Dynamic Output Feedback Consensus Control of Multi-agent Systems with Time-delay

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Abstract. This paper considers the consensus problem of a multi-agent system with time delay and proposes a distributed dynamic output feedback protocol. A sufficient condition in terms of linear matrix inequalities (LMIs) is given to ensure consensus of the multi-agent system.

Introduction

Over the past few decades, the consensus problem in multi-agent systems has received considerable attention from many fields such as physics, biology and control theory. And numerous results have been obtained for consensus problems of multi-agent systems. For example, a simple model is proposed in [1] for phase transition of a group of self-driven particles and complex dynamics of the model are demonstrated numerically. Jadbabaie et al.\cite{2} studied the observed behavior reported in [1] and analyzed the alignment of an undirected network of agents with switching topologies which are periodically connected. Moreau\cite{3} researched the nonlinear discrete-time multi-agent systems with time-dependent communication links, and introduced a novel method based on the notion of convexity. Olfati et al.\cite{4} investigated a systematical framework of consensus problems with directed networks and undirected networks with fixed and switching topologies and time-delay and fixed topology, respectively. Later, Ren et al.\cite{5} extended the conclusions of [2] and [4] and presented more loose conditions for consensus of information under dynamically changing interaction topologies. Owing to constraints on measurement or economic costs in practice, it is sometimes hard to measure the relative information of all states directly. However, only the relative information of all outputs is available. Thus, consensus protocols with dynamic output feedback (dynamic compensator) case are adopted in this paper. Because of each agent has limited capability of collecting information, the consensus protocol for each agent in descriptor multi-agent systems is distributed and only depends on the output information of the agent itself and its neighbours. Some researchers have investigated the consensus problem of the multi-agent systems through dynamic output-feedback approach\cite{6}\cite{7}. However, the results about the consensus problem of the multi-agent systems with time delay through dynamic output-feedback approach are much less. In fact, time delays are ubiquitous phenomenon in multi-agent systems due to the congestion of communication channels and the finite transmission speed, which makes the dynamical behaviors of a network much more complicated. So it is necessary to consider the consensus problem of uncertain multi-agent systems with nonlinear dynamics and time delays.

Preliminaries

Let \(\ell(\nu, \nu, \rho)\) be an undirected graph of order \(n\) with the set of nodes \(\nu = \{s_1, \ldots, s_n\}\), the set of undirected edges \(\nu \times \nu\), and a symmetric adjacency matrix \(\rho = [a_{ij}]\), with nonnegative elements \(a_{ij}\). The node indexes belong to a finite index set \(\{1, \ldots, n\}\). The adjacency elements associated with edges of graph are positive, that is, \(e_{ij} \in \nu\) or \(e_{ji} \in \nu\) if and only if \(a_{ij} = a_{ji} > 0\). Moreover, it is
assumed that $a_{ii} = 0$ for all $i \in \{1, \cdots, n\}$. The Laplacian corresponding to the graph $\ell$ is defined as $L = \left[l_{ij}\right]$, where $l_{ij} = \sum_{j} a_{ij}$, and $l_{ij} = -a_{ij}$, $i \neq j$.

Lemma 1$^{[8]}$: If an undirected graph $\ell$ is connected, then its Laplacian $L$ has the following properties:

1. Zero is a simple eigenvalue of $L$, and $1_n$ is the corresponding eigenvector, that is, $L1_n = 0$.
2. The rest $n-1$ eigenvalues are all positive.

Lemma 2$^{[9]}$: Let $L_c = \left[L_{cj}\right] \in R^{n \times n}$ be a symmetric matrix with $L_{cj} = \begin{cases} (n-1)/n, & i = j \\ -1/n, & i \neq j \end{cases}$\[m]

Then the following statements hold:

1. The eigenvalues of $L_c$ are 1 with multiplicity $n-1$ and 0 with multiplicity 1. The vectors $1_n$ and $1_n$ are the left and the right eigenvectors of $L_c$ associated with the zero eigenvalue, respectively.
2. There exists an orthogonal matrix $U = \left[U_1 \ U_1\right] \in R^{n \times n}$ with $U_1 = \frac{1}{\sqrt{n}}1_n$, such that $U^T L U = \begin{bmatrix} I_{n-1} & 0 \\ 0 & 0 \end{bmatrix}$ holds. Let $L \in R^{n \times n}$ be the Laplacian matrix of any given undirected graph, then $U^T L U = \begin{bmatrix} L_1 & 0 \\ 0 & 0 \end{bmatrix}$ holds, where $L_1 \in R^{(n-1) \times (n-1)}$ is positive definite if and only if the corresponding graph is connected.

Consider the multi-agent system consisting of $n$ identical agents. Suppose the $i$th agent has the following dynamics,

$\dot{x}_i(t) = Ax_i(t) + Bu_i(t)$

$y_i(t) = Cx_i(t), \quad i = 1, \cdots, n$ \[m]

where $x_i(t) \in R^n$, $u_i(t) \in R^p$ and $y_i(t) \in R^q$ are the state, the protocol and the measured output of the agent $i$, respectively.

Define $z_i(t) = x_i(t) - \frac{1}{n} \sum_{j=1}^{n} x_j(t)$, $i = 1, \cdots, n$ as controlled output function to measure the disagreement of $x_i(t)$ to the average state of all agents. It is obvious that consensus of the multi-agent system can be achieved if and only if $\lim_{t \to \infty} z_i(t) = 0$, for all $i = 1, \cdots, n$.

Owing to constraints on measurement, it is sometimes hard to measure the relative information of all states directly. However, only the relative information of all outputs is available. Thus, consensus protocols with dynamic output feedback case are adopted in this paper.

We can design a following distributed dynamic output feedback protocol with time delay.

$\dot{v}_i(t) = A_K v_i(t) + B_K \sum_{j=1}^{n} a_{ij} (y_i(t-\tau) - y_j(t-\tau))$\[m]

$u_i(t) = C_K v_i(t) + D_K \sum_{j=1}^{n} a_{ij} (y_i(t-\tau) - y_j(t-\tau)), \quad i = 1, \cdots, n$ \[m]

where $v_i(t) \in R^p$ is the state of the dynamic output feedback controller, and $a_{ij}$ is the adjacency weights of the interaction graph $\ell$.

Substituting protocol (2) into the system (1), we can obtain the following closed loop system

$\ddot{z}(t) = \bar{A} \dot{\bar{z}}(t) + \bar{A}_1 \bar{z}(t-\tau)$ \[m]

where $\bar{A} = \left[ I_n \otimes A \quad I_n \otimes B \otimes C_K \right]$,

$\bar{A}_1 = \left[ L \otimes B_i \otimes C \quad 0 \right], \quad \bar{C} = \left[ L_c \otimes I_n \quad 0 \right]$
\[ v(t) = \begin{bmatrix} v^+_1(t) & \cdots & v^+_n(t) \end{bmatrix}^T, \quad \bar{v}(t) = \begin{bmatrix} x^T(t) & v^T(t) \end{bmatrix}^T, \text{ } L \text{ is the Laplacian of graph } \ell. \]

We can see that \( L \) is a singular matrix, therefore the closed loop system is uncontrollable if the given matrix \( A \) is unstable. So we must make a model transformation of the closed loop system.

Let \( \dot{x}(t) = x(t) - \frac{1}{n} \sum_{j=1}^n x_j(t) = (L_c \otimes I_m)x(t), \quad \bar{v}(t) = (L_c \otimes I_s)\bar{v}(t), \quad \dot{x}(t) = \dot{x}(t - \tau) = (L_c \otimes I_m)x(t - \tau). \)

Notice \( L_1 = 0, \quad L_1 = 0. \) Then the closed loop system (3) can be rewrite the following form.

\[ \hat{\xi}(t) = \hat{A}\hat{\xi}(t) + \hat{A}_i\hat{\xi}(t - \tau) \]
\[ z(t) = \hat{C}\hat{\xi}(t) \]

where \( \hat{A} = \begin{bmatrix} L_c \otimes A & 0 & L_c \otimes B_cK \end{bmatrix}, \quad \hat{A}_i = \begin{bmatrix} L_cL \otimes B_cD_cK \end{bmatrix}, \quad \hat{C} = \begin{bmatrix} L_c \otimes I_m \end{bmatrix} \)
\[ U^TL_cU = \begin{bmatrix} I_{n-1} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad U^TLU = \begin{bmatrix} L_1 & 0 \end{bmatrix} \]

Let \( \tilde{x}(t) = (U^T \otimes I_m)\hat{x}(t), \quad \tilde{v}(t) = (U^T \otimes I_s)\bar{v}(t), \quad \tilde{z}(t) = (U^T \otimes I_m)\hat{z}(t). \)

Then the closed loop system (4) can be rewrite the following form.

\[ \tilde{\xi}(t) = \tilde{A}\tilde{\xi}(t) + \tilde{A}_i\tilde{\xi}(t - \tau) \]
\[ \tilde{z}(t) = \tilde{C}\tilde{\xi}(t) \]

where \( \tilde{A} = \begin{bmatrix} U^TL_cU \otimes A & U^TL_cU \otimes B_cK \\ 0 & U^TL_cU \otimes A_K \end{bmatrix}, \quad \tilde{A}_i = \begin{bmatrix} L_1 \otimes B_cD_cK \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} U^TL_cU \otimes B_cK \end{bmatrix} \)

\[ \tilde{C} = \begin{bmatrix} U^TL_cU \otimes I_m \end{bmatrix} \]

**Main results**

Then we can derive an reduced-order system as follows.

\[ \ddot{\xi}(t) = A'\dot{\xi}(t) + A'_i\dot{\xi}(t - \tau) \]
\[ z'(t) = C'\dot{\xi}(t) \]

where, \( A' = \begin{bmatrix} I_{n-1} \otimes A & I_{n-1} \otimes B_cK \\ 0 & I_{n-1} \otimes A_K \end{bmatrix}, \quad A'_i = \begin{bmatrix} L_1 \otimes B_cD_cK \end{bmatrix}, \quad C' = [I_{n-1} \otimes I_m] \)

**Theorem 1:** Consider the network with an undirected interaction graph \( \ell \) that is connected. The system (6) is asymptotically stable, if there exist a dynamic output feedback \( u(t) \) and positive definite matrices \( P_i \in R^{(n+1)(n+1)} \), \( i = 1, \cdots, n-1 \), such that the following LMIs are satisfied for \( i = 1, \cdots, n-1 \).

\[ \Xi = \begin{bmatrix} A & B_cK \\ 0 & A_K \end{bmatrix}^T P + P \begin{bmatrix} A & B_cK \\ 0 & A_K \end{bmatrix} + S \begin{bmatrix} \lambda_iB_cD_cK \\ -S \end{bmatrix} < 0 \]

**Proof:** By Lemma 2, the matrix \( L_1 \) is positive definite. Therefore, there exists an orthogonal matrix \( Q \in R^{(n-1)(n-1)} \) such that \( Q^TL_1Q = \text{diag}\{\lambda_1, \cdots, \lambda_{n-1}\} \). Let \( x^*(t) = (Q^T \otimes I_m)x'(t), \quad v^*(t) = (Q^T \otimes I_s)v'(t), \quad z^*(t) = (Q^T \otimes I_m)\bar{z}(t). \) Then the system (6) can be rewritten in the following form.

\[ \ddot{x}^*(t) = A^*x^*(t) + A'_i\dot{x}^*(t - \tau) \]
\[ z^*(t) = C^*\dot{x}^*(t) \]
where, \( A^* = \left[ I_{n-1} \otimes A \quad I_{n-1} \otimes B \otimes C \right], \ A^i_\tau = \left[ \text{diag}\{\lambda_1, \cdots, \lambda_n\} \otimes B \otimes C \quad 0 \right], \ C^\tau = \left[ I_{n-1} \otimes I_m \quad 0 \right]. \)

We can decompose the system (7) to the following \( n-1 \) subsystems.

\[
\begin{align*}
\dot{z}_i(t) &= A^i_\tau z_i(t) + A^i_\tau z_i(t-\tau), \quad i=1,\cdots,n \\
\dot{z}_i^\tau(t) &= C^\tau z_i(t)
\end{align*}
\]

where, \( A^i_\tau = \begin{bmatrix} A & B \otimes C \\ 0 & A \end{bmatrix}, \ A^i_\tau = \begin{bmatrix} \lambda_i \otimes B \otimes C & 0 \\ \lambda_i \otimes B \otimes C & 0 \end{bmatrix}, \ C^\tau = \begin{bmatrix} I_m & 0 \end{bmatrix} \)

If for all \( i=1,\cdots,n \), system (8) is asymptotically stable, system (6) is asymptotically stable.

Consider the following Lyapunov-Kasovskii function,

\[
V(\xi_i^\tau(t)) = \xi_i^\tau(t) P \xi_i(t) + \int_{t-\tau}^{t} \xi_i^\tau(s) S \xi_i(t) ds
\]

If \( \Xi < 0 \), we can obtain that \( \dot{V}(\xi_i^\tau(t)) < 0 \).

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Summary

The consensus problem of a multi-agent system with time delay is dealt with through a distributed dynamic output feedback protocol. A sufficient condition in terms of linear matrix inequalities (LMIs) is obtained to ensure consensus of the multi-agent system.

References