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On an inverse scattering algorithm for the Camassa-Holm equation

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Abstract

We present a clarification of a recent inverse scattering algorithm developed for the Camassa-Holm equation.

1 Introduction

The nonlinear partial differential equation

$$q_t + uq_x + 2qu_x = 0, \quad t > 0, \quad x \in \mathbb{R}, \tag{1.1}$$

where $q = u - u_{xx} + \omega = m + \omega$ and $\omega > 0$, has been derived as a model for the unidirectional propagation of two-dimensional waves in the shallow water approximation, see [4, 20, 19, 12]. The above is called the *Camassa-Holm equation* (CH) and it has attracted a lot of interest due to its complete integrability as an infinite dimensional Hamiltonian system and to the fact that it admits classical solutions that exist for all times and also solutions that develop singularities in finite time [5, 24]. If a singularity develops, the solution remains bounded but its slope becomes unbounded — a feature characterizing breaking waves cf. the discussion in [7]. The CH equation is also a re-expression of the geodesic flow on the Bott-Virasoro group, see [13]. It has been recently suggested that (1.1) might play a rôle in the modeling of tsunamis [22, 9].

The integration of equation (1.1) by means of the scattering problem approach has been undertaken in [6] based on earlier results [5] about global existence or wave-breaking of its solutions. Several difficulties with respect to the classical scattering approach developed for the Korteweg-de Vries (KdV) equation — see [1, 2, 18] — are presented in [15, 14, 6, 10, 11, 21, 23]. As detailed in these references, the inverse scattering transform (IST) problem of the Camassa-Holm equation is a complicated issue. An algorithm has been proposed by Constantin and Lenells in [10] and slightly modificated in [11] to solve the inverse scattering problem for the CH equation. While other approaches were subsequently introduced [8], this method turns out to be quite effective in giving a closed form to the CH solitary waves. We recall that, in contrast to the KdV equation [18], it is very difficult to find explicit expressions for the solitary waves of equation (1.1). It is also important to recall that the solitary waves of the CH equation are stable solitons, cf. [16, 21].

Our intention in this Letter is to clarify a certain fundamental aspect of the Constantin-Lenells algorithm: *it is impossible to reduce the IST computations to solving a linear integral equation with an integral given only over one semi-axis* — see the equation of the auxiliary function w in [10, p. 434] and the equation of the Jost function f in [11, p. 254, Remark]. Based on a simple analytical argument developed in Section 2, we give a rigorous proof of the Constantin-Lenells algorithm.

2 Global existence of solutions to a linear ODE

Our aim in this Section is to provide certain sufficient conditions regarding the continuous function $f : \mathbb{R} \to \mathbb{R}$ such that, given $\gamma > 0$, the ordinary differential equation (ODE) below

$$\frac{d^2x}{dt^2} - [\gamma + f(t)]x = 0, \qquad t \in \mathbb{R}, \tag{2.1}$$

has a positive solution $x(t)$ defined throughout the entire real line and satisfying the restriction

$$\int_{\mathbb{R}} \frac{dt}{x^2(t)} < +\infty. \tag{2.2}$$

Such conditions, in various formulations, are known for the simpler case of *non-principal* solutions, namely solutions $x(t)$ of (2.1) which verify the restriction $\int_{t_0}^{+\infty} \frac{dt}{x^2(t)} < +\infty$ for some t_0 large enough, see [17]. Unfortunately the techniques there cannot be extended to deal with the problem (2.1), (2.2) although they are perfectly capable of dealing with solutions that verify the "mirror" condition $\int_{-\infty}^{-t_0} \frac{dt}{x^2(t)} < +\infty$ for some t_0 large enough – via the obvious change of variable $y(t) = x(-t)$ for all $t \le -t_0$. The latter solutions shall be referred to in the sequel also as non-principal solutions.

To simplify the computations, notice that the substitutions $s = \sqrt{\gamma}t$, $X(s) = x(t)$, $F(s) = \frac{f(t)}{\gamma}$ transform equation (2.1) into $\frac{d^2X}{ds^2} - [1 + F(s)]X = 0$, $s \in \mathbb{R}$. We remark also that $\int_{\mathbb{R}} |F(s)|ds = \frac{1}{\sqrt{\gamma}} \int_{\mathbb{R}} |f(t)|dt$. This is the reason for focusing here only on the case $\gamma = 1$ of the ODE (2.1).

Theorem 1. *Assume that $\gamma = 1$ and $||f||_{L^1(\mathbb{R})} < \frac{1}{4}$. Fix the numbers c, $d > 0$ such that*

$$c > 2\int_0^{+\infty} |f(s)|ds, \quad d > 2\int_{-\infty}^0 |f(s)|ds, \quad c + d < \frac{1}{2}. \tag{2.3}$$

Then, the equation (2.1) has a solution $x(t)$ which verifies (2.2). Furthermore, we have the estimates:

(i) $x(t) \sim \frac{1}{2}[c + x_+(t)]e^t$, with $|x_+(t)| \le \int_0^{+\infty} |f(s)|ds$, when $t \to +\infty$;
(ii) $x(t) \sim \frac{1}{2}[d + x_-(t)]e^{-t}$, with $|x_-(t)| \le \int_{-\infty}^0 |f(s)|ds$ when $t \to -\infty$;
(iii) $x(t) \ge x_\star \cosh t$ for all $t \in \mathbb{R}$, where $x_\star = \min\{c_\star, d_\star\}$, where

$$c_\star = c - 2\int_0^{+\infty} |f(s)|ds, \quad d_\star = d - 2\int_{-\infty}^0 |f(s)|ds.$$

Before proving the theorem, let us remark that the L^1–norm of f cannot be as large as desired. In fact, consider the function

$$f(t) = \begin{cases} 0, \ t \le a-b, \\ -\frac{1+\varepsilon}{b}(t+b-a), \ t \in [a-b, a], \\ -(1+\varepsilon), \ t \in \left[a, a+\frac{2\pi}{\sqrt{\varepsilon}}\right], \\ \frac{1+\varepsilon}{b}(t-a-b-\frac{2\pi}{\sqrt{\varepsilon}}), \ t \in \left[a+\frac{2\pi}{\sqrt{\varepsilon}}, a+\frac{2\pi}{\sqrt{\varepsilon}}+b\right], , \\ 0, \ t \ge a+\frac{2\pi}{\sqrt{\varepsilon}}+b, \end{cases}$$

where b, $\varepsilon > 0$ and a are fixed. The unique (global) solution x of equation (2.1) with the Cauchy data

$$x(a) = \cos(a\sqrt{\varepsilon}), \quad x'(a) = -\sqrt{\varepsilon}\sin(a\sqrt{\varepsilon})$$

verifies the identity

$$x(t) = \cos(t\sqrt{\varepsilon}), \quad t \in I = \left[a, a+\frac{2\pi}{\sqrt{\varepsilon}}\right].$$

Notice that *the preceding formula for $x(t)$ is valid only in I!* It is obvious that the function x has at least two zeros in I. According to Sturm's comparison theorem, all the solutions of equation (2.1) will have at least one zero in I – meaning that (2.2) is out of the question although some of the solutions are non-principal. We have $||f||_{L^1(\mathbb{R})} = (1+\varepsilon)\frac{2\pi}{\sqrt{\varepsilon}}+(1+\varepsilon)b > 4\pi$. Though it is possible to improve the upper bound of the L^1–norm of f from Theorem 1, we have opted for $\frac{1}{4}$ because of the simplicity of our analytical argument.

Proof. Consider the complete metric space $M = (X, d)$ given by the set $X = \{x \in C(\mathbb{R}, \mathbb{R}) : |x(t)| \le \cosh t, \, t \in \mathbb{R}\}$ and the distance $d(x_1, x_2) = \sup\limits_{s\in\mathbb{R}} \frac{|x(s)|}{\cosh s}$. Notice the simple inequality $\mathrm{e}^t \ge \cosh t$ for all $t \ge 0$. This helps establish that, given $x \in X$, one has the estimates $|x(t)|\mathrm{e}^{-t} \le 1$ for all $t \ge 0$ and $|x(t)|\mathrm{e}^{t} \le 1$ for all $t \le 0$.

Introduce the operator $T : X \to C(\mathbb{R}, \mathbb{R})$ with the formula

$$T(x)(t) = \frac{1}{2}\left[c - \int_t^{+\infty} f(s)x(s)\mathrm{e}^{-s}ds\right]\mathrm{e}^t + \frac{1}{2}\left[d - \int_{-\infty}^t f(s)x(s)\mathrm{e}^{s}ds\right]\mathrm{e}^{-t}$$

for $t \in \mathbb{R}$ and $x \in X$.

Several estimates are needed to ensure that $T(X) \subseteq X$. Notice that

$$\begin{aligned} \left|\int_{-\infty}^t f(s)x(s)\mathrm{e}^s ds\right| &\le \int_{-\infty}^0 |f(s)|ds + \int_0^t |f(s)|\mathrm{e}^{2s}ds \\ &\le \int_{-\infty}^0 |f(s)|ds + \frac{1+\mathrm{sgn}\ t}{2}\mathrm{e}^{2t} \cdot \int_0^{+\infty} |f(s)|ds \end{aligned}$$

and

$$\left|\int_t^{+\infty} f(s)x(s)\mathrm{e}^{-s}ds\right| \le \int_0^{+\infty} |f(s)|ds + \frac{1+\mathrm{sgn}\ (-t)}{2}\mathrm{e}^{-2t} \cdot \int_{-\infty}^0 |f(s)|ds$$

and, based upon these two,

$$
\begin{aligned}
|T(x)(t)| &\leq \left[c + d + 2\int_{\mathbb{R}} |f(s)|ds\right] \cdot \cosh t \\
&\leq 2(c+d)\cosh t.
\end{aligned}
$$

The operator T has the set X as an invariant set.

Similarly, we deduce that $d(T(x_1), T(x_2)) \leq \eta \cdot d(x_1, x_2)$ for all $x_1,\ x_2 \in X$, where $\eta = 2\int_{\mathbb{R}} |f(s)|ds < \frac{1}{2}$.

The operator T is a contraction, so it has a fixed point $x = x(t)$ in X. This is the solution of the problem (2.1), (2.2) which we are looking for. To prove that (2.2) holds true, notice that it is a simple consequence of the conclusions (i) – (iii). These conclusions, on the other hand, can be established easily by manipulating the previous computations.

We have the estimate

$$
\begin{aligned}
x(t) &= T(x)(t) \\
&\geq \frac{1}{2}\left[c - \int_0^{+\infty} |f(s)|ds - \mathrm{e}^{-2t}\int_{-\infty}^0 |f(s)|ds\right]\mathrm{e}^t \\
&+ \frac{1}{2}\left[d - \int_{-\infty}^0 |f(s)|ds - \mathrm{e}^{2t}\int_0^{+\infty} |f(s)|ds\right]\mathrm{e}^{-t} \\
&= \frac{c_\star}{2}\cdot \mathrm{e}^t + \frac{d_\star}{2}\cdot \mathrm{e}^{-t} \geq x_\star \cosh t
\end{aligned}
$$

and so $\int_{\mathbb{R}} \frac{dt}{x^2(t)} \leq \frac{1}{x_\star^2}\int_{\mathbb{R}} \frac{dt}{\cosh^2 t} = \frac{2}{x_\star^2}$. The proof is complete. ■

3 The IST algorithm

Using the notations from [6, p. 967], we are concerned here with the nonlinear ODE ($\omega > 0$)

$$
\frac{d^2C}{dy^2} = \left(Q(y) + \frac{1}{4\omega}\right)C - \frac{1}{4C^3}, \quad y \in \mathbb{R}, \tag{3.1}
$$

where the (given) continuous function $Q : \mathbb{R} \to \mathbb{R}$ is restricted by the Faddeev condition $\int_{\mathbb{R}}(1+|y|)|Q(y)|dy < +\infty$, and we look for a solution $C_\star$ such that $\lim\limits_{|y|\to+\infty} C_\star(y) = \sqrt[4]{\omega}$.

The interest in solving (3.1) comes from the next representation, see [10, p. 433], of the potential Q, namely $Q(y) = \frac{1}{4q(y)} + \frac{q''(y)}{4q(y)} - \frac{3{q'}^2(y)}{16q^2(y)} - \frac{1}{4\omega}$ throughout the entire real line, where $q(y) = \sqrt[4]{C_\star(y)}$.

Besides this demanding asymptotic behavior imposed on a solution of (3.1), a further intricacy is provided by the change of variables

$$
\begin{cases}
y = \sqrt{\omega}x + \int_{-\infty}^x [\sqrt{m(\xi)+\omega} - \sqrt{\omega}]d\xi, \\
x = \frac{y}{\sqrt{\omega}} + \int_{-\infty}^y \left[\frac{1}{\sqrt{q(\xi)}} - \frac{1}{\sqrt{\omega}}\right] d\xi,
\end{cases}
$$

where $C^\star = \int_{\mathbb{R}}[\sqrt{m(\xi)+\omega} - \sqrt{\omega}]d\xi = \sqrt{\omega}\cdot\int_{\mathbb{R}}\left[\frac{1}{\sqrt{\omega}} - \frac{1}{\sqrt{q(\xi)}}\right]d\xi$ is a constant of motion (to avoid complications, we refrain from introducing the time notation in our discussion). The change of variables is used for recovering the momentum $m(x) = q(y(x)) + \omega$, $x \in \mathbb{R}$.

The algorithm devised by Constantin and Lenells [10] attacks both issues with success.

The first step consists of giving a *computable representation* to the solutions of (3.1).

This is a particular case of the Bohl-type representation problem, see [3, 25]: if $\psi_1(y)$, $\psi_2(y)$ are linearly independent solutions of the linear ODE

$$[r(y)\psi']' + p(y)\psi = 0, \quad y \in \mathbb{R}, \tag{3.2}$$

with the Wronskian $r(y)[\psi_1'(y)\psi_2(y) - \psi_1(y)\psi_2'(y)] = W > 0$, then the general solution of (3.2) can be written as

$$\psi(y) = AC(y)\cosh\left(\sqrt{\alpha}W \cdot \int_{y_0}^{y} \frac{ds}{r(s)C^2(s)} + B\right),$$

(or, we can use "sinh") where A, $B \in \mathbb{R}$ are the integration constants, $C(y) = \sqrt{\psi_1^2(y) - \alpha\psi_2^2(y)}$ for a fixed $\alpha > 0$ and

$$[r(y)C']' + p(y)C = -\frac{\alpha W^2}{r(y)C^3}, \qquad y \in \mathbb{R}. \tag{3.3}$$

It is now obvious that the equation (3.3) reads exactly as the equation (3.1) for $r(y) = 1$, $p(y) = -\left(Q(y) + \frac{1}{4\omega}\right)$, $\alpha = \frac{1}{4W^2}$.

One notices also that the argument of "cosh" in the representation of $\psi(y)$ from equation (3.2), namely the quantity $\sqrt{\alpha}W \cdot \int_{y_0}^{y} \frac{ds}{r(s)C^2(s)} + B$, is actually an antiderivative of $\frac{\sqrt{\alpha}W}{r(y)C^2(y)}$ and so it can be recast conveniently! In fact, based on this remark, the computable representation of $C(y)$ is provided by the formulas (recall that $\sqrt{\alpha}W = \frac{1}{2}$)

$$V(y) = C(y)\cosh\left(\frac{x(y)}{2}\right), \quad x(y) \in \int \frac{1}{C^2(y)}dy,$$

where

$$\frac{d^2V}{dy^2} = \left(Q(y) + \frac{1}{4\omega}\right)V, \quad y \in \mathbb{R}. \tag{3.4}$$

Obviously, the equation (3.4) plays the rôle of (3.2) and it should have appropriate solutions for the algorithm to work properly. The first step is completed once we have found various solutions to (3.4) – this means we shall look for solutions having the asymptotic properties described by Theorem 1. This will be explained while proceeding with the second step of the algorithm.

The second step of the Constantin-Lenells approach, which regards the change of variables $y = y(x)$, relies on the next formal computations, namely

$$\frac{1}{V^2(y)} \cdot \frac{dy}{dx} = \frac{1}{V^2(y)} \cdot C^2(y) = \frac{1}{\cosh^2\left(\frac{x}{2}\right)}$$

and

$$\int \frac{dy}{V^2(y)} = \int \frac{dx}{\cosh^2\left(\frac{x}{2}\right)} = 2\tanh\left(\frac{x}{2}\right) + \text{a constant of integration.} \tag{3.5}$$

Its task is to find an antiderivative of V^{-2} and then to compute its inverse. Unfortunately, the integral equation technique, based on a certain auxiliary function w, that has been devised in [10, p. 435] to establish the existence of a solution $V(y)$ to (3.4) for which such a task can be accomplished is not satisfactory.

The correct formulation of (3.5) is given by

$$\int_{-\infty}^{y(x)} \frac{ds}{V^2(s)} = \int_{-\infty}^{x} \frac{ds}{\cosh^2\left(\frac{s}{2}\right)}, \quad x \in \mathbb{R}. \tag{3.6}$$

It supposes, as one notices immediately, that the solution V is non-principal. Such a feature of V follows from the inferences of the w-function approach described in [10]: $V(y) \sim \cosh\left(\frac{y}{2} - \frac{C^\star}{4}\right)$ when $y \to -\infty$. But this is the best outcome of the technique.

The main difficulty one faces with regard to the formula (3.6) is that it does not ensure that $\lim_{x\to+\infty} y(x) = +\infty$! In fact, the formula leads us only to

$$\int_{-\infty}^{\lim_{x\to+\infty} y(x)} \frac{ds}{V^2(s)} = \int_{-\infty}^{+\infty} \frac{ds}{\cosh^2\left(\frac{s}{2}\right)} = 4.$$

To bypass this obstacle, we can employ Theorem 1. In fact, we propose the following approach: **first**, compute a solution V_0 of the ODE (3.4), where $\int_{\mathbb{R}}(1+|y|)|Q(y)|dy < +\infty$, which verifies the condition $I = \int_{\mathbb{R}} \frac{ds}{V_0^2(s)} < +\infty$. **Second**, normalize the solution: consider the new solution V given by $V = \eta \cdot V_0$, with $\eta = \frac{\sqrt{I}}{2}$. In this way, we have $\int_{-\infty}^{+\infty} \frac{ds}{V^2(s)} = \int_{-\infty}^{+\infty} \frac{ds}{\cosh^2\left(\frac{s}{2}\right)}$ together with $\int_{-\infty}^{\tau} \frac{ds}{V^2(s)} < \int_{-\infty}^{+\infty} \frac{ds}{\cosh^2\left(\frac{s}{2}\right)}$ for all $\tau \in \mathbb{R}$. Now, formula (3.6) yields a function $y(x)$ such that $\lim_{x\to+\infty} y(x) = +\infty$ and we can write also that

$$y(x) = H^{-1}\left(2 + 2\tanh\left(\frac{x}{2}\right)\right), \quad H(y) = \int_{-\infty}^{y} \frac{ds}{V^2(s)}.$$

With this clarification, the Constantin-Lenells algorithm can be implemented to integrate the Camassa-Holm equation (1.1).

A commentary about the modification of the algorithm introduced in [11] must be made. Here, for the solution V of equation (3.4) it is required that

$$\int_{-\infty}^{y} \frac{ds}{V^2(s)} < +\infty, \quad \int_{y}^{+\infty} \frac{ds}{V^2(s)} = +\infty, \quad y \in \mathbb{R}.$$

We encounter again the need to evaluate V throughout the entire real line by an appropriately designed variant of Theorem 1.

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References

[1] Beals R, Deift P, Tomei C, Direct and Inverse Scattering on the Line, Mathematics Surveys and Monographs, 28, AMS, Rhode Island, 1988.

[2] Beals R, Sattinger D H, Szmigielski J, Acoustic scattering and the extended Korteweg-de Vries hierarchy, *Adv. Math.* **10** (1998), 190–206.

[3] Bohl P, Über eine Differentialgleichung der Störungstheorie, *J. Reine Angew. Math.* **131** (1906), 268–231.

[4] Camassa R, Holm D D, An integrable shallow wave equation with peaked solitons, *Phys. Rev. Lett.* **71** (1993), 1661–1664.

[5] Constantin A, Existence of permanent and breaking waves for a shallow water equation: a geometric approach, *Ann. Inst. Fourier (Grenoble)* **50** (2000), 321–362.

[6] Constantin A, On the scattering problem for the Camassa-Holm equation, *Proc. R. Soc. Lond. A* **457** (2001), 953–970.

[7] Constantin A, Escher J, Wave breaking for nonlinear nonlocal shallow water equations, *Acta Mathematica* **181** (1998), 229–243.

[8] Constantin A, Gerdjikov V, Ivanov R, Inverse scattering transform for the Camassa-Holm equation, *Inverse Problems* **22** (2006), 2197–2207.

[9] Constantin A, Johnson R S, Propagation of very long water waves, with vorticity, over variable depth, with applications to tsunamis, *Fluid Dynamics Research* **40** (2008), 175–211.

[10] Constantin A, Lenells J, On the inverse scattering approach for an integrable shallow water wave equation, *Phys. Lett. A* **308** (2003), 432–436.

[11] Constantin A, Lenells J, On the inverse scattering approach to the Camassa-Holm equation, *J. Nonlinear Math. Phys.* **10** (2003), 252–255.

[12] Constantin A, Lannes D, The hydrodynamical relevance of the Camassa-Holm and Degasperis-Procesi equations, *Arch. Rat. Mech. Anal.*, in print, DOI: 10.1007/s00205-008-0128-2

[13] Constantin A, Kappeler T, Kolev B, Topalov P, On geodesic exponential maps of the Virasoro group, *Ann. Global Anal. Geom.* **31** (2007), 155–180.

[14] Constantin A, McKean H P, A shallow water equation on the circle, *Comm. Pure Appl. Math.* **52** (1999), 949–982.

[15] Constantin A, On the inverse spectral problem for the Camassa-Holm equation, *J. Funct. Anal.* **155** (1998), 352–363.

[16] Constantin A, Strauss W A, Stability of the Camassa-Holm solitons, *J. Nonlinear Sci.* **12** (2002), 415–422.

[17] Coppel W A, Disconjugacy, Lecture Notes in Mathematics, 220, Springer-Verlag, Berlin, 1971.

[18] Drazin P G, Johnson R S, Solitons. An Introduction, Cambridge University Press, Cambridge, 1992.

[19] Ionescu-Kruse D, Variational derivation of the Camassa-Holm shallow water equation, *J. Nonlinear Math. Phys.* **14** (2007), 303–312.

[20] Johnson R S, Camassa-Holm, Korteweg-de Vries and related models for water waves, *J. Fluid Mech.* **455** (2002), 63–82.

[21] Johnson R S, On solutions of the Camassa-Holm equation, *Proc. R. Soc. Lond. A* **459** (2003), 1687–1708.

[22] Lakshmanan M, Integrable nonlinear wave equations and possible connections to tsunami dynamics, in "Tsunami and Nonlinear Waves", pp. 31–49, Springer, Berlin, 2007.

[23] Li Y, Zhang J E, The multiple-soliton solution of the Camassa-Holm equation, *Proc. R. Soc. Lond. A* **460** (2004), 2617–2627.

[24] Mustafa O G, On the Cauchy problem for a generalized Camassa-Holm equation, *Nonlinear Anal. TMA* **64** (2006), 1382–1399.

[25] Willett D, Classification of second order linear differential equations with respect to oscillation, *Adv. Math.* **3** (1969), 594–623.