Journal of Nonlinear Mathematical Physics Volume 15, Number 3 (2008), 299–315 Article

Polynomial integrals for third- and fourth-order ordinary difference equations

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Received May 15, 2008; Accepted September 1, 2008

Abstract

A direct method to construct polynomial integrals for third order ordinary difference equation ($O\Delta E$) $w(n+3) = F(w(n), w(n+1), w(n+2))$ and fourth order $O\Delta E$ $w(n+4) = F(w(n), w(n+1), w(n+2), w(n+3))$ is presented. The effectiveness of the method to construct more than one polynomial integral for N-th order $O\Delta E$ is also briefly discussed.

1 Introduction

The discrete nonlinear systems governed by both ordinary difference equations or mappings and partial difference equations or lattice equations have drawn much attention by researchers working under different areas of applied science [1, 2, 3, 5, 7, 9, 10]. Since discrete systems governed by difference equations are more fundamental than the continuous ones described by differential equations their study becomes essential which will lead to the development of a general theory of discrete and in particular nonlinear difference equations. Even though there exists no unique definition of integrability considerable number of analytical methods have been formulated by different groups in recent years to deal with integrability [4, 7, 8, 10, 11, 12, 13, 16, 18, 19, 20, 21, 22, 24, 25] and significant advancement has already been made for the second order both for autonomous and nonautonomous cases [5, 7, 9, 10, 17, 19, 20, 25]. We take the working definition of integrability, here, the one which is related with the existence of sufficient number of integrals of an $O\Delta E$. An integral (also referred to as conserved quantity) of $O\Delta E$ is a function that is not identically constant but is constant on all solutions of it. An autonomous N-th order nonlinear $O\Delta E$ is said to be integrable if it admits (N-1) functionally independent integrals. Note that if a difference equation is measure preserving in dimension N and has (at least) $(N-2)$ independent integrals then it has a (degenerate) Poisson structure, so defines a symplectic map on each 2D level set of these integrals [6]. Given an autonomous N-th order nonlinear $O\Delta E$ there exists no systematic analytic technique to derive its integrals enabling one to investigate its integrability. Recently a direct method was proposed for N-th order autonomous difference equation to construct rational integrals and several new integrable difference equations of higher order were identified [23]. In this article we present a method to construct polynomial integrals through third and fourth order $O\Delta E$ [see 14,15 for a different method]. The effectiveness of the method is also discussed for N -th order $O\Delta E$.

The plan of the article is as follows. In §2 we describe the method, which involves factorisation, through third order autonomous $O\Delta E$ $w_{n+3} = F(w_n, w_{n+1}, w_{n+2})$, $w_n = w(n)$,$w_{n+1} = w(n+1)$ etc. to derive 2 polynomial integrals. In §3 we extend it for fourth-order autonomous $O\Delta E$ $w_{n+4} = F(w_n, w_{n+1}, w_{n+2}, w_{n+3})$ and identify F admitting 2 independent integrals. In §4 we present a brief summary of our results. The effectiveness of the method is also discussed for N -th order $O\Delta E$ in the Appendix.

2 Construction of integrals for third-order autonomous difference equation

Consider an autonomous third order $O\Delta E$ having the form

$$w_{n+3} = F(w_n, w_{n+1}, w_{n+2}) \text{ or } w_3 = F(w_0, w_1, w_2). \tag{2.1}$$

Hereafter, we denote $w_0 = w_n, w_1 = w_{n+1}, ..., w_N = w_{n+N}$ unless otherwise specified. Assume that equation (2.1) admits an integral $I(w_0, w_1, w_2)$ having the form

$$I(w_0, w_1, w_2) = \sum_{j=1}^{3}[A_{1j}(w_1)w_2^2 + A_{2j}(w_1)w_2 + A_{3j}(w_1)]w_0^{3-j}. \tag{2.2}$$

The integrability condition $I(w_0, w_1, w_2) - I(w_1, w_2, w_3) = 0$ leads to a quadratic equation in w_3

$$F_1(w_1, w_2)w_3^2 + F_2(w_1, w_2)w_3 - [F_4(w_1, w_2)w_0^2 + F_5(w_1, w_2)w_0 + F_6(w_1, w_2) - F_3(w_1, w_2)] = 0. \tag{2.3}$$

where

$$F_i(w_1, w_2) = \sum_{j=1}^{3} A_{ij}(w_2)w_1^{3-j}, \; i = 1, 2, 3,$$

$$F_{k+3}(w_1, w_2) = \sum_{j=1}^{3} A_{jk}(w_1)w_2^{3-j}, \; k = 1, 2, 3.$$

Hereafter we denote $F_i(w_1, w_2)$ as F_i for the remaining section unless otherwise specified. Equation (2.3) can be factorised as

$$\left(w_3 + w_0 + \frac{F_5 + F_2}{2F_1}\right)\left(w_3 - w_0 + \frac{F_2 - F_5}{2F_1}\right) = 0 \tag{2.4}$$

provided

$$A_{11}(w_2)w_1^2 + A_{12}(w_2)w_1 + A_{13}(w_2) = A_{11}(w_1)w_2^2 + A_{21}(w_1)w_2 + A_{31}(w_1) \tag{2.5a}$$

and

$$(F_5 + F_2)(F_5 - F_2) = 4(F_6 - F_3)F_1 \tag{2.5b}$$

Thus we obtain,

$$w_3 = -w_0 - \frac{F_2 + F_5}{2F_1}, \tag{2.6}$$

$$w_3 = w_0 - \frac{F_2 - F_5}{2F_1}. \tag{2.7}$$

Solving equation(2.5a) yields

$$A_{11}(w_2) = a_1w_2^2 + a_2w_2 + a_3, \tag{2.8a}$$

$$A_{21}(w_2) = a_2w_2^2 + b_2w_2 + c_2, \tag{2.8b}$$

$$A_{31}(w_2) = a_3w_2^2 + b_3w_2 + c_3, \tag{2.8c}$$

$$A_{12}(w_2) = a_2w_2^2 + b_2w_2 + b_3, \tag{2.8d}$$

$$A_{13}(w_2) = a_3w_2^2 + c_2w_2 + c_3, \tag{2.8e}$$

where $a_1, a_2, a_3, b_2, b_3, c_2$ and c_3 are arbitrary constants. Equation (2.5b) suggests that it can be solved for two distinct possibilities:

$$(i) \qquad F_5 = F_2 \quad \text{and} \quad F_6 = F_3, \tag{2.9}$$

$$(ii) \qquad F_5 \neq F_2 \quad \text{and} \quad F_6 \neq F_3. \tag{2.10}$$

The conditions given in equation (2.9) can be rewritten respectively as

$$A_{12}(w_1)w_2^2 - A_{21}(w_2)w_1^2 = A_{22}(w_2)w_1 - A_{22}(w_1)w_2 + A_{23}(w_2) - A_{32}(w_1), \tag{2.11}$$

$$A_{13}(w_1)w_2^2 - A_{31}(w_2)w_1^2 = A_{32}(w_2)w_1 - A_{23}(w_1)w_2 + A_{33}(w_2) - A_{33}(w_1). \tag{2.12}$$

Making use of the forms for $A_{11}(w_2), A_{12}(w_2), A_{13}(w_2), A_{21}(w_2)$ and $A_{31}(w_2)$ given in equations (2.8a)-(2.8e) in (2.11) and (2.12) we obtain

$$A_{22}(w_2) = b_2w_2^2 + e_2w_2 + e_3, \tag{2.13a}$$

$$A_{23}(w_2) = b_3w_2^2 + e_3w_2 + f_3, \tag{2.13b}$$

$$A_{32}(w_2) = c_2w_2^2 + e_3w_2 + f_3, \tag{2.13c}$$

$$A_{33}(w_2) = c_3w_2^2 + f_3w_2 + j_3, \tag{2.13d}$$

where e_2, e_3, f_3 and j_3 are arbitrary constants. Thus we obtain a third order difference equation

$$w_3 = -w_0 - \frac{(a_2w_2^2 + b_2w_2 + c_2)w_1^2 + b_2w_1w_2^2 + e_2w_1w_2 + e_3w_1 + b_3w_2^2 + e_3w_2 + f_3}{(a_1w_2^2 + a_2w_2 + a_3)w_1^2 + a_2w_1w_2^2 + b_2w_1w_2 + b_3w_1 + a_3w_2^2 + c_2w_2 + c_3} \tag{2.14}$$

with one integral

$$
\begin{aligned}
I_1 &= [(a_1w_1^2+a_2w_1+a_3)w_0^2+(a_2w_1^2+b_2w_1+b_3)w_0+a_3w_1^2+c_2w_1+c_3]w_2^2\\
&+[(a_2w_1^2+b_2w_1+c_2)w_0^2+(b_2w_1^2+e_2w_1+e_3)w_0+b_3w_1^2+e_3w_1+f_3]w_2\\
&+(a_3w_1^2+b_3w_1+c_3)w_0^2+(c_2w_1^2+e_3w_1+f_3)w_0+c_3w_1^2+f_3w_1. \qquad (2.15)
\end{aligned}
$$

We would like to mention that equation (2.14) can also be rewritten as

$$
w_3=-w_0-\frac{(\alpha_1w_1w_2+\alpha_2w_1+\alpha_3w_2+\alpha_4)(\gamma_1w_1w_2+\gamma_2w_1+\gamma_3w_2+\gamma_4)}{(\alpha_1w_1w_2+\alpha_2w_1+\alpha_3w_2+\alpha_4)(\beta_1w_1w_2+\beta_2w_1+\beta_3w_2+\beta_4)} \qquad (2.16)
$$

by choosing the constants $a_1,a_2,a_3,b_2,b_3,c_2,c_3,e_3$ and f_3 appropriately. For example, equation (2.14) becomes

$$
w_3=-w_0-\frac{2\beta_3w_1w_2+\gamma_2(w_1+w_2)+\gamma_4}{\beta_1w_1w_2+\beta_3(w_1+w_2)+\gamma_2-\alpha_4\beta_3} \qquad (2.17)
$$

provided

$$
\begin{aligned}
&a_1=\beta_1^2,\ a_2=2\beta_1\beta_3,\ a_3=\beta_3^2,\\
&b_2=(\beta_1\gamma_2+2\beta_3^2),\ c_2=b_3=\gamma_2\beta_3,\ c_3=\alpha_4\beta_3(\gamma_2-\alpha_4\beta_3),\\
&e_2=(2\gamma_2\beta_3+\beta_1\gamma_4+2\alpha_4\beta_3^2),\ e_3=\beta_3(\alpha_4\gamma_2+\gamma_4),\ f_3=\alpha_4\gamma_4\beta_3,\ j_3\text{-arbitrary}
\end{aligned}
$$

and the integral (2.15) becomes

$$
\begin{aligned}
I_1 &= (\beta_1w_0w_1+\beta_3(w_0+w_1)+\alpha_4\beta_3)[(\beta_1w_0w_1+\beta_3(w_0+w_1)+\gamma_2-\alpha_4\beta_3)w_2\\
&+2\beta_3w_0w_1+\gamma_2(w_0+w_1)+\gamma_4]w_2+\beta_3(w_1+\alpha_4)(\beta_3w_0w_1+\gamma_2w_1\\
&+(\gamma_2-\alpha_4\beta_3)w_0+\gamma_4)w_0+\alpha_4\beta_3(\gamma_2-\beta_3\alpha_4)w_1^2+\alpha_4\beta_3\gamma_4w_1. \qquad (2.18)
\end{aligned}
$$

In order to construct second integral I_2 we use the other possibility given in equation (2.10). It is clear from equation (2.5 a) that $A_{11}(w_1)$, $A_{12}(w_1)$, $A_{13}(w_1)$, $A_{21}(w_1)$ and $A_{31}(w_1)$ are quadratic polynomials. However, equation (2.5b) suggests that $A_{22}(w_1)$, $A_{23}(w_1)$, $A_{32}(w_1)$ and $A_{33}(w_1)$ may be quartic polynomials. Thus we consider

$$A_{11}(w_1)=\tilde a_1w_1^2+\tilde a_2w_1+\tilde a_3, \qquad (2.19a)$$

$$A_{21}(w_1)=\tilde a_2w_1^2+\tilde b_2w_1+\tilde c_2, \qquad (2.19b)$$

$$A_{31}(w_1)=\tilde a_3w_1^2+\tilde b_3w_1+\tilde c_3, \qquad (2.19c)$$

$$A_{12}(w_1)=\tilde a_2w_1^2+\tilde b_2w_1+\tilde b_3, \qquad (2.19d)$$

$$A_{13}(w_1)=\tilde a_3w_1^2+\tilde c_2w_1+\tilde c_3, \qquad (2.19e)$$

$$A_{22}(w_1)=\epsilon_1w_1^4+\epsilon_2w_1^3+\epsilon_3w_1^2+\epsilon_4w_1+\epsilon_5, \qquad (2.19f)$$

$$A_{23}(w_1)=\epsilon_6w_1^4+\epsilon_7w_1^3+\epsilon_8w_1^2+\epsilon_9w_1+\epsilon_{10}, \qquad (2.19g)$$

$$A_{32}(w_1)=\epsilon_{11}w_1^4+\epsilon_{12}w_1^3+\epsilon_{13}w_1^2+\epsilon_{14}w_1+\epsilon_{15}, \qquad (2.19h)$$

$$A_{33}(w_1)=\epsilon_{16}w_1^4+\epsilon_{17}w_1^3+\epsilon_{18}w_1^2+\epsilon_{19}w_1+\epsilon_{20}, \qquad (2.19i)$$

where $\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \tilde{b}_2, \tilde{b}_3, \tilde{c}_2, \tilde{c}_3$ and $\epsilon_i, i = 1, 2...20$ are unknown constants to be determined. Then equation (2.6) reads

$$w_3 = -w_0 - \frac{\tilde{F}}{2\tilde{G}} \tag{2.20}$$

where,

$$\begin{aligned}
\tilde{F} &= \epsilon_{11}w_1^4 + \epsilon_6 w_2^4 + \epsilon_{12}w_1^3 + \epsilon_7 w_2^3 + \epsilon_1(w_1^3 + w_2^3)w_1w_2 + (\epsilon_3 + b_2)(w_1 + w_2)w_1w_2 \\
&\quad + \epsilon_2(w_1^2 + w_2^2)w_1w_2 + 2a_2w_2^2w_1^2 + (\epsilon_{13} + c_2)w_1^2 + (b_3 + \epsilon_8)w_2^2 + 2\epsilon_4 w_2w_1 \\
&\quad + (\epsilon_5 + \epsilon_{14})w_1 + (\epsilon_5 + \epsilon_9)w_2 + \epsilon_{10} + \epsilon_{15}, \\
\tilde{G} &= (\tilde{a}_1w_1^2w_2^2 + \tilde{a}_2w_2w_1^2 + \tilde{a}_3w_1^2 + \tilde{a}_2w_1w_2^2 + \tilde{b}_2w_1w_2 + \tilde{b}_3w_1 + \tilde{a}_3w_2 + \tilde{c}_2w_2 + \tilde{c}_3).
\end{aligned}$$

We would like to mention here that both $\tilde{F}$ and $\tilde{G}$ can be factored with a common factor by choosing the constants $\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \tilde{b}_2, \tilde{b}_3, \tilde{c}_2, \tilde{c}_3$ and $\epsilon_i, i = 1, 2...20$, appropriately. That is,

$$\begin{aligned}
\tilde{F} &= (\tilde{\alpha}_1w_1w_2 + \tilde{\alpha}_2w_1 + \tilde{\alpha}_3w_2 + \tilde{\alpha}_4)[\tilde{\gamma}_1w_1^2w_2^2 + \tilde{\gamma}_2w_1^2w_2 + \tilde{\gamma}_3w_1w_2^2 + \tilde{\gamma}_4w_1^2 + \tilde{\gamma}_5w_2^2 \\
&\quad + \tilde{\gamma}_6w_1w_2 + \tilde{\gamma}_7w_1 + \tilde{\gamma}_8w_2 + \tilde{\gamma}_9] \\
\tilde{G} &= (\tilde{\alpha}_1w_1w_2 + \tilde{\alpha}_2w_1 + \tilde{\alpha}_3w_2 + \tilde{\alpha}_4)(\tilde{\beta}_1w_1w_2 + \tilde{\beta}_2w_1 + \tilde{\beta}_3w_2 + \tilde{\beta}_4)
\end{aligned}$$

For example, equation (2.20) reduces into

$$w_3 = -w_0 - \frac{\tilde{\alpha}_4(4\tilde{\beta}_2w_1w_2 + \tilde{\gamma}_8(w_1 + w_2) + \tilde{\gamma}_9)}{2\tilde{\beta}_1\tilde{\alpha}_4w_1w_2 + 2\tilde{\alpha}_4\tilde{\beta}_2(w_1 + w_2) + \tilde{\alpha}_4\tilde{\gamma}_8 - 2\epsilon_9} \tag{2.21}$$

provided

$$\begin{aligned}
&\tilde{a_1} = \tilde{a_2} = \tilde{a_3} = 0,\ \tilde{b}_2 = \tilde{\alpha}_4\tilde{\beta}_1,\ \tilde{c}_2 = \tilde{b}_3 = \tilde{\alpha}_4\tilde{\beta}_2,\ \tilde{c}_3 = \frac{\tilde{\alpha}_4\tilde{\gamma}_8}{2} - \epsilon_9, \\
&\epsilon_3 = -\tilde{b}_2,\ \epsilon_4 = 2\tilde{b}_3,\ \epsilon_5 = -\epsilon_9 + \tilde{\alpha}_4\tilde{\gamma}_8,\ \epsilon_8 = -\tilde{b}_3,\ \epsilon_9\ \text{ -- arbitrary}, \\
&\epsilon_{10} = \epsilon_{15} = \frac{\tilde{\alpha}_4\tilde{\gamma}_9}{2},\ \epsilon_{13} = -\tilde{b}_3,\ \epsilon_{14} = \epsilon_9,\ \epsilon_{18} = -\epsilon_9, \\
&\epsilon_1 = \epsilon_2 = \epsilon_6 = \epsilon_7 = \epsilon_{11} = \epsilon_{12} = \epsilon_{16} = \epsilon_{17} = \epsilon_{19} = \epsilon_{20} = 0.
\end{aligned}$$

Also the conditions (2.5a,b) and (2.10) are identically satisfied for the above parametric restrictions which in turn leads to the existence of second integral I_2. Furthermore equation (2.21) can be rewritten as

$$w_{n+3} = -w_n - \frac{2\lambda_1w_{n+1}w_{n+2} + \lambda_2(w_{n+1} + w_{n+2}) + \lambda_3}{\lambda_4w_{n+1}w_{n+2} + \lambda_1(w_{n+1} + w_{n+2}) + \lambda_5}, \tag{2.22}$$

where

$$\tilde{\beta}_2 = \frac{\lambda_1}{2\tilde{\alpha}_4}, \tilde{\gamma}_8 = \frac{\lambda_2}{\tilde{\alpha}_4}, \tilde{\gamma}_9 = \frac{\lambda_3}{\tilde{\alpha}_4}, \tilde{\beta}_1 = \frac{\lambda_4}{2\tilde{\alpha}_4},\ \epsilon_9 = \frac{\alpha_4\gamma_8 - \lambda_5}{2}$$

and so the integral I_2 is

$$\begin{aligned}I_2 = &[\lambda_4 w_{n+1}w_n + \lambda_1 w_n + \lambda_1 w_{n+1} + \lambda_5]w_{n+2}^2 + [\lambda_4 w_{n+1}w_n^2 + \lambda_1 w_n^2 - \lambda_4 w_{n+1}^2 w_n \\ &+2\lambda_1 w_n w_{n+1} + \lambda_2 w_n + \lambda_5 w_n - \lambda_1 w_{n+1}^2 + (\lambda_2 - \lambda_5)w_{n+1} + \lambda_3]w_{n+2} + \lambda_1 w_{n+1}w_n^2 \\ &+\lambda_5 w_n^2 - \lambda_1 w_{n+1}^2 w_n + (\lambda_2 - \lambda_5)w_{n+1}w_n + \lambda_3 w_n + (\lambda_5 - \lambda_2)w_{n+1}^2. \qquad (2.23)\end{aligned}$$

It is easy to see that the third order $O\Delta E$ (2.17) admitting the integral I_1 assumes exactly the same form as (2.22) by choosing the parameters $\beta_1, \beta_3, \gamma_2$ and γ_4 as

$$\alpha_4 = \frac{\lambda_2 - \lambda_5}{\lambda_1},\ \beta_1 = \lambda_4,\ \beta_3 = \lambda_1,\ \gamma_2 = \lambda_2,\ \gamma_4 = \lambda_3.$$

and the integral I_1 becomes

$$\begin{aligned}I_1 = &(\lambda_4 w_n w_{n+1} + \lambda_1 w_n + \lambda_1 w_{n+1} + \lambda_2 - \lambda_5)[(\lambda_4 w_n w_{n+1} + \lambda_1 w_n + \lambda_1 w_{n+1} + \lambda_5)w_{n+2} \\ &+2\lambda_1 w_n w_{n+1} + \lambda_2 w_n + \lambda_2 w_{n+1} + \lambda_3]w_{n+2} + (\lambda_1 w_{n+1} + \lambda_2 - \lambda_5)[\lambda_1 w_n w_{n+1} + \lambda_5 w_n \\ &+\lambda_2 w_{n+1} + \lambda_3]w_n + (\lambda_5 w_{n+1} + \lambda_3)(\lambda_2 - \lambda_5)w_{n+1}. \qquad (2.24)\end{aligned}$$

Thus we conclude that the third order $O\Delta E$ (2.22) admits 2 independent integrals I_1 and I_2.

Proceeding along the similar lines described above one can identify more than one third order $O\Delta E$ possessing 2 independent integrals since the factorisation of $\tilde{F}$ and $\tilde{G}$ explained earlier is not unique. For example $\tilde{F}$ and $\tilde{G}$ can also be factored as

$$\begin{aligned}\tilde{F} &= 2(\tilde{\beta}_3 w_1 + \tilde{\beta}_3 w_2 + \tilde{\beta}_4)[\tilde{\beta}_3(w_1 + w_2)^2 + \tilde{\gamma}_7(w_1 + w_2) + \tilde{\gamma}_9] \\ \tilde{G} &= (\tilde{\beta}_3 w_1 + \tilde{\beta}_3 w_2 + \tilde{\beta}_4)^2\end{aligned}$$

provided

$$\begin{aligned}&\tilde{a}_3 = \tilde{\beta}_3^2,\ \tilde{b}_2 = 2\tilde{a}_3,\ \tilde{b}_3 = \tilde{c}_2 = 2\tilde{\beta}_3\tilde{\beta}_4,\ \tilde{c}_3 = \tilde{\beta}_4^2, \\ &\epsilon_3 = 4\tilde{a}_3,\ \epsilon_4 = 2\tilde{\beta}_3(\tilde{\beta}_4 + \tilde{\gamma}_7),\ \epsilon_5 = \tilde{\beta}_3\tilde{\gamma}_9 + 2\tilde{\beta}_4^2, \epsilon_7 = 2\tilde{a}_3, \epsilon_8 = 2\tilde{\beta}_3\tilde{\gamma}_7, \\ &\epsilon_9 = 2(\tilde{\beta}_4\tilde{\gamma}_7 + \tilde{\beta}_3\tilde{\gamma}_9) - \epsilon_5,\ \epsilon_{10} = \tilde{\beta}_4\tilde{\gamma}_9, \epsilon_{12} = 2\tilde{\beta}_3^2,\ \epsilon_{13} = 2\tilde{\beta}_3\tilde{\gamma}_7, \\ &\epsilon_{14} = 2(\tilde{\beta}_4\tilde{\gamma}_7 + \tilde{\beta}_3\tilde{\gamma}_9) - \epsilon_5,\ \epsilon_{15} = \tilde{\beta}_4\tilde{\gamma}_9,\ \epsilon_{16} = \tilde{\beta}_3^2,\ \epsilon_{17} = 2\tilde{\beta}_3(\tilde{\gamma}_7 - \tilde{\beta}_4), \\ &\epsilon_{18} = -2\tilde{\beta}_4\tilde{\gamma}_7 + \tilde{\beta}_3\tilde{\gamma}_9 + \tilde{\gamma}_7^2 + \tilde{\beta}_4^2,\ \epsilon_{19} = \tilde{\gamma}_9(\tilde{\gamma}_7 - \tilde{\beta}_4), \\ &\tilde{a}_1 = \tilde{a}_2 = \epsilon_1 = \epsilon_2 = \epsilon_6 = \epsilon_{11} = \epsilon_{20} = 0.\end{aligned}$$

Here again the conditions (2.5ab) and (2.10) are identically satisfied for the above parametric restrictions. In this case equation (2.20) becomes

$$w_{n+3} = -w_n - \frac{\lambda_1(w_{n+1} + w_{n+2})^2 + \lambda_2(w_{n+1} + w_{n+2}) + \lambda_3}{\lambda_1(w_{n+1} + w_{n+2}) + \lambda_4}, \qquad (2.25)$$

where

$$\tilde{\beta}_3 = \lambda_1,\ \tilde{\gamma}_7 = \lambda_2, \tilde{\gamma}_9 = \lambda_3, \tilde{\beta}_4 = \lambda_4$$

and admits second integral I_2 as

$$I_2 = [\lambda_1 w_n w_{n+2} + \lambda_4 w_{n+2} + \lambda_1 w_n w_{n+1} + \lambda_4 w_n + \lambda_1 w_{n+1}^2 + \lambda_1 w_{n+1} w_{n+2} \\ +(\lambda_2 - \lambda_4) w_{n+1}][\lambda_1 w_n w_{n+2} + \lambda_4 w_{n+2} + \lambda_1 w_n w_{n+1} + \lambda_4 w_n + \lambda_1 w_{n+1}^2 \\ +\lambda_1 w_{n+1} w_{n+2} + (\lambda_2 - \lambda_4) w_{n+1} + \lambda_3]. \tag{2.26}$$

It is easy to see that equation (2.14) possessing an integral I_1 takes exactly the same form as (2.25) for the parametric restrictions

$$c_2 = b_3 = \lambda_1,\ e_2 = 2\lambda_1,\ e_3 = \lambda_2,\ f_3 = \lambda_3,\ c_3 = \lambda_4,\ a_1 = a_2 = b_2 = a_3 = 0$$

and the integral I_1 becomes

$$I_1 = [\lambda_1 w_n + \lambda_1 w_{n+1} + \lambda_4] w_{n+2}^2 + [\lambda_1 w_n^2 + 2\lambda_1 w_n w_{n+1} + \lambda_2 w_n + \lambda_1 w_{n+1}^2 + \lambda_2 w_{n+1} \\ +\lambda_3] w_{n+2} + \lambda_1 w_{n+1} w_n^2 + \lambda_4 w_n^2 + \lambda_1 w_{n+1}^2 w_n + \lambda_2 w_{n+1} w_n + \lambda_3 w_n + \lambda_4 w_{n+1}^2 + \lambda_3 w_{n+1}. \tag{2.27}$$

A detailed calculation shows that there exists no other third order difference equation possessing 2 independent polynomial integrals. We wish to mention that the identified third order difference equations (2.22) and (2.25) admitting 2 integrals are also measure preserving with measure 1 and hence they are integrable. Equations (2.22) and (2.25) were also obtained by Iatrou(2003a) using a different method. However, our analysis shows that there exists a third order difference equation involving atleast 10 parameters possessing one cyclic integral I_1.

3 Construction of integrals for fourth order autonomous ordinary difference equation

Consider an autonomous fourth order $O\Delta E$ having the form

$$w_4 = F(w_0, w_1, w_2, w_3). \tag{3.1}$$

Assume that equation (3.1) admits an integral $I(w_0, w_1, w_2, w_3)$ having the form

$$I(w_0, w_1, w_2, w_3) = \sum_{j=1}^{3} [A_{1j}(w_1, w_2) w_3^2 + A_{2j}(w_1, w_2) w_3 + A_{3j}(w_1, w_2)] w_0^{3-j}. \tag{3.2}$$

Then the integrability condition $I(n+1) - I(n) = 0$ leads to a quadratic equation in w_4

$$F_1(w_1, w_2, w_3) w_4^2 + F_2(w_1, w_2, w_3) w_4 \\ -[F_4(w_1, w_2, w_3) w_0^2 + F_5(w_1, w_2, w_3) w_0 + (F_6(w_1, w_2, w_3) - F_3(w_1, w_2, w_3))] = 0, \tag{3.3}$$

where

$$F_i(w_1, w_2, w_3) = \sum_{j=1}^{3} A_{ij}(w_2, w_3) w_1^{3-j}, i = 1, 2, 3,$$

$$F_{k+3}(w_1, w_2, w_3) = \sum_{j=1}^{3} A_{jk}(w_1, w_2) w_3^{3-j}, k = 1, 2, 3.$$

Hereafter we denote $F_i(w_1, w_2, w_3)$ as F_i for the remaining section unless otherwise specified. Equation (3.3) can be factorised as

$$\left(w_4 + w_0 + \frac{F_5 + F_2}{2F_1}\right)\left(w_4 - w_0 + \frac{F_2 - F_5}{2F_1}\right) = 0 \tag{3.4}$$

provided

$$\begin{aligned} &A_{11}(w_2, w_3)w_1^2 + A_{12}(w_2, w_3)w_1 + A_{13}(w_2, w_3) \\ &\quad = A_{11}(w_1, w_2)w_3^2 + A_{21}(w_1, w_2)w_3 + A_{31}(w_1, w_2) \end{aligned} \tag{3.5a}$$

and

$$(F_5 + F_2)(F_5 - F_2) = 4(F_6 - F_3)F_1. \tag{3.5b}$$

Thus we obtain,

$$w_4 = -w_0 - \frac{F_2 + F_5}{2F_1}, \tag{3.6}$$

$$w_4 = w_0 - \frac{F_2 - F_5}{2F_1}. \tag{3.7}$$

It is easy to check that equation (3.5a) is satisfied with

$$A_{11}(w_1, w_2) = (a_1w_1^2 + a_2w_1 + a_3)w_2^2 + (a_2w_1^2 + a_5w_1 + a_6)w_2 + a_3w_1^2 + a_8w_1 + a_9 \tag{3.8a}$$

$$A_{12}(w_1, w_2) = (a_2w_1^2 + a_5w_1 + a_8)w_2^2 + A_{122}(w_1)w_2 + A_{123}(w_1) \tag{3.8b}$$

$$A_{13}(w_1, w_2) = (a_3w_1^2 + a_6w_1 + a_9)w_2^2 + A_{132}(w_1)w_2 + A_{133}(w_1) \tag{3.8c}$$

$$A_{21}(w_1, w_2) = (a_2w_2^2 + a_5w_2 + a_6)w_1^2 + A_{122}(w_2)w_1 + A_{132}(w_2) \tag{3.8d}$$

$$A_{31}(w_1, w_2) = (a_3w_2^2 + a_8w_2 + a_9)w_1^2 + A_{123}(w_2)w_1 + A_{133}(w_2), \tag{3.8e}$$

where a_i's are constants while $A_{122}, A_{123}, A_{132}$ and A_{133} are arbitrary functions. As pointed out for third order difference equations, equation (3.5b) can be solved for two distinct possibilities:

$$(i) \qquad F_5 = F_2 \text{ and } F_6 = F_3, \tag{3.9}$$

$$(ii) \qquad F_5 \neq F_2 \text{ and } F_6 \neq F_3. \tag{3.10}$$

Considering the condition (3.9) we have

$$A_{11}(w_1, w_2) = (a_1w_1^2 + a_2w_1 + a_3)w_2^2 + (a_2w_1^2 + a_5w_1 + a_6)w_2 + a_3w_1^2 + a_8w_1 + a_9 \tag{3.11a}$$

$$A_{12}(w_1, w_2) = (a_2w_1^2 + a_5w_1 + a_8)w_2^2 + (b_4w_1^2 + b_5w_1 + b_6)w_2 + b_7w_1^2 + b_8w_1 + b_9 \tag{3.11b}$$

$$A_{13}(w_1, w_2) = (a_3w_1^2 + a_6w_1 + a_9)w_2^2 + (b_7w_1^2 + c_5w_1 + c_6)w_2 + c_7w_1^2 + c_8w_1 + c_9 \tag{3.11c}$$

$$A_{21}(w_1, w_2) = (a_2w_1^2 + b_4w_1 + b_7)w_2^2 + (a_5w_1^2 + b_5w_1 + c_5)w_2 + a_6w_1^2 + b_6w_1 + c_6 \tag{3.11d}$$

$$A_{22}(w_1,w_2) = (a_5w_1^2+b_5w_1+b_8)w_2^2+(b_5w_1^2+e_5w_1+e_6)w_2+c_5w_1^2+e_6w_1+e_9 \tag{3.11e}$$

$$A_{23}(w_1,w_2) = (a_8w_1^2+b_6w_1+b_9)w_2^2+(b_8w_1^2+e_6w_1+e_9)w_2+c_8w_1^2+f_8w_1+f_9 \tag{3.11f}$$

$$A_{31}(w_1,w_2) = (a_3w_1^2+b_7w_1+c_7)w_2^2+(a_8w_1^2+b_8w_1+c_8)w_2+a_9w_1^2+b_9w_1+c_9 \tag{3.11g}$$

$$A_{32}(w_1,w_2) = (a_6w_1^2+c_5w_1+c_8)w_2^2+(b_6w_1^2+e_6w_1+f_8)w_2+c_6w_1^2+e_9w_1+f_9 \tag{3.11h}$$

$$A_{33}(w_1,w_2) = (a_9w_1^2+c_6w_1+c_9)w_2^2+(b_9w_1^2+e_9w_1+f_9)w_2+c_9w_1^2+f_9w_1+j_9 \tag{3.11i}$$

where a_i's, b_i's, c_i's, e_i's, f_i's and j_9 are arbitrary constants. Thus we obtain a fourth order difference equation

$$w_4 = -w_0 - \frac{F_2+F_5}{2F_1} = -w_0 - \frac{F}{G} \tag{3.12}$$

where

$$\begin{aligned} F = {} & [(a_2w_1^2+a_5w_1+a_8)w_2^2+(b_4w_1^2+b_5w_1+b_6)w_2+b_7w_1^2+b_8w_1+b_9]w_3^2 \\ & +[(a_5w_1^2+b_5w_1+b_8)w_2^2+(b_5w_1^2+e_5w_1+e_6)w_2+c_5w_1^2+e_6w_1+e_9]w_3 \\ & +[(a_6w_1^2+c_5w_1+c_8)w_2^2+(b_6w_1^2+e_6w_1+f_8)w_2+c_6w_1^2+e_9w_1+f_9] \end{aligned}$$

$$\begin{aligned} G = {} & [(a_1w_1^2+a_2w_1+a_3)w_2^2+(a_2w_1^2+a_5w_1+a_6)w_2+a_3w_1^2+a_8w_1+a_9]w_3^2 \\ & +[(a_5w_1^2+b_5w_1+c_5)w_2+c_6+(b_4w_1+b_7+a_2w_1^2)w_2^2+b_6w_1+a_6w_1^2]w_3 \\ & +(a_3w_1^2+b_7w_1+c_7)w_2^2+c_9+a_9w_1^2+(a_8w_1^2+b_8w_1+c_8)w_2+b_9w_1 \end{aligned}$$

with an integral I_1 given by

$$\begin{aligned} I_1 = {} & [[(a_1w_1^2+a_2w_1+a_3)w_2^2+(a_2w_1^2+a_5w_1+a_6)w_2+a_3w_1^2+a_8w_1+a_9]w_0^2 \\ & +[(a_2w_1^2+a_5w_1+a_8)w_2^2+(b_4w_1^2+b_5w_1+b_6)w_2+b_7w_1^2+b_8w_1+b_9]w_0 \\ & +(a_3w_1^2+a_6w_1+a_9)w_2^2+(b_7w_1^2+c_5w_1+c_6)w_2+c_7w_1^2+c_8w_1+c_9]w_3^2 \\ & +[[(a_2w_1^2+b_4w_1+b_7)w_2^2+(a_5w_1^2+b_5w_1+c_5)w_2+a_6w_1^2+b_6w_1+c_6]w_0^2 \\ & +[(a_5w_1^2+b_5w_1+b_8)w_2^2+(b_5w_1^2+e_5w_1+e_6)w_2+c_5w_1^2+e_6w_1+e_9]w_0 \\ & +(a_8w_1^2+b_6w_1+b_9)w_2^2+(b_8w_1^2+e_6w_1+e_9)w_2+c_8w_1^2+f_8w_1+f_9]w_3 \\ & +[(a_3w_1^2+b_7w_1+c_7)w_2^2+(a_8w_1^2+b_8w_1+c_8)w_2+a_9w_1^2+b_9w_1+c_9]w_0^2 \\ & +[(a_6w_1^2+c_5w_1+c_8)w_2^2+(b_6w_1^2+e_6w_1+f_8)w_2+c_6w_1^2+e_9w_1+f_9]w_0 \\ & +(a_9w_1^2+c_6w_1+c_9)w_2^2+(b_9w_1^2+e_9w_1+f_9)w_2+c_9w_1^2+f_9w_1 \end{aligned} \tag{3.13}$$

Proceeding further along the lines described for third order $O\Delta E$ we find that there exist six fourth order $O\Delta E$ possessing 2 independent integrals. They are

$$(i) \quad w_{n+4} = -w_n - \frac{\lambda_1(w_{n+2}^2+w_{n+2}(w_{n+1}+w_{n+3}))+\lambda_3(w_{n+1}+w_{n+3})+\lambda_2w_{n+2}+\lambda_4}{\lambda_1w_{n+2}+\lambda_2},$$

$$(3.14)$$

$$\begin{aligned}I_1 &= (\lambda_1 w_{n+1} + \lambda_3)w_{n+3}^2 + ((2\lambda_1 w_{n+1} + 2\lambda_3)w_{n+2} + \lambda_1 w_{n+1}^2 + \lambda_2 w_{n+1} + \lambda_4)w_{n+3}\\ &+(\lambda_1 w_{n+1} + 2\lambda_3)w_{n+2}^2 + (\lambda_1 w_{n+1}^2 + (\lambda_2 + 2\lambda_3)w_{n+1} + 2\lambda_4)w_{n+2} + 2\lambda_3 w_{n+1}^2\\ &+2\lambda_4 w_{n+1} + (\lambda_1 w_{n+2} + \lambda_3)w_n^2 + ((2\lambda_1 w_{n+2} + 2\lambda_3)w_{n+1} + \lambda_1 w_{n+2}^2 + \lambda_2 w_{n+2} + \lambda_4)w_n.\end{aligned}$$

$$(ii) \qquad w_{n+4} = -w_n - \frac{\lambda_1(w_{n+1} + w_{n+2} + w_{n+3})^2 + \lambda_2(w_{n+1} + w_{n+2} + w_{n+3}) + \lambda_3}{\lambda_1(w_{n+1} + w_{n+2} + w_{n+3}) + \lambda_4}, \tag{3.15}$$

$$\begin{aligned}I_1 &= (\lambda_1 w_n + \lambda_1 w_{n+2} + \lambda_1 w_{n+1} + \lambda_4)w_{n+3}^2 + (\lambda_1 w_n^2 + (2\lambda_1 w_{n+2} + 2\lambda_1 w_{n+1} + \lambda_2)w_n\\ &+\lambda_1 w_{n+2}^2 + (2\lambda_1 w_{n+1} + \lambda_2)w_{n+2} + \lambda_1 w_{n+1}^2 + \lambda_2 w_{n+1} + \lambda_3)w_{n+3} + (\lambda_1 w_{n+1}\\ &+\lambda_4)w_{n+2}^2 + (\lambda_1 w_{n+1}^2 + \lambda_2 w_{n+1} + \lambda_3)w_{n+2} + \lambda_4 w_{n+1}^2 + \lambda_3 w_{n+1} + (\lambda_1 w_{n+2}\\ &+\lambda_1 w_{n+1} + \lambda_4)w_n^2 + (\lambda_1 w_{n+1}^2 + (2\lambda_1 w_{n+2} + \lambda_2)w_{n+1} + \lambda_1 w_{n+2}^2 + \lambda_2 w_{n+2} + \lambda_3)w_n.\end{aligned}$$

$$(iii) \qquad w_{n+4} = -w_n - \frac{\lambda_1(w_{n+1}w_{n+2} + w_{n+2}w_{n+3}) + \lambda_2(w_{n+1} + w_{n+3}) + \lambda_4 w_{n+2} + \lambda_3}{\lambda_1(w_{n+1} + w_{n+3}) + \lambda_4}, \tag{3.16}$$

$$\begin{aligned}I_1 &= (\lambda_1 w_n + \lambda_2 - \lambda_4)(\lambda_1 w_{n+2} + \lambda_2 - \lambda_4)(\lambda_1 w_{n+1} + \lambda_2 - \lambda_4)[(\lambda_1 w_n + \lambda_1 w_{n+2} + \lambda_4)w_{n+3}\\ &+\lambda_1 w_n w_{n+1} + \lambda_1 w_{n+1}w_{n+2} + \lambda_4 w_{n+1} + \lambda_2 w_{n+2} + \lambda_2 w_n + \lambda_3]w_{n+3} + (\lambda_2 - \lambda_4)\\ &\times(\lambda_1 w_{n+2} + \lambda_2 - \lambda_4)(\lambda_1 w_{n+1} + \lambda_2 - \lambda_4)(\lambda_1 w_{n+1}w_{n+2} + \lambda_1 w_n w_{n+1} + \lambda_2 w_{n+1}\\ &+\lambda_4 w_{n+2} + \lambda_4 w_n + \lambda_3)w_n + (\lambda_2 - \lambda_4)^2(\lambda_2^2 w_{n+1}w_{n+2} + \lambda_2\lambda_4 w_{n+2}^2 + \lambda_2\lambda_3 w_{n+2}\\ &-\lambda_2\lambda_4 w_{n+1}w_{n+2} + \lambda_1\lambda_2 w_{n+1}w_{n+2}^2 + \lambda_1\lambda_2 w_{n+1}^2 w_{n+2} + \lambda_2\lambda_4 w_{n+1}^2 + \lambda_2\lambda_3 w_{n+1}\\ &-\lambda_4\lambda_3 w_{n+2} + \lambda_1^2 w_{n+1}^2 w_{n+2}^2 - \lambda_4^2 w_{n+2}^2 + \lambda_1\lambda_3 w_{n+1}w_{n+2} - \lambda_4^2 w_{n+1}^2 - \lambda_3\lambda_4 w_{n+1}).\end{aligned}$$

$$(iv) \quad w_{n+4} = -w_n - \frac{(\lambda_1 w_{n+2} + 2\lambda_3)(w_{n+1} + w_{n+3}) + \lambda_1(w_{n+2}^2 + w_{n+1}w_{n+3}) + \lambda_2 w_{n+2} + \lambda_4}{\lambda_1 w_{n+2} + \lambda_3}, \tag{3.17}$$

$$\begin{aligned}I_1 &= (\lambda_1 w_{n+1} + \lambda_3)w_{n+3}^2 + ((\lambda_1 w_{n+2} + \lambda_1 w_{n+1} + 2\lambda_3)w_n + (\lambda_1 w_{n+1} + 2\lambda_3)w_{n+2}\\ &+\lambda_1 w_{n+1}^2 + \lambda_2 w_{n+1} + \lambda_4)w_{n+3} + \lambda_3 w_{n+2}^2 + (2\lambda_3 w_{n+1} + \lambda_4)w_{n+2} + \lambda_3 w_{n+1}^2\\ &+\lambda_4 w_{n+1} + (\lambda_1 w_{n+2} + \lambda_3)w_n^2 + ((\lambda_1 w_{n+2} + 2\lambda_3)w_{n+1} + \lambda_1 w_{n+2}^2 + \lambda_2 w_{n+2} + \lambda_4)w_n.\end{aligned}$$

$$\begin{aligned}(v) \quad w_{n+4} &= -w_n - \frac{\lambda_1(w_{n+1}^2 + 2w_{n+2}^2 + w_{n+3}^2 + 4(w_{n+1}w_{n+2} + w_{n+2}w_{n+3} + w_{n+3}w_{n+1}))}{\lambda_1 w_{n+1} + \lambda_1 w_{n+3} + 2\lambda_1 w_{n+2} + \lambda_3}\\ &-\frac{2(\lambda_2 - \lambda_3)w_{n+2} + \lambda_2(w_{n+1} + w_{n+3}) + \lambda_4}{\lambda_1 w_{n+1} + \lambda_1 w_{n+3} + 2\lambda_1 w_{n+2} + \lambda_3},\end{aligned} \tag{3.18}$$

$$I_1 = (\lambda_1 w_n + \lambda_1 w_{n+2} + 2\lambda_1 w_{n+1} + \lambda_3) w_{n+3}^2 + (\lambda_1 w_n^2 + (4\lambda_1 w_{n+2} + 4\lambda_1 w_{n+1} + \lambda_2) w_n + \lambda_1 w_{n+2}^2 + (4\lambda_1 w_{n+1} + \lambda_2) w_{n+2} + 2\lambda_1 w_{n+1}^2 + (2\lambda_2 - 2\lambda_3) w_{n+1} + \lambda_4) w_{n+3} + (\lambda_1 w_{n+1} + \lambda_3) w_{n+2}^2 + (\lambda_1 w_{n+1}^2 + \lambda_2 w_{n+1} + \lambda_4) w_{n+2} + \lambda_3 w_{n+1}^2 + \lambda_4 w_{n+1} + (\lambda_1 w_{n+1} + 2\lambda_1 w_{n+2} + \lambda_3) w_n^2 + (\lambda_1 w_{n+1}^2 + (4\lambda_1 w_{n+2} + \lambda_2) w_{n+1} + 2\lambda_1 w_{n+2}^2 + (2\lambda_2 - 2\lambda_3) w_{n+2} + \lambda_4) w_n.$$

(vi)
$$w_{n+4} = -w_n - \frac{\lambda_1 w_{n+1} w_{n+3} + 2\lambda_4 w_{n+2} + (\lambda_4 + \lambda_5)(w_{n+1} + w_{n+3}) + \lambda_3}{\lambda_1 w_{n+2} + \lambda_4 - \lambda_5}, \quad (3.19)$$

$$I_1 = (\lambda_1 w_{n+1} + \lambda_4 - \lambda_5) w_{n+3}^2 + ((\lambda_1 w_{n+2} + \lambda_1 w_{n+1} + 2\lambda_4) w_n + (2\lambda_5 - \lambda_1 w_{n+1}) w_{n+2} + 2\lambda_4 w_{n+1} + \lambda_3) w_{n+3} - 2\lambda_5 (w_{n+1}^2 + w_{n+2}^2) + 2(\lambda_5 - \lambda_4) w_{n+1} w_{n+2} + (\lambda_1 w_{n+2} + \lambda_4 - \lambda_5) w_n^2 + ((2\lambda_5 - \lambda_1 w_{n+2}) w_{n+1} + 2\lambda_4 w_{n+2} + \lambda_3) w_n.$$

Since the calculations are tedious and cumbersome the explicit forms of expressions $A_{ij}(w_1, w_2)$ associated with second integral I_2 for each of the above identified six cases are given in Appendix A.

Finally we would like to mention that with the limited software facilities available we have identified the above fourth order $O\Delta E$ possessing 2 independent polynomial integrals and hence we do not claim that the identified list of difference equations is an exhaustive one.

4 Summary

In this article a method is described to construct polynomial integrals for third order and fourth order $O\Delta E$. We have identified two distinct third order equations (2.22) and (2.25), each of them admits 2 independent integrals. It is appropriate to mention here that one of the integrals in both the cases is cyclic invariant. Also the identified equations (2.22) and (2.25) are measure preserving with measure 1 and hence they are integrable. Similar observation has also been noted by Iatrou (2003a). For fourth order $O\Delta E$ we have identified six distinct equations (3.14), (3.15), (3.16), (3.17), (3.18) and (3.19) admitting 2 independent integrals. Here again one of the two integrals is cyclic invariant. Also they are reversible as well as measure preserving with measure 1. We would like to mention that the fourth order equations given in equations (3.14), (3.15) and (3.16) are also symplectic and hence they are integrable in the Liouville sense.This was also observed by Iatrou(2003b). We wish to recall that Byrnes *et al* (1999) has proved that if an N -th order autonomous difference equation is measure preserving and has at least $N-2$ integrals then it has a degenerate Poisson structure, so defines a symplectic map on each $2D$ level set of these integrals. Difference equations given by (3.17), (3.18), (3.19) possessing 2 independent integrals belong to this category and they are expected to be integrable. The integrability of the above difference equations can be established(hopefully) by other means, for example through a criterion developed for discrete equations (Halburd,2005). The effectiveness of the method is also discussed for N -th order $O\Delta E$ in the Appendix.

Acknowledgments. *The work of R.S. forms part of the research project funded by CSIR, New Delhi. The work of C.U. is supported by UGC, New Delhi.*

Appendix A

From the analysis we observe that $A_{21}(w_{n+1}, w_{n+2}) = A_{12}(w_{n+2}, w_{n+1})$, $A_{32}(w_{n+1}, w_{n+2}) = A_{23}(w_{n+2}, w_{n+1})$, $A_{31}(w_{n+1}, w_{n+2}) = A_{13}(w_{n+2}, w_{n+1})$ in all the six fourth order difference equations and therefore we presented only the forms of $A_{11}(w_{n+1}, w_{n+2})$, $A_{12}(w_{n+1}, w_{n+2})$, $A_{13}(w_{n+1}, w_{n+2})$, $A_{22}(w_{n+1}, w_{n+2})$, $A_{23}(w_{n+1}, w_{n+2})$, $A_{33}(w_{n+1}, w_{n+2})$.
Explicit forms for $A_{ij}(w_{n+1}, w_{n+2})$ associated with fourth order difference equation (3.14):

$$A_{11} = 0,$$

$$A_{12} = \lambda_1(\lambda_1 w_{n+1} + \lambda_3)(\lambda_1 w_{n+2} + \lambda_3),$$

$$A_{13} = (\lambda_1 w_{n+1} + \lambda_3)((\lambda_1\lambda_3 w_{n+2} + \lambda_1\lambda_4) + 2\lambda_3^2 - \lambda_2\lambda_3),$$

$$\begin{aligned} A_{22} = {} & (2\lambda_3^2 - \lambda_2\lambda_3 + \lambda_1\lambda_3 w_{n+2} + \lambda_1\lambda_4)(\lambda_1 w_{n+1} + \lambda_3) + \lambda_1(\lambda_1 w_{n+2} + \lambda_3) \\ & \times(\lambda_4 + \lambda_1 w_{n+1}^2 + \lambda_1 w_{n+1}w_{n+2} + \lambda_2 w_{n+1} + \lambda_3 w_{n+2}), \end{aligned}$$

$$A_{23} = (\lambda_1\lambda_3 w_{n+2} + \lambda_1\lambda_4 + 2\lambda_3^2 - \lambda_3\lambda_2)(\lambda_4 + \lambda_1 w_{n+1}^2 + \lambda_1 w_{n+1}w_{n+2} + \lambda_2 w_{n+1} + \lambda_3 w_{n+2}),$$

$$\begin{aligned} A_{33} = {} & (2\lambda_3^2 - \lambda_3\lambda_2 + \lambda_1\lambda_4)(\lambda_3 w_{n+2}^2 + \lambda_4 w_{n+2} + \lambda_4 w_{n+1}) + \lambda_1\lambda_3(\lambda_3 w_{n+2} + \lambda_3 w_{n+1} \\ & +\lambda_4)w_{n+1}w_{n+2} + \lambda_3(2\lambda_3^2 - \lambda_3\lambda_2 + \lambda_1\lambda_4)(w_{n+2} + w_{n+1})w_{n+1}. \end{aligned}$$

Explicit forms for $A_{ij}(w_{n+1}, w_{n+2})$ associated with fourth order difference equation (3.15):

$$A_{11} = \lambda_1^2(\lambda_1 w_{n+2} + \lambda_1 w_{n+1} - 2\lambda_4 + \lambda_2),$$

$$\begin{aligned} A_{12} = {} & \lambda_1(\lambda_1 w_{n+1} + \lambda_1 w_{n+2} + \lambda_4)(\lambda_1 w_{n+2} + \lambda_1 w_{n+1} - 2\lambda_4 + \lambda_2) \\ & +\lambda_1^2((2\lambda_2 - 4\lambda_4)w_{n+1} + \lambda_1 w_{n+1}w_{n+2} + \lambda_1 w_{n+1}^2 + \lambda_2 w_{n+2} - 2\lambda_4 w_{n+2}), \end{aligned}$$

$$\begin{aligned} A_{13} = {} & \lambda_1(\lambda_1 w_{n+1} + \lambda_1 w_{n+2} + \lambda_4)(2\lambda_2 w_{n+1} - 4\lambda_4 w_{n+1} + \lambda_1 w_{n+1}w_{n+2} + \lambda_1 w_{n+1}^2 \\ & +\lambda_2 w_{n+2} - 2\lambda_4 w_{n+2}), \end{aligned}$$

$$\begin{aligned} A_{22} = {} & 2\lambda_1^3 w_{n+2}^3 + \lambda_1(\lambda_1(\lambda_3 - 16\lambda_4 w_{n+1} + 7\lambda_1 w_{n+1}^2 + 11\lambda_2 w_{n+1}) + 2\lambda_2(\lambda_2 - 2\lambda_4))w_{n+2} \\ & +\lambda_1^2(5\lambda_2 - 7\lambda_4 + 7\lambda_1 w_{n+1})w_{n+2}^2 + 2\lambda_1^3 w_{n+1}^3 + \lambda_1^2(5\lambda_2 - 7\lambda_4)w_{n+1}^2 + \lambda_1(\lambda_1\lambda_3 \\ & +2\lambda_2^2 - 4\lambda_2\lambda_4)w_{n+1} + \lambda_1\lambda_2\lambda_3 + \lambda_4(4\lambda_2\lambda_4 - \lambda_2^2 - 4\lambda_4^2 - 2\lambda_1\lambda_3), \end{aligned}$$

$$\begin{aligned} A_{23} = {} & 2\lambda_1^2(\lambda_1 w_{n+1} + \lambda_2 - 2\lambda_4)w_{n+2}^3 + (\lambda_1^2(5\lambda_1 w_{n+1} + 8\lambda_2 - 13\lambda_4)w_{n+1} + \lambda_1(2\lambda_4^2 \\ & +2\lambda_2^2 - 5\lambda_2\lambda_4))w_{n+2}^2 + (\lambda_1(\lambda_1(4\lambda_1 w_{n+1} + 9\lambda_2 - 13\lambda_4)w_{n+1} + 4\lambda_2^2 - 8\lambda_2\lambda_4 \\ & +\lambda_1\lambda_3)w_{n+1} + \lambda_4(4\lambda_4^2 - 4\lambda_2\lambda_4 - 2\lambda_1\lambda_3 + \lambda_2^2) + \lambda_1\lambda_2\lambda_3)w_{n+2} + \lambda_1^3 w_{n+1}^4 \\ & +\lambda_1^2(3\lambda_2 - 4\lambda_4)w_{n+1}^3 + \lambda_1(\lambda_1\lambda_3 + 2\lambda_2^2 - 4\lambda_2\lambda_4)w_{n+1}^2 + 2\lambda_1\lambda_3(\lambda_2 - 2\lambda_4)w_{n+1}, \end{aligned}$$

$$\begin{aligned}A_{33} &= \lambda_1^2(\lambda_1 w_{n+1}+\lambda_2-2\lambda_4)w_{n+2}^4+(3\lambda_1^3w_{n+1}^2+\lambda_1^2(5\lambda_2-8\lambda_4)w_{n+1}+\lambda_1(2\lambda_2^2+4\lambda_4^2\\&\quad-6\lambda_2\lambda_4))w_{n+2}^3+(3\lambda_1^3w_{n+1}^3+4\lambda_1^2(2\lambda_2-3\lambda_4)w_{n+1}^2+\lambda_1(6\lambda_4^2-13\lambda_2\lambda_4+5\lambda_2^2\\&\quad+\lambda_1\lambda_3)w_{n+1}+\lambda_4(4\lambda_2\lambda_4-4\lambda_2^2-2\lambda_1\lambda_3)+\lambda_2(\lambda_2^2+\lambda_1\lambda_3))w_{n+2}^2+(\lambda_1^3w_{n+1}^4\\&\quad+\lambda_1^2(5\lambda_2-8\lambda_4)w_{n+1}^3+\lambda_1(6\lambda_4^2-13\lambda_2\lambda_4+5\lambda_2^2+\lambda_1\lambda_3)w_{n+1}^2+(4\lambda_4^3-6\lambda_1\lambda_3\lambda_4\\&\quad-3\lambda_2^2\lambda_4+3\lambda_1\lambda_2\lambda_3+\lambda_2^3)w_{n+1}+\lambda_3(4\lambda_4^2-4\lambda_2\lambda_4+\lambda_2^2))w_{n+2}\\&\quad+w_{n+1}(\lambda_2-2\lambda_4)(\lambda_1w_{n+1}+\lambda_2-2\lambda_4)(\lambda_1w_{n+1}^2+\lambda_2w_{n+1}+\lambda_3).\end{aligned}$$

Explicit forms for $A_{ij}(w_{n+1},w_{n+2})$ associated with fourth order difference equation (3.16):

$$A_{11}=\lambda_1^3(\lambda_1w_{n+1}w_{n+2}+(\lambda_2-\lambda_4)(w_{n+1}+w_{n+2})),$$

$$\begin{aligned}A_{12}&=\lambda_1^2(\lambda_1w_{n+2}+\lambda_4)(\lambda_1w_{n+1}w_{n+2}+(\lambda_2-\lambda_4)(w_{n+2}+w_{n+1}))\\&\quad+\lambda_1(\lambda_2-\lambda_4)(\lambda_1^2w_{n+1}w_{n+2}+(\lambda_2-\lambda_4)(\lambda_1w_{n+1}-\lambda_2+\lambda_4)),\end{aligned}$$

$$A_{13}=(\lambda_1w_{n+2}+\lambda_4)(\lambda_2-\lambda_4)(\lambda_1^2w_{n+1}w_{n+2}+(\lambda_2-\lambda_4)(\lambda_1w_{n+1}+\lambda_4-\lambda_2)),$$

$$\begin{aligned}A_{22}&=\lambda_1^2(\lambda_1w_{n+1}+\lambda_2)(\lambda_2-\lambda_4+\lambda_1w_{n+1})w_{n+2}^2+\lambda_1^2(\lambda_1(2\lambda_2-\lambda_4)w_{n+1}^2\\&\quad+(\lambda_1\lambda_3+2\lambda_2^2-\lambda_2\lambda_4-\lambda_4^2)w_{n+1}+\lambda_3(\lambda_2-\lambda_4))w_{n+2}\\&\quad+(\lambda_2-\lambda_4)(\lambda_1^2(\lambda_2w_{n+1}+\lambda_3)w_{n+1}+(\lambda_2-\lambda_4)(\lambda_4^2-\lambda_2^2)),\end{aligned}$$

$$\begin{aligned}A_{23}&=\lambda_1(\lambda_2-\lambda_4)(\lambda_1^2w_{n+1}^2+\lambda_1(2\lambda_2-\lambda_4)w_{n+1}+(\lambda_2-\lambda_4)^2)w_{n+2}^2\\&\quad+(\lambda_2-\lambda_4)(\lambda_1^2(\lambda_2w_{n+1}+\lambda_3)w_{n+1}-(\lambda_2-\lambda_4)((\lambda_2-\lambda_4)^2-2\lambda_1\lambda_4w_{n+1}))w_{n+2}\\&\quad+(\lambda_2-\lambda_4)^2(\lambda_4w_{n+1}+\lambda_3)(\lambda_1w_{n+1}-\lambda_2+\lambda_4),\end{aligned}$$

$$\begin{aligned}A_{33}&=(\lambda_2-\lambda_4)^2((\lambda_1^2w_{n+1}^2+\lambda_1\lambda_2w_{n+1}+(\lambda_2-\lambda_4)^2)w_{n+2}^2+(\lambda_1\lambda_2w_{n+1}^2\\&\quad-(\lambda_2^2-3\lambda_2\lambda_4+2\lambda_4^2-\lambda_1\lambda_3)w_{n+1})w_{n+2}+(\lambda_2-\lambda_4)^2w_{n+1}^2).\end{aligned}$$

Explicit forms for $A_{ij}(w_{n+1},w_{n+2})$ associated with fourth order difference equation (3.17):

$$A_{11}=0,$$

$$A_{12}=\alpha_{11}\lambda_1(\lambda_1w_{n+1}+\lambda_3),$$

$$A_{13}=\lambda_1(\lambda_1w_{n+1}+\lambda_3)(\alpha_{11}(w_{n+1}+w_{n+2})+\alpha_{12}),$$

$$\begin{aligned}A_{22}&=\alpha_{12}\lambda_1(\lambda_1(w_{n+1}+w_{n+2})+2\lambda_3)+\alpha_{11}(\lambda_1(\lambda_1(w_{n+1}^2+w_{n+2}^2)\\&\quad+(3\lambda_1w_{n+1}+4\lambda_3)w_{n+2}+4\lambda_3w_{n+1}+\lambda_4)+\lambda_3(3\lambda_3-\lambda_2)),\end{aligned}$$

$$A_{23} = \alpha_{12}\lambda_1((\lambda_1 w_{n+1} + 2\lambda_3)w_{n+2} + (\lambda_1 w_{n+1} + \lambda_2)w_{n+1} + \lambda_4)$$
$$+\alpha_{11}(\lambda_1((3\lambda_3 + 2\lambda_1 w_{n+1})w_{n+2}^2 + (3\lambda_1 w_{n+1}^2 + 2\lambda_2 w_{n+1} + \lambda_4)w_{n+2}$$
$$+\lambda_1 w_{n+1}^3 + \lambda_2 w_{n+1}^2 + \lambda_4 w_{n+1}) + \lambda_3(\lambda_2 - 3\lambda_3)w_{n+2}),$$

$$A_{33} = \alpha_{12}\lambda_1(\lambda_3 w_{n+2}^2 + (2\lambda_3 w_{n+1} + \lambda_4)w_{n+2} + (\lambda_3 w_{n+1} + \lambda_4)w_{n+1}) + \alpha_{11}(\lambda_1(\lambda_1 w_{n+1}$$
$$+2\lambda_3)w_{n+2}^3 + (2\lambda_1(\lambda_1 w_{n+1} + \lambda_2)w_{n+1} + \lambda_1\lambda_4 + 2\lambda_3(\lambda_2 - 3\lambda_3))w_{n+2}^2 + (\lambda_1^2 w_{n+1}^3$$
$$+2\lambda_1\lambda_2 w_{n+1}^2 + (\lambda_2^2 + 2\lambda_1\lambda_4 - 2\lambda_2\lambda_3 - 3\lambda_3^2)w_{n+1} + \lambda_4(\lambda_2 - 3\lambda_3))w_{n+2}$$
$$+2\lambda_1\lambda_3 w_{n+1}^3 + (\lambda_1\lambda_4 + 2\lambda_2\lambda_3 - 6\lambda_3^2)w_{n+1}^2 + \lambda_4(\lambda_2 - 3\lambda_3)w_{n+1}).$$

Explicit forms for $A_{ij}(w_{n+1}, w_{n+2})$ associated with fourth order difference equation (3.18):

$$A_{11} = \alpha_8\lambda_1^2,$$

$$A_{12} = \lambda_1(\alpha_8(4\lambda_1 w_{n+1} + 2\lambda_1 w_{n+2} + \lambda_3) + 2\alpha_{12}\lambda_1),$$

$$A_{13} = \lambda_1(2\lambda_1 w_{n+1} + \lambda_1 w_{n+2} + \lambda_3)(2\alpha_8 w_{n+1} + \alpha_8 w_{n+2} + 2\alpha_{12}),$$

$$A_{22} = \alpha_{12}(2\lambda_1\lambda_2 + 8\lambda_1^2(w_{n+2} + w_{n+1})) + \alpha_8(2\lambda_1\lambda_2(w_{n+1} + w_{n+2})$$
$$+2\lambda_1^2(8w_{n+2}w_{n+1} + 3(w_{n+1}^2 + w_{n+2}^2)) + \lambda_1\lambda_4 + 2\lambda_3^2 - \lambda_2\lambda_3),$$

$$A_{23} = \alpha_{12}(2\lambda_1^2(2w_{n+1}^2 + 4w_{n+2}w_{n+1} + w_{n+2}^2) + 2\lambda_1\lambda_2 w_{n+2} + 4\lambda_1(\lambda_2 - \lambda_3)w_{n+1}$$
$$+2\lambda_1\lambda_4) + \alpha_8(2\lambda_1\lambda_2 w_{n+2}^2 + 2\lambda_1^2(2w_{n+1}^3 + w_{n+2}^3 + 7w_{n+2}w_{n+1}^2 + 5w_{n+2}^2 w_{n+1})$$
$$+6\lambda_1\lambda_2 w_{n+1}w_{n+2} - \lambda_1\lambda_3(4w_{n+1}^2 + w_{n+2}^2 + 4w_{n+2}w_{n+1}) + \lambda_1\lambda_4(w_{n+2} + 2w_{n+1})$$
$$+\lambda_2\lambda_3 w_{n+2} - 2\lambda_3^2 w_{n+2} + 4\lambda_1\lambda_2 w_{n+1}^2),$$

$$A_{33} = \alpha_{12}(2\lambda_1\lambda_4(w_{n+1} + w_{n+2}) + 2\lambda_1\lambda_2 w_{n+1}w_{n+2} + 2\lambda_1\lambda_3(w_{n+1}^2 + w_{n+2}^2)$$
$$+2\lambda_1^2 w_{n+1}w_{n+2}(w_{n+1} + w_{n+2})) + \alpha_8(-2\lambda_2\lambda_3 w_{n+1}^2 + 2\lambda_3^2 w_{n+2}w_{n+1}$$
$$+2\lambda_2^2 w_{n+2}w_{n+1} - 2\lambda_2\lambda_3 w_{n+2}^2 + \lambda_1^2 w_{n+2}^4 + \lambda_1^2 w_{n+1}^4 - 2\lambda_4\lambda_3 w_{n+1} + \lambda_1\lambda_4 w_{n+2}^2$$
$$+11\lambda_1^2 w_{n+1}^2 w_{n+2}^2 + 8\lambda_1\lambda_2 w_{n+1}w_{n+2}^2 - 9\lambda_1\lambda_3 w_{n+1}w_{n+2}^2 + 8\lambda_1\lambda_2 w_{n+1}^2 w_{n+2}$$
$$-9\lambda_1\lambda_3 w_{n+1}^2 w_{n+2} + 3\lambda_1\lambda_4 w_{n+1}w_{n+2} - 5\lambda_2\lambda_3 w_{n+2}w_{n+1} + 6\lambda_1^2 w_{n+2}^3 w_{n+1}$$
$$+2\lambda_1\lambda_2 w_{n+2}^3 - 2\lambda_1\lambda_3 w_{n+2}^3 + 6\lambda_1^2 w_{n+1}^3 w_{n+2} + \lambda_4\lambda_2 w_{n+2} - 2\lambda_3\lambda_4 w_{n+2}$$
$$+2\lambda_1\lambda_2 w_{n+1}^3 - 2\lambda_1\lambda_3 w_{n+1}^3 + \lambda_1\lambda_4 w_{n+1}^2 + \lambda_4\lambda_2 w_{n+1} + \lambda_2^2 w_{n+2}^2 + \lambda_2^2 w_{n+1}^2).$$

Explicit forms for $A_{ij}(w_{n+1}, w_{n+2})$ associated with fourth order difference equation (3.19):

$$A_{11} = A_{12} = 0,$$

$$A_{13} = (\lambda_1 w_{n+1} + \lambda_4 + \lambda_5)(\lambda_4 + \lambda_1 w_{n+1} - \lambda_5),$$

$$A_{22} = (\lambda_1 w_{n+1} + \lambda_4 + \lambda_5)(\lambda_1 w_{n+2} + \lambda_4 + \lambda_5),$$

$$A_{23} = (\lambda_1 w_{n+1} + \lambda_4 + \lambda_5)(2\lambda_4 w_{n+1} + \lambda_5 w_{n+2} + \lambda_4 w_{n+2} + \lambda_3),$$

$$A_{33} = (\lambda_4 + \lambda_5)((\lambda_4 + \lambda_5)w_{n+2}w_{n+1} + \lambda_3(w_{n+1} + w_{n+2}) + \lambda_4(w_{n+2}^2 + w_{n+1}^2) - \lambda_5(w_{n+2}^2 + w_{n+1}^2)).$$

Appendix B

Consider an Nth-order autonomous $O\Delta E$ having the form

$$w_{n+N} = F(w_n, ...w_{n+N-1}). \tag{B1}$$

Let us assume that equation (B1) admits a polynomial integral $I(n)$ having the form

$$I(n) = \sum_{j=1}^{3}[A_{1j}(n)w_{n+N-1}^2 + A_{2j}(n)w_{n+N-1} + A_{3j}(n)]w_n^{3-j}, \tag{B2}$$

where

$$A_{ij}(n) = A_{ij}(w_{n+1},\ w_{n+2},\ ...,\ w_{n+N-2}),\ i,\ j = 1, 2, 3$$

are unknown functions. Then the integrability condition $I(n+1) - I(n) = 0$ leads to a quadratic equation in w_{n+N}

$$F_1(n)w_{n+N}^2 + F_2(n)w_{n+N} + F_3(n) - (F_4(n)w_n^2 + F_5(n)w_n + F_6(n)) = 0, \tag{B3}$$

where
$F_i(n) = \sum\limits_{j=1}^{3} A_{ij}(n+1)w_{n+1}^{3-j},\ i = 1, 2, 3,\ F_k(n) = \sum\limits_{j=1}^{3} A_{jk}(n)w_{n+N-1}^{3-j},\ k = 1, 2, 3.$
It is straightforward to check that equation (B3) can be factorised as

$$\left(w_{n+N} + w_n + \frac{F_2(n) + F_5(n)}{2F_1(n)}\right)\left(w_{n+N} - w_n + \frac{F_2(n) - F_5(n)}{2F_1(n)}\right) = 0 \tag{B4}$$

provided

$$A_{11}(n+1)w_{n+1}^2 - A_{11}(n)w_{n+N-1}^2 + A_{12}(n+1)w_{n+1} - A_{21}(n)w_{n+N-1} = A_{31}(n) - A_{13}(n+1) \tag{B5a}$$

and

$$[F_5(n) + F_2(n)][F_5(n) - F_2(n)] = 4[F_6(n) - F_3(n)]F_1(n). \tag{B5b}$$

Thus we obtain,

$$w_{n+N} = -w_n - \frac{F_2(n) + F_5(n)}{2F_1(n)}, \tag{B6}$$

$$w_{n+N} = w_n - \frac{F_2(n) - F_5(n)}{2F_1(n)}. \tag{B7}$$

Obviously (B5b) is satisfied for distinct possibilities

(1) $F_5 = F_2$ and $F_6 = F_3$,

(2) $F_5 \neq F_2$ and $F_6 \neq F_3$.

Demanding that one of the integrals of the given $O\Delta E$ (B1), say, $I_1(n)$ is cyclic invariant, that is

$$\begin{aligned} I_1(w_n, w_{n+1}, \cdots, w_{n+N-2}, w_{n+N-1}) &= I_1(w_{n+1}, w_{n+2}, \cdots, w_{n+N-1}, w_n) \\ = I_1(w_{n+2}, w_{n+3}, \cdots, w_n, w_{n+1}) &= \cdots = I_1(w_{n+N-1}, w_n, \cdots, w_{n+N-3}, w_{n+N-2}) \end{aligned}$$

it is easy to verify that both the conditions given in equation (B5a) as well as $F_5 = F_2$ and $F_6 = F_3$ satisfy. In this case the integral I_1 will take the form

$$I_1(n) = \sum_{0 \le i_1 \le i_2 \le \ldots \le i_n \le 2} a_{i_1 i_2 \ldots i_n} w_{n+1}^{i_1} w_{n+2}^{i_2} \ldots w_{n+N+1}^{i_n},$$

where

$$a_{i_1, i_2, \cdots, i_{n-1}, i_n} = a_{i_2, i_3, \cdots, i_n, i_1} = \cdots = a_{i_n, i_1, \cdots, i_{n-2}, i_{n-1}}, \text{ provided } i_1 \neq i_2 \neq \cdots \neq i_n.$$

Remaining integrals $I_2(n), I_3(n) \cdots I_{N-1}(n)$ may be constructed by demanding them not to be cyclic invariant. We find that the integrals (2.24) of the third order difference equation (2.22) and (2.27) of (2.25) are cyclic invariant.

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