endpoint then calculate $f'(x_i + 0)$ (resp. $f'(x_i - 0)$) instead of $f'(x_i)$ in μ_γ . Whether $\mu_\gamma = 1$? Yes, Output: ND, Quit; No, go to next step.

Step2: $\mu_\gamma > 1$? Yes, Output: T, Quit; No, go to next step.

Step3: Take a positive ε small enough and construct a neighbourhood of L_γ as (6). Take a point $y_0 \in L_\gamma$ and compute $f(y_0), \dots, f^{s-1}(y_0), f^s(y_0)$.

Whether $f(y_0), \dots, f^{s-1}(y_0), f^s(y_0) \subset L_\gamma$? Yes, Output: NT, Quit; No, Output: T, Quit.

Conclusion

In this paper, by computing periodic orbit of nonlinear updating function, we consider the termination of nonlinear loop P_1 and present the corresponding decision algorithms. For the general programs having several program variables, the termination problem will be considered in another paper.

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