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A Hidden Group Structure for the Integrals of the Benney System

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Abstract

The integrals of the Benney system are shown to possess a group structure. The KP hierarchy breaks the group law down.

1 Introduction

The remarkable $2+1$-dimensional *integro-differential* free-surface hydrodynamical system derived by Benney [1],

$$u_t = uu_x + gh_x - u_y \int_0^y u_x dy' \tag{1.1a}$$

$$h_t = (\int_0^h udy)_x, \tag{1.1b}$$

was shown by him to result in a purely *differential* evolutionary system

$$A_{n,t} = A_{n+1,x} + gnA_{n-1}A_{0,x}, \quad n \in \mathbf{Z}_{\geq 0}, \tag{1.2}$$

for the moments

$$A_n = \int_0^h u^n dy, \quad n \in \mathbf{Z}_{\geq 0}. \tag{1.3}$$

Here $u = u(x, y, t)$ is the horizonal component of velocity at time t, at the point, $x, y;\ 0 \leq y \leq h = h(x, t), h$ being the elevation of the free surface over the flat bottom $\{y = 0\}$; subscripts x and t denote the corresponding partial derivatives.

Benney showed that the system (1.2) is *integrable*: it has an infinite sequence of polynomial conserved densities $H_n \in A_n + \mathbf{Z}[A_0, ..., A_{n-2}, g]$:

$$H_0 = A_0, \tag{1.4a}$$

$$H_1 = A_1, \tag{1.4b}$$

$$H_2 = A_2 + gA_0^2, \tag{1.4c}$$

$$H_3 = A_3 + 3gA_0A_1, \tag{1.4d}$$

$$H_4 = A_4 + 4gA_0A_2 + 2gA_1^2 + 2g^2A_0^3, \tag{1.4e}$$

$$H_5 = A_5 + 5gA_0A_3 + 5gA_1A_2 + 10g^2A_0^2A_1, ... \tag{1.4f}$$

Subsequently, Manin and myself showed [5,6] that the Benney integrals are in involution with respect to the Hamiltonian structure

$$B_{nm} = nA_{n+m-1}\partial + \partial m A_{n+m+1}, \quad \partial = \partial/\partial x. \tag{1.5}$$

Denote the map sending the A's to the H's by φ^g:

$$\boldsymbol{H} = \varphi^g(\boldsymbol{A}) \tag{1.6}$$

The purpose of this note is to show that the map φ^g represents, in fact, a *group law*:

$$\varphi^h \circ \varphi^g = \varphi^{g+h}. \tag{1.7}$$

This formula is proven in the next Section. Section 3 discusses a more general perspective, in particular why the law breaks down when one allows the dispersion in, i.e., for the KP flow #2, whose quasiclassical limit is the Benney system (1.2).

2 The group law

Let u be the generator of the ring $K((u^{-1}))$,

$$K = \boldsymbol{Q}[g][\boldsymbol{A}], \tag{2.1}$$

and consider the equation on v:

$$g^{-1}u = v + \sum_{i=0}^{\infty}(gv)^{-i-1}A_i. \tag{2.2a}$$

Solving this equation, let's write

$$v = g^{-1}u - \sum_{i=0}^{\infty} H_i u^{-i-1}. \tag{2.2b}$$

Thus,

$$v_t = -\sum H_{i,t}u^{-i-1}. \tag{2.3}$$

Let us verify that the H_n's are exactly the conserved densities of the Benney system (1.2). Proceeding exactly like in [5], we first verify that

$$H_n \in A_n + \boldsymbol{Q}[A_0, ..., A_{n-1}, g]. \tag{2.4}$$

Next, differentiating (2.2a) with respect to u, t, and x, and combining the results, we find that

$$v_t = (g\frac{v^2}{2} + A_0)_x, \tag{2.5}$$

as expected.

We now change perspective and look at the formulae (2.2), as simply a pair of inversion formulae: the A_i's and the H_i's are considered as free variables, without any dependence upon parameters such as x and t. Thus, $\boldsymbol{H} = \varphi^g(\boldsymbol{A})$.

Set, for the map $\varphi^h(\boldsymbol{H})$, the pair of inversion formulae

$$h^{-1}x = z + \sum_{i=0}^{\infty} (hz)^{-i-1} H_i, \tag{2.6a}$$

$$z = h^{-1}x - \sum_{i=0}^{\infty} x^{-i-1} G_i, \tag{2.6b}$$

and for the map $\varphi^{h+g}(\boldsymbol{A})$, the pair

$$(h+g)^{-1}x = y + \sum_{i=0}^{\infty} [(h+g)y]^{-i-1} A_i, \tag{2.7a}$$

$$y = (h+g)^{-1}x - \sum_{i=0}^{\infty} x^{-i-1} \tilde{G}_i. \tag{2.7b}$$

We want to show that

$$G_i = \tilde{G}_i, \quad i \in \mathbf{Z}_{\geq 0}. \tag{2.8}$$

Comparing formula (2.6b) and (2.7b), (2.8) becomes:

$$z - h^{-1}x = y - (h+g)^{-1}x, \tag{2.9}$$

or

$$z = y + \frac{g}{h(h+g)}x. \tag{2.10}$$

Thus, our original system $\{(2.2, 6, 7)\}$ is reduced to $\{2.2, 6a, 7a, 10)\}$.
Next, set

$$u = hz \text{ [by (2.10)]} = hy + \frac{g}{h+g}x. \tag{2.11}$$

The equations (2.2b, 6a) are then replaced by

$$g^{-1}u - v = h^{-1}x - z, \tag{2.12}$$

or, with the help of formula (2.10), by

$$v = \frac{h+g}{g}y. \tag{2.13}$$

Our new system is now $\{(2.2a, 7a, 11, 13)\}$.

Since, by formula (2.13),

$$gv = (h+g)y, \tag{2.14}$$

the equations (2.2a) and (2.7a) are compatible iff

$$g^{-1}u - v = (h+g)^{-1}x - y, \tag{2.15}$$

which formulae (2.11, 13) imply at once. We are done.

Remark 2.16. With hindsight available, the symmetry $\boldsymbol{A} \to -\boldsymbol{H}$ described in §1.4 in [5] is a particular case

$$g = -h = 1 \tag{2.17}$$

of our general group law (1.6).

3 Generalizations

As shown by Manin and Lebedev in [7], the Benney hierarchy constructed in [5,6], is the zero-dispersion limit of the KP hierarchy (see, e.g., [2]). The Benney flow (1.2) is the quasiclassical limit of the second KP flow (g-scaled)

$$A_{n,t} = \frac{1}{2} g A_n^{(2)} + A_{n+1}^{(1)} + \sum_{r \geq 0} \binom{n}{r+1} (-1)^r A_{n-1-r} A_0^{(r+1)} g^{r+1}, \tag{3.1}$$

$$(\cdot)^m = \partial^m(\cdot). \tag{3.2}$$

Surprisingly enough, the group law (1.6) breaks down under the influence of dispersion. (This can be seen already for the integral H_4). I conjecture that the underlying reason is that the KP hierarchy is not the right hierarchy whose quasiclassical limit is the Benney hierarchy: the KP ansatz is stripped so much off its natural gauge degrees of freedoms as to become too rigid. This conclusion can be arrived at by analyzing Wilson's second construction [8].

On the other hand, in purely hydrodynamic (zero-dispersion) case, the group law appears to hold (sometimes in modified form) for many hydrodynamical chains described in [3,4]. This subject is now under development.

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