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Assessment of C_{pm} in the presence of measurement errors

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Abstract

With the advent of Taguchi's loss function, the concept of target and sticking to the target to achieve better process performance has become widely accepted. In practice, while estimating process performance, the gauge measurement error is not taken into consideration. In the real world scenario this hardly happens since measurement errors cannot be avoided in most of the manufacturing processes. Ignoring this measurement error while estimating the process capability may often lead to unreliable/wrong decision about the capability of the process under study. Therefore, in this work we apply the method of Generalized Confidence Interval (GCI) to measure the process capability index C_{pm} in presence of measurement errors. In this study, an exhaustive set of simulation has been conducted to assess the performance of the GCI method in terms of expected value of generalized lower confidence limit (L_{pm}) and Coverage Probability (CP). The efficacy of dealing with the measurement error has been found satisfactory in this model. Finally it can be concluded that GCI method seems to be quite satisfactory for measuring process capability when the measurement errors are present; as well as when measurement error is negligible.

Keywords: Process Capability, Generalized Confidence Intervals, General Pivotal Quantity, Coverage Probability.

1. Introduction

Process Capability Indices (PCIs) have been widely used in quality assurance for quite sometimes. It is viewed as an aid to Total Quality Management. See Ref. 41 for details. The successes of the six-sigma programs in organizations like General Motors and Motorola have given a fillip to the study and use of PCIs. This can be gauged by the fact that there has been a very large body of literature on this subject. See Ref. 42 and 47 for a bibliography of the recent papers. Papers dealing with PCIs have appeared in Computer Science, Industrial Engineering, Management Science, Operations Research, Quality Engineering, Statistics and TQM journals.

The first PCI appearing in the literature was the potential index C_p, introduced by Kane [22]. In order to account for the deviation of the process mean from the midpoint of the specification limits, the C_{pk} index was

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proposed by Kane [22]. For details of C_p and C_{pk}, see Ref. 1 for example. The C_p and C_{pk} are relevant measures of progress for quality improvement in which reduction of variability is the guiding principle and process yield is the primary measure of success. Taguchi [43] has advocated a different approach to quality improvement in which reduction of variation from the target value is the guiding principle. Chan et al. [8] introduced the so-called Taguchi Capability Index (C_{pm}) that is measurable and directly related to the quadratic loss of the measured feature. When the target (T) is at the mid-point of the specification limits, it is defined by:

$$C_{pm} = \frac{USL - LSL}{6\sqrt{\sigma^2 + (\mu - T)^2}} \tag{1}$$

where USL is the upper specification limit, LSL is the lower specification limit, σ^2 is the process variance and μ is the process mean. Later Boyles [1991] gave the general statistical methodology for capability index C_{pm} without the restrictive assumption of $\mu = T$ which is as follows:

$$C_{pm}^{*} = min\left(\frac{USL - T}{3\sqrt{\sigma^2 + (\mu - T)^2}}, \frac{T - LSL}{3\sqrt{\sigma^2 + (\mu - T)^2}}\right) \tag{2}$$

Johnson [21] exploited the relationship between the capability index C_{pm} and the expected squared loss to provide an intuitive interpretation of C_{pm} in terms of percentage loss. Denniston [15] presents strong motivation for using C_{pm} and shows that it can indicate the probability of meeting the customers' product specification. He argues that it can be used to provide a better estimate of the cost of poor quality and hence it can be used to better manage products' quality to the customer.

While there is a large body of literature on PCIs, most of them do not take into account gauge measurement error. Measurement error cannot be avoided, even when carried out with very sophisticated and precise measuring instruments. McNesse and Klein [28], to the best of our knowledge, were the first researchers to point out that the variability is inherent in the measurement systems and sampling techniques add variability to the input from a process, and hence affect the process capability. Porter and Oakland [39] also emphasize that process capability assessment is dependent upon the measurement method. Montgomery and Runger [31] also dwell upon the importance of gauge capability. Persijn and Nuland [38] argue that process capability is meaningful only if the measurement system is capable.

Levinson [25] should be credited for bringing the focus on gauge. However, it was Mittag [29] who first discussed the effects of measurement errors on the performance of the four most basic process capability indices. He showed that random and constant measurement errors can considerably falsify the results of process capability analysis. He argued that the accuracy of a capability analysis could be significantly influenced by the accuracy of the gauges. Hence, it is important to ensure gauge capability before assessing process capability.

However, Mittag's analysis is restricted to considering the effects of measurement errors only in the behavior of theoretical capability indices. Bordignon and Scagliarini [3] considered the effect of measurement errors on the properties of the index $C_p$and $C_{pk}$when estimated from sample data. Scagliarini [40] studied the properties of the estimator of C_p for autocorrelated data contaminated with measurement errors. Pearn and Liao [33] considered estimating and testing of C_{pk} in the presence of gauge measurement error. Later Bordignon and Scagliarini [4] focused their attention on the behavior of the estimator of C_{pm} in presence of measurement error. Pearn and Liao [35] studied the estimation and testing of C_{pu} and C_{pl} in the presence of measurement error and obtained adjusted lower confidence bounds and critical values for true process capability. In a similar vein Hsu et al. [18] studied the third generation index C_{pmk} in the presence of measurement errors.

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The result of these analyses clearly emphasize that the presence of measurement errors in the data severely affects the assessment of process capability. In fact, the capability is underestimated. Conclusions inferred from process capability analysis without taking into consideration measurement errors are, therefore, *incorrect*.

In this paper, we have used the concept of Generalized Confidence Interval (GCI) to determine process capability of the Taguchi Index C_{pm} in the presence of measurement errors. The rest of the paper is organized as follows. We briefly introduce the Taguchi index C_{pm} in Section 2. Section 3 is concerned with the estimation of process capability in the presence of measurement errors. Next in Section 4, we turn our attention to the construction of the confidence interval for the index C_{pm} based on GCI method in presence of the measurement error. Section 5 deals with an exhaustive simulation study which has been done to assess the performance of the GCI method in terms of the coverage probability (CP) and the expected value of the generalized lower confidence limits. A sensitivity analysis is also carried out to analyze the effects of ignoring the measurement errors on the performance of the GCI method with various parameter settings. Numerical results are also analyzed and discussed in this section. A case study, discussed in Section 6, demonstrates the application of the proposed method. Finally some conclusions are drawn in Section 7.

2. The C_{pm} index

The index C_{pm} has been introduced in the published literature by Chan et al. [8]. However, it was proposed by Hsiang and Taguchi [17], though they did not use the symbol C_{pm}. Taguchi derived this index from his famous theory where he first introduced the concept of target and declares that deviation of the process characteristics from the target value (which is virtually deviation from quality) results in monetary loss. As Taguchi could relate the process performance with its monetary loss, different processes have become comparable. Taguchi's philosophy has widely been accepted in the domain of quality and process improvement as in most of the cases his philosophy has produced unexpectedly excellent results in terms of process improvement. The index which is sometimes called the Taguchi index or loss-based capability index, which was also proposed independently by Chan et al. [8] has already been defined in Eq. (1) above. When the target is not at the midpoint of USL and LSL then the Taguchi capability index is modified as in Eq. (2) above. This index is based on the idea of the squared error loss. It attempts to measure the ability of the process to cluster around the target. The C_{pm} index highlights on measuring the ability of the process on hitting the target, which therefore reflects the degree of process targeting (centering).

The point estimation of C_{pm} has been studied by Chan et al. [8] and Boyles [5]. The construction of (approximate) confidence bounds is more complicated and has been attempted by Marcucci and Beazley [26], Chan et al. [9] and Boyles [5].

3. Process Capability measures with measurement errors

Till now we have been concerned with the inbuilt process variability. This variability is inevitable in any manufacturing process. But if we are interested in estimating the capability of any manufacturing process, then we should be concerned about one more class of error namely the measurement error. In case of measurement error, the measurement system of measuring the process characteristics values takes a vital role along with the capability of the gauge by which the measurement is done. Though we may sometimes feel that gauge capability can be increased by doing regular calibration, it may not always be the case. Gauge capability reflects the gauge's precision, i.e. lack of variation, but it should not be confused with calibration, which assures the gauge's accuracy. As stated earlier, process capability is the measure of quantifying the ability of a manufacturing process to meet pre-assigned specifications. Nowadays, many customers use process capability to judge supplier's ability to deliver quality products. In this connection, reference may be made to Chen et al. [13], Chen and Chen [11], Pearn et al. [36] and Chen et al. [12]. Suppliers ought to be aware of how gauges affect various process capability estimates. It might happen that while estimating process capability from a sample, the process is erroneously classified as inefficient. This can happen since the process variance is inflated by the measurement errors, thereby causing a decrease in the value of the index.

The gauge capability includes two components of measurement systems' variability – repeatability and reproducibility. Repeatability represents the variability from the gauge or measurement instrument when it is used to measure the same specimen (with the same operator or setup or in the same time period). Reproducibility reflects the variability arising from different operators, setups time periods or in general, different conditions. These studies are often referred to as gauge repeatability and reproducibility (Gauge R&R) studies. To summarize we have:

$$\sigma^2_{gauge\ measurement\ error}(\sigma_G^2) = \sigma^2_{repeatability} + \sigma^2_{reproducibility}\,.$$

Estimates of $\sigma^2_{repeatability}$ and $\sigma^2_{reproducibility}$ come from a Gauge R&R study. Barrentine [2], Levinson [25] and Montgomery [30] have provided useful procedures for Gauge R&R studies.

The main objective of Gauge R&R study is to quantify the measurement errors. Two popular approaches to gauge R&R studies are the Range Method (Montgomery and Runger [31]) and the ANOVA Method (Montgomery and Runger [32] and Budrick et al. [6]). Both these methods assume that the distribution of the measurement errors is normal with a mean error of 0. Let the measurement errors be represented by a random variable, $G{\sim}N(\mu_G, \sigma_G^2)$. The mean μ_G is referred to as the bias of the measurement system. Typically, the bias can be eliminated by proper calibration of the system. Thus, we assume $\mu_G = 0$, so $G{\sim}N(0, \sigma_G^2)$.

Montgomery and Runger [31] presented the gauge capability λ by the following formula:

$$\lambda = \frac{6\sigma_G}{(USL - LSL)} \times 100\%\,.$$

The ratio of $6\sigma_G$ to the tolerance width has been referred by Montgomery [30] as the precision to tolerance (or P/T) ratio. For the measurement system to be considered acceptable, the variability in the measurements due to the measurement system should be less than a predetermined percentage of the manufacturing tolerance.

Let X be the quality characteristic of interest of a manufacturing process. Assume that $X \sim N(\mu, \sigma^2)$. Let C_{pm} measure the true process capability based on the random variable X. However in practice, instead of X, we estimate the process capability with the observed value $Y(= X + G)$. Assume that the quality characteristic X and the random variable G (which models the measurement error) are stochastically independent. Hence, we have $Y \sim N(\mu_Y,\ \sigma_Y^2)$ where, $\sigma_Y^2 = \sigma^2 + \ \sigma_G^2$ in this case. However, since the error distribution has mean zero, it follows that $Y \sim N(0,\ \sigma_Y^2)$. The empirical process capability index C_{pm}^Y will be obtained after substituting σ_Y^2 for $\sigma 2$. The relationship between true process capability and the empirical process capability C_{pm}^Y can be expressed as:

$$\frac{C_{pm}^Y}{C_{pm}} = \sqrt{\frac{\sigma^2 + (\mu - T)^2}{\sigma_Y^2 + (\mu - T)^2}} = \sqrt{\frac{\sigma^2 + (\mu - T)^2}{\sigma^2 + \sigma_G^2 + (\mu - T)^2}}$$

Since the variation of the observed data is larger than the variation of the original data, the denominator of the index C_{pm} becomes larger and the true capability of the process will be underestimated if calculations of PCI are based on the empirical data Y (the observed quantity contaminated with measurement errors). In case of C_{pk}, Pearn and Liao [34] indicated that the true process capability would be severely underestimated if instead of σ^2, σ_Y^2 (or its estimate S_Y^2) is used. Thus the risk,α, and the power of the test will decrease when the gauge measurement error increases. Since the true capability of a process is severely underestimated and the power becomes small, producers cannot firmly state that their processes meet the capability requirement even if their processes are indeed sufficiently capable. Adequate and even superior product units (or lots) could be incorrectly rejected in this case. In order to account for the measurement error while estimating the process capability index C_{pm}, Generalized Confidence Interval (GCI) technique can be applied. This technique is discussed in the next section.

4. Generalized Confidence Intervals (GCI) for index C_{pm}

We now turn our attention to the notions of the generalized pivotal quantity (GPQ) and generalized confidence interval (GCI) which are used in our paper. Tsui and Weerahandi [44] introduced the concept of Generalized Inference for testing hypothesis when exact method does not exist. Weerahandi [45] extended this concept to construct generalized confidence intervals. Construction of GCI requires a GPQ with a distribution that is free of the parameters under study. Approximate Confidence Intervals are then constructed by computing desired percentiles of GPQ using either numerical integration or simulation. The idea of GCI and generalized tests has been successfully exploited to obtain meaningful inference procedures in non-standard problems; see e.g. Ref. 10, 16, 14, 23, 24 & 27. Iyer and Patterson [2002] provided a generalized recipe for the construction of generalized test variable and GPQs. Hsu et al. [19] used this idea to construct generalized confidence intervals for the C_{pm} index. Pearn et al. [37] used the idea of GCI to assess process capability based on the C_{pmk} index; while Wu [46] used similar idea to study the C_{pk} index.

Before proceeding further, we list down the notations that will be subsequently used.

GCI	Generalized Confidence Interval
GPQ	Generalized Pivotal Quantity
R_μ	GPQ of μ
$R_{\sigma_Y^2}$	GPQ of σ_Y^2
$R_{C_{pm}}$	GPQ of C_{pm}(ignoring the measurement error)
$R_{C_{pm}^Y}$	GPQ of C_{pm}^Y (considering the measurement error)
L_{pm}	95% lower confidence interval of C_{pm}(ignoring the measurement error)

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L_{pm}^{Y} 95% lower confidence interval of C_{pm}^{Y} (considering the measurement error)
CP Coverage Probability
CP_{pm} Coverage Probability of L_{pm}
CP_{pm}^{Y} Coverage Probability of L_{pm}^{Y}

We shall now give some basic concepts of Generalized Confidence Interval (GCI) and Generalized Pivotal Quantity (GPQ). For details, see Ref. 20 and 7. These definitions and the above mentioned notations below will be used throughout this paper, unless otherwise stated. Construction of a GCI requires a GPQ with a distribution that is free of the model parameters.

Suppose we wish to construct a GCI for a model parameter θ. Let X represent a random variable with distribution function $F(x;\ \theta, \eta)$ where the observed value of X is x, θ is the parameter of interest and η is a nuisance parameter.

Definition 1: *Let $R = R(X, x; \theta, \eta)$ be a function of X, the observed value x, parameters θ and η. Then R is said to be a GPQ if it satisfies following conditions: –*

a) *For fixed x, the distribution of R is free from any unknown parameters;*
b) *The observed value of R, $R_{obs} = R(x, x; \theta, \eta)$, does not depend on nuisance parameters.*

Definition 2: *Let Θ be the parameter space of θ. If a subset $C_{1-\alpha}$ of the sample space of R satisfies $P\{R(X, x; \theta, \eta) \in C_{1-\alpha}\} = 1 - \alpha$, then the subset Θ_C of the parameter space given by $\Theta_{C(1-\alpha)} = \{\theta \in \Theta | R_{obs} \in C_{1-\alpha}\}$ is said to be a $100(1-\alpha)\%$ GCI for θ.*

As we have already stated that $Y \sim N(\mu, \sigma_Y^2 = \sigma^2 + \sigma_G^2)$, then $\bar{Y}$ and S_Y^2 are independent random variables with $\bar{Y} \sim N(\mu, \sigma_Y^2/n)$ and $(n-1)\, S_Y^2/\sigma_Y^2 \sim \chi_{n-1}^2(n-1)$ where μ and σ_Y^2 are unknown constants. The pair $(\bar{Y}, S_Y^2)$ may be viewed as the sample mean and sample variance (complete and sufficient statistics) from an i.i.d. $N(\mu, \sigma_Y^2)$ sample of size n. Set $\theta_1 = \mu$ and $\theta_2 = \sigma_y^2$. Consider the functions f_1 and f_2 defined by:

$$Z = f_1(\bar{Y}, S_Y^2;\ \mu, \sigma_Y^2) = \frac{\bar{Y} - \mu}{\sqrt{\sigma_Y^2/n}},$$

and

$$V = f_2(\bar{Y}, S_Y^2; \mu, \sigma_Y^2) = \frac{(n-1)S_Y^2}{\sigma_Y^2}.$$

Clearly, Z and V are independent, where Z has a standard normal distribution and V has a χ^2 distribution with $n-1$ degrees of freedom. Thus, the joint distribution of (Z, V) is free from any model parameters. Inverting the functions f_1 and f_2, we get:

$$\theta_1 = \mu = g_1(\bar{Y}, S_Y^2;\ Z, V) = \bar{Y} - Z\sqrt{\frac{(n-1)S_Y^2}{nV}},$$

and

$$\theta_2 = \sigma_Y^2 = g_2(\bar{Y}, S_Y^2; Z, V) = \frac{(n-1)S_Y^2}{V}.$$

In order to construct the GCI for μ and σ_Y^2, we first obtain the GPQs, R_μ and $R_{\sigma_Y^2}$ which are given by:

$$R_\mu = \bar{y} - Z\sqrt{\frac{(n-1)s_Y^2}{nV}} = \bar{y} - Z\sqrt{\frac{R_{\sigma_Y^2}}{n}} \tag{3}$$

and

$$R_{\sigma_Y^2} = \frac{(n-1)s_Y^2}{V} \tag{4}$$

where $\bar{y}$ and s_Y^2 are the observed value of $\bar{Y}$ and S_Y^2 respectively. Since $\sigma_G^2 = \{(USL - LSL)\lambda/6\}^2$is known, the GPQ of σ^2 (denoted by R_{σ^2}) can be obtained by:

$$R_{\sigma^2} = \max\left\{R_{\sigma_Y^2} - \sigma_G^2, \varepsilon\right\},$$

where ε is a small positive quantity used to ensure positive value of R_{σ^2}. Here $R_\mu$and $R_{\sigma_Y^2}$ are free from any unknown parameters. Thus, a GPQ for C_{pm} is given by:

$$R_{C_{pm}} = \frac{min(USL - T, T - LSL)}{3\sqrt{R_{\sigma^2} + (R_\mu - T)^2}}. \tag{5}$$

Considering measurement error GPQ of C_{pm}^Y is given by:

$$R_{C_{pm}^Y} = \frac{min(USL - T, T - LSL)}{3\sqrt{R_{\sigma_Y^2} + (R_\mu - T)^2}}. \tag{6}$$

And a $100(1-\alpha)\%$ generalized confidence limit of C_{pm}can be obtained by calculating the $100\alpha^{th}$ percentile of $R_{C_{pm}}$, $R_{C_{pm}}[\alpha]$,which satisfies $P\left(R_{C_{pm}} < R_{C_{pm}}[\alpha]\right) = \alpha$.

5. Numerical Results and Calculations

Here we have considered as USL = 20, LSL = 5 with a target T = 12.5 for our entire simulation study. For each degree of measurement errors (or gauge capability (λ = {0, 0.1, 0.2, 0.3, 0.4})), nine different combinations were run for the combinations of values μ = {12.5, 13, 13.5} and C_{pm} = {1, 1.25, 1.5}. For each simulation, a sample size of n = {25, 50, 75, 100} is drawn and 2000 samples are randomly generated. With the input values of $[C_{pm}, \mu, \lambda, n]$, using Eqn.(1) σ_Y^2 is calculated and a sample of size n is generated which follows $N(\mu, \sigma^2)$.

Corresponding to each sample, 5000 simulated values of (Z, V) are generated and $R_\mu$and $R_{\sigma_Y^2}$ are computed using Eqn.(3) & Eqn.(4). As $R_{\sigma^2} = \max\left\{R_{\sigma_Y^2} - \sigma_G^2, \varepsilon\right\}$, putting $\varepsilon = 0.001$ and value of σ_G^2 (recall that $\sigma_G^2 = \{(USL - LSL)\lambda/6\}^2$) we get the value of R_{σ^2}. Putting the value of Rμ, R_{σ^2} (or $R_{\sigma_Y^2}$) into the Eqn.(4) {or Eqn.(5)} we obtain the value of $R_{C_{pm}}$ or $R_{C_{pm}^Y}$. Thus 5000 $R_{C_{pm}}$ or $R_{C_{pm}^Y}$ values are obtained. L_{pm} is the lower $100\alpha^{th}$ percentile of these 5000 $R_{C_{pm}}$ values. L_{pm}-s are obtained by arranging these 5000 values of $R_{C_{pm}}$ in an increasing order and selecting the 251th of the sorted list. The process is repeated for 2000 times with different set of values of $(\bar{y}, s_Y^2)$. Thus 2000 L_{pm} (or L_{pm}^Y) are obtained corresponding to each run. Expected value of L_{pm}, i.e.$E(L_{pm})$is obtained by simply taking the average of these 2000 L_{pm} values for a single set of $[C_{pm}, \mu, \lambda, n]$. These 2000 L_{pm} values are arranged in descending order. The proportion of times L_{pm}< True C_{pm} is computed. This gives the Coverage Probability (CP).

The simulations have been carried out by using the Minitab 16 on PC platform.

Table 1: Expected value of Generalized Lower Confidence Limit for 95% confidence limit

Expected value of Generalized Lower Confidence Limit for 95% confidence limit											
C_{pm}	μ	n	$\lambda = 0$	$\lambda = 0.1$		$\lambda = 0.2$		$\lambda = 0.3$		$\lambda = 0.4$	
			$\mathrm{E}(L_{pm})$[a]	$\mathrm{E}(L_{pm})$	$E(L_{pm}^{y})$	$\mathrm{E}(L_{pm})$	$E(L_{pm}^{y})$	$\mathrm{E}(L_{pm})$	$E(L_{pm}^{y})$	$\mathrm{E}(L_{pm})$	$E(L_{pm}^{y})$
1	12.5	25	0.7563	0.7525	0.7502	0.7443	0.7356	0.7417	0.7227	0.7279	0.6969
		50	0.8286	0.8267	0.8239	0.8226	0.8113	0.8162	0.7920	0.8114	0.7704
		75	0.8621	0.8594	0.8562	0.8549	0.8424	0.8534	0.8262	0.8474	0.8016
		100	0.8819	0.8769	0.8735	0.8761	0.8627	0.8727	0.8438	0.8670	0.8184
	13	25	0.7580	0.7597	0.7574	0.7541	0.7451	0.7419	0.7230	0.7372	0.7049
		50	0.8284	0.8286	0.8257	0.8224	0.8111	0.8194	0.7949	0.8115	0.7705
		75	0.8617	0.8619	0.8586	0.8594	0.8467	0.8524	0.8253	0.8453	0.7999
		100	0.8813	0.8805	0.8771	0.8778	0.8644	0.8730	0.8441	0.8646	0.8163
	13.5	25	0.7687	0.7653	0.7629	0.7608	0.7516	0.7556	0.7357	0.7436	0.7108
		50	0.8372	0.8371	0.8341	0.8316	0.8200	0.8246	0.7996	0.8175	0.7756
		75	0.8682	0.8650	0.8617	0.8623	0.8495	0.8562	0.8288	0.8501	0.8039
		100	0.8845	0.8830	0.8796	0.8808	0.8672	0.8758	0.8466	0.8718	0.8225
1.25	12.5	25	0.9416	0.9372	0.9329	0.9278	0.9112	0.9149	0.8799	0.9026	0.8450
		50	1.0354	1.0277	1.0222	1.0248	1.0032	1.0117	0.9667	0.9982	0.9247
		75	1.0729	1.0728	1.0665	1.0656	1.0417	1.0585	1.0078	1.0457	0.9629
		100	1.1008	1.0996	1.0929	1.0924	1.0668	1.0876	1.0331	1.0713	0.9833
	13	25	0.9510	0.9492	0.9446	0.9365	0.9195	0.9223	0.8867	0.9023	0.8450
		50	1.0402	1.0415	1.0358	1.0323	1.0103	1.0181	0.9723	1.0019	0.9275
		75	1.0778	1.0760	1.0697	1.0731	1.0487	1.0609	1.0098	1.0477	0.9645
		100	1.1029	1.1001	1.0934	1.0941	1.0684	1.0844	1.0304	1.0751	0.9862
	13.5	25	0.9701	0.9636	0.9590	0.9577	0.9396	0.9434	0.9055	0.9241	0.8630
		50	1.0521	1.0545	1.0486	1.0412	1.0187	1.0298	0.9825	1.0194	0.9414
		75	1.0891	1.0842	1.0778	1.0850	1.0598	1.0707	1.0184	1.0550	0.9702
		100	1.1090	1.1067	1.0999	1.1018	1.0756	1.0967	1.0409	1.0808	0.9907
1.5	12.5	25	1.1294	1.1313	1.1237	1.1125	1.0842	1.0887	1.0308	1.0613	0.9706
		50	1.2425	1.2423	1.2325	1.2277	1.1911	1.2117	1.1364	1.1806	1.0638
		75	1.2946	1.2879	1.2771	1.2763	1.2358	1.2621	1.1786	1.2430	1.1102
		100	1.3199	1.3174	1.3059	1.3105	1.2670	1.2938	1.2047	1.2732	1.1321
	13	25	1.1401	1.1370	1.1292	1.1244	1.0952	1.1000	1.0408	1.0739	0.9803
		50	1.2480	1.2496	1.2397	1.2348	1.1976	1.2111	1.1357	1.1952	1.0743
		75	1.2989	1.2922	1.2813	1.2839	1.2427	1.2631	1.1794	1.2446	1.1113
		100	1.3209	1.3228	1.3113	1.3071	1.2640	1.2941	1.2050	1.2761	1.1342
	13.5	25	1.1911	1.1741	1.1657	1.1552	1.1238	1.1392	1.0742	1.1019	1.0020
		50	1.2775	1.2737	1.2632	1.2551	1.2163	1.2413	1.1610	1.2110	1.0861
		75	1.3155	1.3125	1.3012	1.2978	1.2555	1.2800	1.1933	1.2598	1.1221
		100	1.3373	1.3391	1.3271	1.3282	1.2831	1.3110	1.2188	1.2892	1.1438

[a] For $\lambda = 0, E(L_{pm}^{y}) = E(L_{pm})$

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Table 2: Coverage Probabilities of Generalized Lower Confidence Limit for 95% confidence limit

Coverage Probabilities of Generalized Lower Confidence Limit for 95% confidence limit											
C_{pm}	μ	n	$\lambda = 0$	$\lambda = 0.1$		$\lambda = 0.2$		$\lambda = 0.3$		$\lambda = 0.4$	
			CP_{pm}[a]	CP_{pm}	CP_{pm}^{y}	CP_{pm}	CP_{pm}^{y}	CP_{pm}	CP_{pm}^{y}	CP_{pm}	CP_{pm}^{y}
1	12.5	25	0.9675	0.9650	0.9655	0.9680	0.9765	0.9750	0.9865	0.9755	0.9925
		50	0.9605	0.9730	0.9765	0.9660	0.9775	0.9680	0.9870	0.9590	0.9920
		75	0.9630	0.9650	0.9690	0.9640	0.9730	0.9610	0.9875	0.9605	0.9970
		100	0.9510	0.9670	0.9710	0.9725	0.9825	0.9610	0.9895	0.9610	0.9970
	13	25	0.9670	0.9690	0.9700	0.9725	0.9785	0.9735	0.9875	0.9710	0.9880
		50	0.9680	0.9645	0.9665	0.9675	0.9800	0.9625	0.9870	0.9700	0.9945
		75	0.9650	0.9695	0.9725	0.9605	0.9765	0.9705	0.9875	0.9690	0.9965
		100	0.9640	0.9610	0.9655	0.9660	0.9800	0.9615	0.9900	0.9600	0.9960
	13.5	25	0.9675	0.9670	0.9690	0.9670	0.9750	0.9670	0.9795	0.9715	0.9945
		50	0.9560	0.9625	0.9660	0.9610	0.9745	0.9600	0.9805	0.9590	0.9890
		75	0.9535	0.9605	0.9655	0.9545	0.9750	0.9595	0.9900	0.9595	0.9940
		100	0.9555	0.9585	0.9665	0.9570	0.9760	0.9520	0.9895	0.9580	0.9970
1.25	12.5	25	0.9745	0.9715	0.9745	0.9750	0.9830	0.9710	0.9870	0.9735	0.9935
		50	0.9705	0.9720	0.9745	0.9640	0.9825	0.9715	0.9930	0.9665	0.9980
		75	0.9690	0.9660	0.9705	0.9700	0.9865	0.9660	0.9950	0.9695	0.9985
		100	0.9535	0.9625	0.9705	0.9625	0.9825	0.9655	0.9945	0.9580	0.9995
	13	25	0.9625	0.9630	0.9675	0.9705	0.9805	0.9725	0.9930	0.9760	0.9960
		50	0.9620	0.9605	0.9665	0.9630	0.9790	0.9620	0.9895	0.9655	0.9970
		75	0.9615	0.9710	0.9750	0.9635	0.9815	0.9585	0.9935	0.9655	0.9990
		100	0.9605	0.9660	0.9730	0.9650	0.9860	0.9630	0.9965	0.9560	0.9985
	13.5	25	0.9695	0.9725	0.9755	0.9655	0.9805	0.9705	0.9880	0.9680	0.9940
		50	0.9645	0.9550	0.9625	0.9715	0.9840	0.9610	0.9925	0.9525	0.9945
		75	0.9525	0.9590	0.9680	0.9495	0.9805	0.9625	0.9940	0.9520	0.9985
		100	0.9570	0.9655	0.9710	0.9605	0.9840	0.9510	0.9945	0.9610	0.9995
1.5	12.5	25	0.9720	0.9685	0.9735	0.9750	0.9855	0.9705	0.9910	0.9690	0.9975
		50	0.9660	0.9705	0.9775	0.9675	0.9845	0.9645	0.9955	0.9720	0.9990
		75	0.9625	0.9615	0.9735	0.9645	0.9875	0.9660	0.9980	0.9625	0.9995
		100	0.9640	0.9650	0.9720	0.9640	0.9875	0.9570	0.9985	0.9620	0.9995
	13	25	0.9700	0.9665	0.9690	0.9690	0.9840	0.9675	0.9905	0.9650	0.9995
		50	0.9710	0.9625	0.9700	0.9570	0.9790	0.9645	0.9935	0.9610	0.9985
		75	0.9615	0.9580	0.9725	0.9585	0.9860	0.9690	0.9975	0.9620	1.0000
		100	0.9625	0.9650	0.9700	0.9690	0.9925	0.9535	0.9990	0.9580	1.0000
	13.5	25	0.9655	0.9610	0.9660	0.9630	0.9800	0.9615	0.9890	0.9680	0.9965
		50	0.9560	0.9580	0.9635	0.9610	0.9865	0.9525	0.9945	0.9585	0.9985
		75	0.9520	0.9625	0.9750	0.9605	0.9860	0.9565	0.9975	0.9530	0.9995
		100	0.9590	0.9585	0.9670	0.9560	0.9880	0.9565	0.9995	0.9630	1.0000

[a]For $\lambda = 0, CP_{pm}^{y} = CP_{pm}$

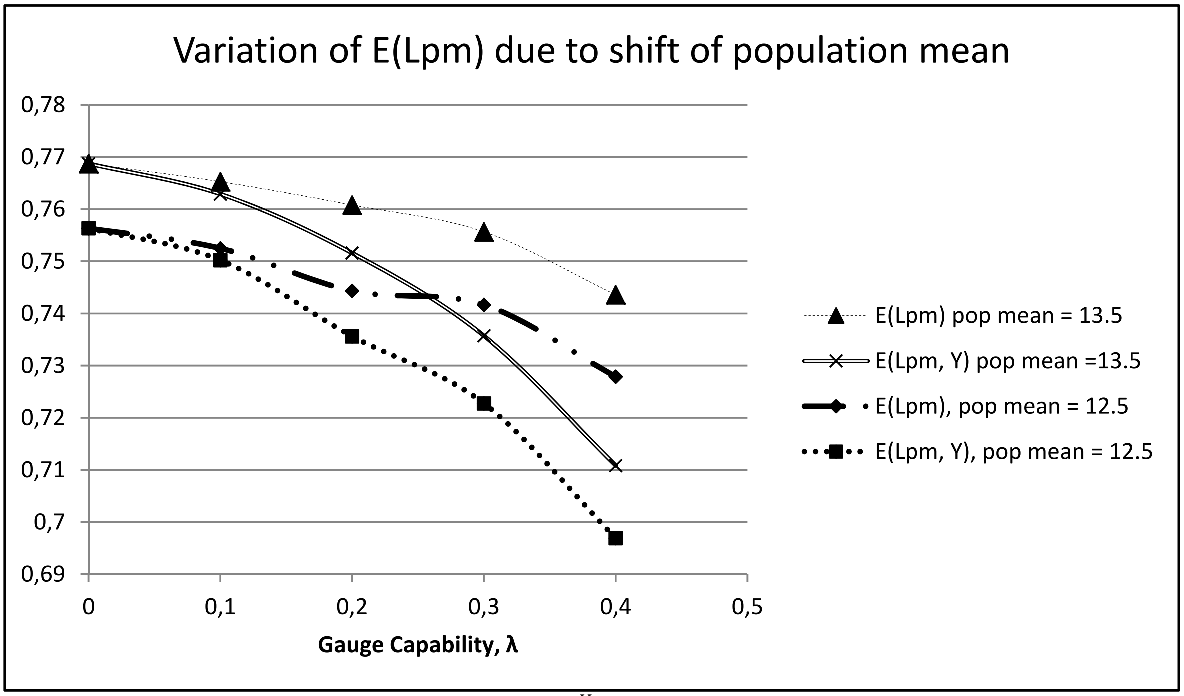

Fig 1: E(L_{pm}) and E(L_{pm}^{Y}) for $C_{pm} = 1$, n = 25

From Fig. 1 it is clear that as the population mean deviates more from the Target value (T), the expected value of C_{pm} (or L_{pm}^{Y}) tends to increase. This can be explained by the lower variance of the process to maintain the same value of true C_{pm} as it deviates from the target value.

It is also very clear that both $E\left(L_{pm}\right)$ and $E\left(L_{pm}^{Y}\right)$ decrease with the increase of λ. This can be explained as follows:

From Eq. (2) we get:

$$C_{pm} = min\left(\frac{USL - T}{3\sqrt{\sigma_Y^2 - \sigma_G^2 + (\mu - T)^2}}, \frac{T - LSL}{3\sqrt{\sigma_Y^2 - \sigma_G^2 + (\mu - T)^2}}\right)$$

or

$$\sqrt{\sigma_Y^2 - \sigma_G^2 + (\mu - T)^2} = \frac{\min(USL - T, T - LSL)}{3C_{pm}}$$

or

$$\sigma_Y^2 = \left[\frac{\min(USL - T, T - LSL)}{3C_{pm}}\right]^2 + \sigma_G^2 - (\mu - T)^2$$

or

$$\sigma_Y^2 = \left[\frac{\min(USL-T, T-LSL)}{3C_{pm}}\right]^2 + \left[\frac{(USL-LSL)\lambda}{6}\right]^2 - (\mu - T)^2$$

From the above equation it is very clear that as λ increases σ_Y^2 also increases. Since $Y \sim N(\mu,\ \sigma_Y^2)$, with the increase of λ, the variability of the process increases leading to lower C_{pm}. Hence lower $E(L_{pm})$ is obtained.

From Fig.1, it is been also observed that as the λ increases, the difference between the $E(L_{pm})$ and $E(L_{pm}^Y)$ also increases. Maximum attainable value for $E(L_{pm}^Y)$ is $E(L_{pm})$ at $\lambda = 0$. That can be explained as follows:

$$\frac{R_{C_{pm}^Y}}{R_{C_{pm}}} = \frac{\sqrt{R_{\sigma^2} + (R_\mu - T)^2}}{\sqrt{R_{\sigma_Y^2} + (R_\mu - T)^2}} = \frac{\sqrt{R_{\sigma^2} - \sigma_G^2 + (R_\mu - T)^2}}{\sqrt{R_{\sigma_Y^2} + (R_\mu - T)^2}}.$$

As λ increases, σ_G^2 also increases which in turn decreases ratio $R(C_{pm}^Y)/R(C_{pm})$.Thus with the increase of λ, $E(L_{pm}^Y)/E(L_{pm})$decreases.We also observe from Fig. 2 that as λ increases $E(L_{pm})$ decreases steadily. Also we can observe that with the increase of sample size(n) the value of $E(L_{pm})$ increases.

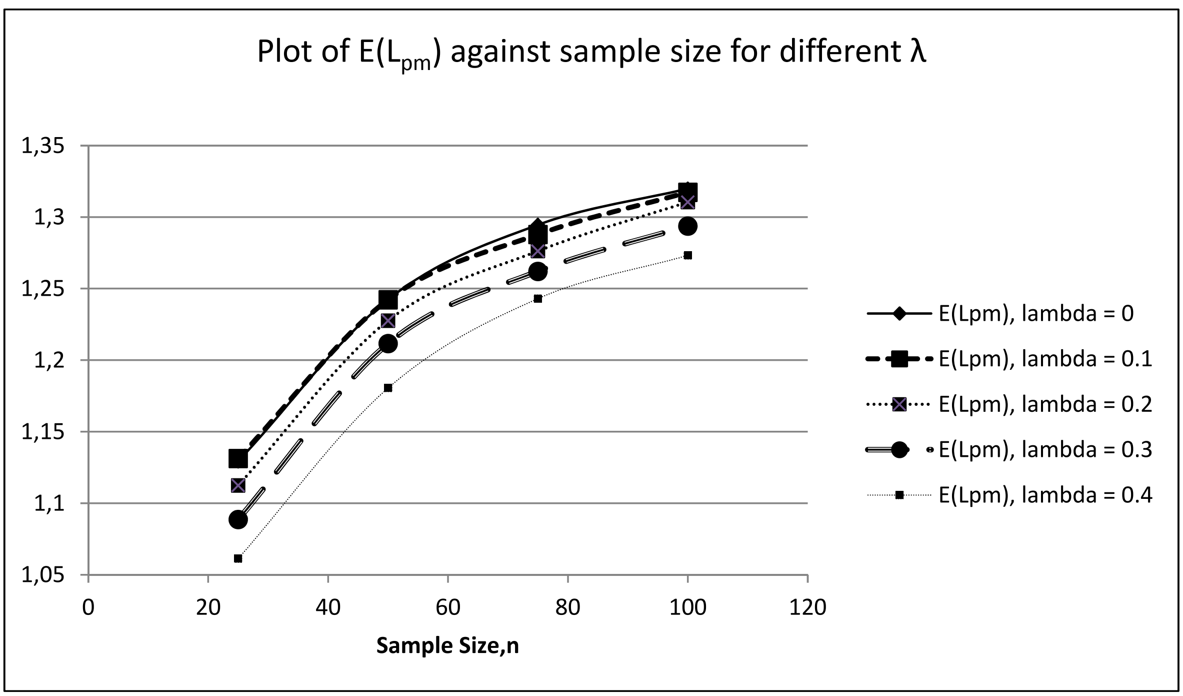

Fig 2: E(L_{pm}) for C_{pm} = 1.5, μ = 12.5

6. Case Study

A real-world problem is implemented with data collected from a manufacturing factory, which is located on the Science-Based Industrial Park in Taiwan. The data is reported in Pearn et al. [2009]. This manufacturing company produces a variety of optoelectronic devices including the Light Emitting Diodes (LEDs). An essential product characteristic for LEDs is the luminous intensity, which has a significant impact on product quality. For luminous intensity of a particular model of LED, the upper specification (USL) is set to 13,800 mcd [Milli-candela (mcd) is a measurement unit of brightness or light intensity]; the lower specification (LSL) is set to 6,200 mcd. The target (T) is set to 10,000 mcd. Consider the LED manufacturing process has process mean (μ) =10.1 and process standard deviation $(\sigma) = 0.5.$ Also consider the luminous intensity follows normal distribution. A total of 120 observations collected from a stable process in the factory are displayed in Table 3.

In this case suppose that the capability requirement is defined 'satisfactory' if $C_{pm} \geq$ 1.3. Also consider that gauge capability (λ) is equal to 0.2. The above table shows a sample data of 120 observations. The above data has a sample mean of 10.6462 and standard deviation = 0.5254.

Table 3: Data for Case Study

10.19	10.75	9.51	10.31	10.77	11.74	11.13	10.42	11.14	12.14	10.19	10.12
11.02	9.84	10.45	10.93	10.47	11.34	11.76	11.41	10.79	11.46	10.94	10.74
11.45	10.76	10.69	9.84	11.24	10.01	10.66	10.85	10.59	11.25	10.84	10.45
10.46	8.85	10.24	10.92	10.99	10.5	11.03	10.24	11.02	10.65	10.87	10.14
10.29	9.89	11.81	10.72	11.08	10.62	10.78	11.03	9.95	10.52	11.04	10.58
10.87	11.14	10.48	10.28	10.44	9.85	10.19	10.12	11.22	11.23	10.74	10.41
10.86	10.76	10.78	11.35	11.21	10.86	10.85	10.39	10.03	9.99	10.61	10.34
10.55	10.79	10.75	10.15	10.97	9.98	10.42	11.37	10.94	10.4	11.33	11.15
10.33	10.78	11.07	10.05	10.11	9.85	10.69	9.92	10.31	9.79	11.16	10.36
9.9	10.71	10.96	10.96	10.4	11.24	10.94	9.75	11.19	10.03	10.38	10.45

In this case suppose that the capability requirement is defined 'satisfactory' if $C_{pm} \geq$ 1.3. Also consider that gauge capability (λ) is equal to 0.2. The above table shows a sample data of 120 observations. The above data has a sample mean of 10.6462 and standard deviation = 0.5254.

Under the hypothesis that the above data follows a normal distribution the Anderson-Darling, Kolmogorov-Smirnov & Ryan-Joiner tests yields the following results:

Anderson-Darling Test	Kolmogorov-Smirnov Test	Ryan-Joiner Test
AD = 0.267	KS = 0.054	RJ = 0.994
P = 0.682	p>=0.15	p>= 0.10

So each of the above test confirms our normality our assumption. The MLE estimates of μ and σ are given by $\hat{\mu} = 10.65$ and $\hat{\sigma} = 0.5254.$ Hence, the point estimation of C_{pm} based on the empirical data Y (the observed values contaminated with measurement errors) is $\hat{C}_{pm}$= 1.27624. The true C_{pm} of this manufacturing process ($\mu = 10.1$ and $\sigma = 0.5$) would be 2.13333(>1.3). Clearly the process is fairly satisfactory. But as the true process mean and process variance is never known, the true process C_{pm} can never be computed. Using GCI methods for 5000 replications for the GPQs of C_{pm} , if we ignore the gauge measurement errors, the process would be considered *incapable* since the calculated lower 95% confidence limit of C_{pm} is 1.26535(<1.3). That is the process is determined to be '*unsatisfactory*' since the lower confidence of C_{pm} is lower than the preset process capability requirement of 1.3. In contrast to that, if we consider the measurement error, and compute the 95% generalized confidence limit of C_{pm}(i.e. L_{pm}) we get L_{pm}= 1.30793(>1.3). Then we will be able to correctly judge the process as '*satisfactory*'. Therefore the process would have been incorrectly rejected in this case if we did not take gauge measurement errors into consideration.

7. Conclusion

A substantial majority of research works on process capability analysis that has appeared in the literature do not take into account measurement errors. Unfortunately, measurement errors cannot be avoided in many branches of manufacturing industry but they can be reduced by improving the gauge measurement and properly training the operators. If the producers do not take into account the effects of the gauge capability on estimating and testing process capability, it may often lead to incorrect decisions and resulting serious loss. This paper applies the GCI concept to measure process capability based on the second generation index C_{pm} in the presence of measurement errors. An exhaustive simulation has been conducted to assess the performance of GCI methods in terms of the CP and the expected value of the generalized lower confidence limit. Moreover, a sensitivity study was also carried out to analyze the effects of ignoring measurement errors on the performance of the GCI with various parameter settings. The result indicate that GCI method appears quite satisfactory for assessing process capability when measurement errors are present.

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