Polynomial integrals for third- and fourth-order ordinary difference equations

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Abstract

A direct method to construct polynomial integrals for third order ordinary difference equation (O\(\Delta\)E) \(w(n + 3) = F(w(n), w(n + 1), w(n + 2))\) and fourth order O\(\Delta\)E \(w(n + 4) = F(w(n), w(n + 1), w(n + 2), w(n + 3))\) is presented. The effectiveness of the method to construct more than one polynomial integral for N-th order O\(\Delta\)E is also briefly discussed.

1 Introduction

The discrete nonlinear systems governed by both ordinary difference equations or mappings and partial difference equations or lattice equations have drawn much attention by researchers working under different areas of applied science [1, 2, 3, 5, 7, 9, 10]. Since discrete systems governed by difference equations are more fundamental than the continuous ones described by differential equations their study becomes essential which will lead to the development of a general theory of discrete and in particular nonlinear difference equations. Even though there exists no unique definition of integrability considerable number of analytical methods have been formulated by different groups in recent years to deal with integrability [4, 7, 8, 10, 11, 12, 13, 16, 18, 19, 20, 21, 22, 24, 25] and significant advancement has already been made for the second order both for autonomous and nonautonomous cases [5, 7, 9, 10, 17, 19, 20, 25]. We take the working definition of integrability, here, the one which is related with the existence of sufficient number of integrals of an O\(\Delta\)E. An integral (also referred to as conserved quantity) of O\(\Delta\)E is a function that is not identically constant but is constant on all solutions of it. An autonomous N-th order nonlinear O\(\Delta\)E is said to be integrable if it admits (N-1) functionally independent integrals. Note that if a difference equation is measure preserving in dimension \(N\) and has (at least) \((N - 2)\) independent integrals then it has a (degenerate) Poisson structure, so defines a symplectic map on each 2D level set of these integrals [6]. Given an autonomous N-th order nonlinear O\(\Delta\)E there exists no systematic analytic technique to derive its integrals enabling one to investigate its integrability. Recently a direct method was proposed...

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for $N$-th order autonomous difference equation to construct rational integrals and several new integrable difference equations of higher order were identified [23]. In this article we present a method to construct polynomial integrals through third and fourth order $O\Delta E$ [see 14,15 for a different method]. The effectiveness of the method is also discussed for $N$-th order $O\Delta E$.

The plan of the article is as follows. In §2 we describe the method, which involves factorisation, through third order autonomous $O\Delta E$ $w_{n+3} = F(w_n, w_{n+1}, w_{n+2})$, $w_n = w(n), w_{n+1} = w(n+1)$ etc. to derive 2 polynomial integrals. In §3 we extend it for fourth-order autonomous $O\Delta E$ $w_{n+4} = F(w_n, w_{n+1}, w_{n+2}, w_{n+3})$ and identify $F$ admitting 2 independent integrals. In §4 we present a brief summary of our results. The effectiveness of the method is also discussed for $N$-th order $O\Delta E$ in the Appendix.

2 Construction of integrals for third-order autonomous difference equation

Consider an autonomous third order $O\Delta E$ having the form

$$w_{n+3} = F(w_n, w_{n+1}, w_{n+2}) \text{ or } w_3 = F(w_0, w_1, w_2). \quad (2.1)$$

Hereafter, we denote $w_0 = w_n, w_1 = w_{n+1}, \ldots, w_N = w_{n+N}$ unless otherwise specified. Assume that equation (2.1) admits an integral $I(w_0, w_1, w_2)$ having the form

$$I(w_0, w_1, w_2) = \sum_{j=1}^{3} [A_{1j}(w_1)w_2^2 + A_{2j}(w_1)w_2 + A_{3j}(w_1)]w_0^{3-j}. \quad (2.2)$$

The integrability condition $I(w_0, w_1, w_2) - I(w_1, w_2, w_3) = 0$ leads to a quadratic equation in $w_3$

$$F_1(w_1, w_2)w_3^2 + F_2(w_1, w_2)w_3 - [F_4(w_1, w_2)w_0^2 + F_5(w_1, w_2)w_0 + F_6(w_1, w_2) - F_3(w_1, w_2)] = 0. \quad (2.3)$$

where

$$F_i(w_1, w_2) = \sum_{j=1}^{3} A_{ij}(w_2)w_1^{3-j}, \quad i = 1, 2, 3,$$

$$F_{k+3}(w_1, w_2) = \sum_{j=1}^{3} A_{jk}(w_1)w_2^{3-j}, \quad k = 1, 2, 3.$$

Hereafter we denote $F_i(w_1, w_2)$ as $F_i$ for the remaining section unless otherwise specified. Equation (2.3) can be factorised as

$$\left( w_3 + w_0 + \frac{F_5 + F_2}{2F_1} \right) \left( w_3 - w_0 + \frac{F_2 - F_5}{2F_1} \right) = 0 \quad (2.4)$$
Solving equation (2.5a) yields provided
\[ A_{11}(w_2)w_1^2 + A_{12}(w_2)w_1 + A_{13}(w_2) = A_{11}(w_1)w_2^2 + A_{21}(w_1)w_2 + A_{31}(w_1) \] (2.5a)
and
\[ (F_5 + F_2)(F_5 - F_2) = 4(F_6 - F_3)F_1 \] (2.5b)
Thus we obtain,
\[ w_3 = -w_0 - \frac{F_2 + F_5}{2F_1}, \] (2.6)
\[ w_3 = w_0 - \frac{F_2 - F_5}{2F_1}. \] (2.7)
Solving equation (2.5a) yields
\[ A_{11}(w_2) = a_1w_2^2 + a_2w_2 + a_3, \] (2.8a)
\[ A_{21}(w_2) = a_2w_2^2 + b_2w_2 + c_2, \] (2.8b)
\[ A_{31}(w_2) = a_3w_2^2 + b_3w_2 + c_3, \] (2.8c)
\[ A_{12}(w_2) = a_2w_2^2 + b_2w_2 + b_3, \] (2.8d)
\[ A_{15}(w_2) = a_3w_2^2 + c_2w_2 + c_3, \] (2.8e)
where \(a_1, a_2, a_3, b_2, b_3, c_2\) and \(c_3\) are arbitrary constants. Equation (2.5b) suggests that it can be solved for two distinct possibilities:
\[ (i) \quad F_5 = F_2 \quad \text{and} \quad F_6 = F_3, \] (2.9)
\[ (ii) \quad F_5 \neq F_2 \quad \text{and} \quad F_6 \neq F_3. \] (2.10)
The conditions given in equation (2.9) can be rewritten respectively as
\[ A_{12}(w_1)w_2^2 - A_{21}(w_2)w_1^2 = A_{22}(w_2)w_1 - A_{21}(w_1)w_2 + A_{32}(w_2) - A_{32}(w_1), \] (2.11)
\[ A_{15}(w_1)w_2^2 - A_{31}(w_2)w_1^2 = A_{32}(w_2)w_1 - A_{21}(w_1)w_2 + A_{33}(w_2) - A_{33}(w_1). \] (2.12)
Making use of the forms for \(A_{11}(w_2), A_{12}(w_2), A_{13}(w_2), A_{21}(w_2)\) and \(A_{31}(w_2)\) given in equations (2.8a)-(2.8e) in (2.11) and (2.12) we obtain
\[ A_{22}(w_2) = b_2w_2^2 + e_2w_2 + e_3, \] (2.13a)
\[ A_{23}(w_2) = b_1w_2^2 + e_3w_2 + f_3, \] (2.13b)
\[ A_{32}(w_2) = c_2w_2^2 + e_3w_2 + f_3, \] (2.13c)
\[ A_{33}(w_2) = c_3w_2^2 + f_3w_2 + j_3, \] (2.13d)
where \(e_2, e_3, f_3\) and \(j_3\) are arbitrary constants. Thus we obtain a third order difference equation
\[ w_3 = -w_0 \frac{(a_2w_2^3 + b_2w_2 + c_2)w_1^2 + b_2w_1w_2^2 + e_2w_1w_2 + e_3w_1 + b_3w_2^2 + e_3w_2 + f_3}{(a_1w_2^2 + a_2w_2 + a_3)w_1^2 + a_2w_1w_2^2 + b_2w_1w_2 + b_3w_1 + a_3w_2^2 + c_2w_2 + c_3} \] (2.14)
with one integral

\[ I_1 = [(a_1 w_1^2 + a_2 w_1 + a_3) w_0^2 + (a_2 w_1^2 + b_2 w_1 + b_3) w_0 + a_3 w_1^2 + c_2 w_1 + c_3] w_2^2 + [(a_2 w_1^2 + b_2 w_1 + c_2) w_0^2 + (b_2 w_1^2 + e_2 w_1 + e_3) w_0 + b_3 w_1^2 + e_3 w_1 + f_3] w_2 + (a_3 w_1^2 + b_3 w_1 + c_3) w_0^2 + (c_2 w_1^2 + e_3 w_1 + f_3) w_0 + c_3 w_1^2 + f_3 w_1. \]  
\tag{2.15} \]

We would like to mention that equation (2.14) can also be rewritten as

\[ w_3 = -w_0 - \left( \frac{a_1 w_1 w_2 + a_2 w_1 + a_3 w_0 + a_4)(\gamma_1 w_1 w_2 + \gamma_2 w_1 + \gamma_3 w_2 + \gamma_4)}{(\alpha_1 w_0 w_2 + a_2 w_1 + a_3 w_2 + \beta_2 w_1 + \beta_2 w_2 + \beta_4)} \right) \]  
\tag{2.16} \]

by choosing the constants \(a_1, a_2, a_3, b_2, b_3, c_2, c_3, e_2\) and \(f_3\) appropriately. For example, equation (2.14) becomes

\[ w_3 = -w_0 - \frac{2\beta_3 w_1 w_2 + \gamma_2(w_1 + w_2) + \gamma_4}{\beta_1 w_1 w_2 + \beta_3(w_1 + w_2) + \gamma_2 - \alpha_4 \beta_3} \]  
\tag{2.17} \]

provided

\[ a_1 = \beta_1^3, \quad a_2 = 2\beta_1 \beta_3, \quad a_3 = \beta_2^3, \]
\[ b_2 = (\beta_1 \gamma_2 + 2\beta_2^2), \quad c_2 = b_3 = \gamma_2 \beta_3, \quad c_3 = \alpha_4 \beta_3 (\gamma_2 - \alpha_4 \beta_3), \]
\[ e_2 = (2\gamma_2 \beta_3 + \beta_1 \gamma_4 + 2\alpha_4 \beta_3^2), \quad e_3 = \beta_3 (\alpha_4 \gamma_2 + \gamma_4), \quad f_3 = \alpha_4 \gamma_4 \beta_3, \quad j_3\text{-arbitrary} \]

and the integral (2.15) becomes

\[ I_1 = (\beta_1 w_0 w_1 + \beta_3 (w_0 + w_1) + \alpha_4 \beta_3)(\beta_1 w_0 w_1 + \beta_3 (w_0 + w_1) + \gamma_2 - \alpha_4 \beta_3) w_2 + 2\beta_3 w_0 w_1 + \gamma_2 (w_0 + w_1) + \gamma_4 w_2 + \beta_3 (w_1 + \alpha_4)(\beta_3 w_0 w_1 + \gamma_2 w_1) + (\gamma_2 - \alpha_4 \beta_3) w_0 + \alpha_4 \beta_3 (\gamma_2 - \beta_3 \alpha_4) w_1^2 + \alpha_4 \beta_3 \gamma_4 w_1. \]  
\tag{2.18} \]

In order to construct second integral \(I_2\) we use the other possibility given in equation (2.10). It is clear from equation (2.5 a) that \(A_{11}(w_1), A_{12}(w_1), A_{13}(w_1), A_{21}(w_1)\) and \(A_{31}(w_1)\) are quadratic polynomials. However, equation (2.5b) suggests that \(A_{22}(w_1), A_{23}(w_1), A_{32}(w_1)\) and \(A_{33}(w_1)\) may be quartic polynomials. Thus we consider

\[ A_{11}(w_1) = \tilde{a}_1 w_1^2 + \tilde{a}_2 w_1 + \tilde{a}_3, \]  
\tag{2.19a} \]
\[ A_{21}(w_1) = \tilde{a}_2 w_1^2 + \tilde{b}_2 w_1 + \tilde{c}_2, \]  
\tag{2.19b} \]
\[ A_{31}(w_1) = \tilde{a}_3 w_1^2 + \tilde{b}_3 w_1 + \tilde{c}_3, \]  
\tag{2.19c} \]
\[ A_{12}(w_1) = \tilde{a}_2 w_1^2 + \tilde{b}_2 w_1 + \tilde{b}_3, \]  
\tag{2.19d} \]
\[ A_{13}(w_1) = \tilde{a}_3 w_1^2 + \tilde{c}_2 w_1 + \tilde{c}_3, \]  
\tag{2.19e} \]
\[ A_{22}(w_1) = \epsilon_1 w_1^4 + \epsilon_2 w_1^2 + \epsilon_3 w_1^2 + \epsilon_4 w_1 + \epsilon_5, \]  
\tag{2.19f} \]
\[ A_{23}(w_1) = \epsilon_6 w_1^4 + \epsilon_7 w_1^2 + \epsilon_8 w_1^2 + \epsilon_9 w_1 + \epsilon_{10}, \]  
\tag{2.19g} \]
\[ A_{32}(w_1) = \epsilon_{11} w_1^4 + \epsilon_{12} w_1^2 + \epsilon_{13} w_1^2 + \epsilon_{14} w_1 + \epsilon_{15}, \]  
\tag{2.19h} \]
\[ A_{33}(w_1) = \epsilon_{16} w_1^4 + \epsilon_{17} w_1^2 + \epsilon_{18} w_1^2 + \epsilon_{19} w_1 + \epsilon_{20}, \]  
\tag{2.19i} \]
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where \( \tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \tilde{b}_2, \tilde{b}_3, \tilde{c}_2, \tilde{c}_3 \) and \( \epsilon_i, i = 1, 2, \ldots, 20 \) are unknown constants to be determined. Then equation (2.6) reads

\[ w_3 = -w_0 - \frac{\tilde{F}}{2G} \tag{2.20} \]

where,

\[
\tilde{F} = \epsilon_{11} w_1^4 + \epsilon_6 w_2^3 + \epsilon_{12} w_3^3 + \epsilon_7 w_1^3 + \epsilon_1 (w_1^3 + w_2^3) w_1 w_2 + (\epsilon_3 + b_2) (w_1 + w_2) w_1 w_2 + (\epsilon_5 + \epsilon_9) w_1 + \epsilon_{10} + \epsilon_{15},
\]

\[
\tilde{G} = (\tilde{a}_1 w_1^2 + \tilde{a}_2 w_2^2 + \tilde{a}_3 w_3^2 + \tilde{a}_4) (\tilde{b}_1 w_1 + \tilde{b}_2 w_2 + \tilde{b}_3 w_3 + \tilde{c}_3).
\]

We would like to mention here that both \( \tilde{F} \) and \( \tilde{G} \) can be factored with a common factor by choosing the constants \( \tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \tilde{b}_2, \tilde{b}_3, \tilde{c}_2, \tilde{c}_3 \) and \( \epsilon_i, i = 1, 2, \ldots, 20 \), appropriately. That is,

\[
\tilde{F} = (\tilde{a}_1 w_1 + \tilde{a}_2 + \tilde{a}_3 + \tilde{a}_4) [\tilde{b}_1 w_1 + \tilde{b}_2 + \tilde{b}_3 w_2 + \tilde{b}_4]
\]

For example, equation (2.20) reduces into

\[ w_3 = -w_0 - \frac{\tilde{a}_4 (4\tilde{b}_2 w_1 + \tilde{b}_3 w_2 + \tilde{c}_8 (w_1 + w_2))}{2\tilde{b}_2 \tilde{a}_4 w_1 + 2\tilde{a}_4 \tilde{b}_2 (w_1 + w_2) + \tilde{a}_4 \tilde{c}_8 - 2\epsilon_9} \tag{2.21} \]

provided

\[
\tilde{a}_1 = \tilde{a}_2 = \tilde{a}_3 = 0, \quad \tilde{b}_2 = \tilde{a}_4 \tilde{b}_1, \quad \tilde{c}_2 = \tilde{b}_3 = \tilde{a}_4 \tilde{b}_2, \quad \tilde{c}_3 = \frac{\tilde{a}_4 \tilde{c}_8}{2} - \epsilon_9,
\]

\[
\tilde{\epsilon}_3 = -\tilde{b}_2, \quad \tilde{\epsilon}_1 = 2\tilde{b}_3, \quad \tilde{\epsilon}_4 = -\tilde{b}_3 + \tilde{a}_4 \tilde{c}_8, \quad \tilde{\epsilon}_8 = -\tilde{b}_3, \quad \epsilon_9 - \text{arbitrary},
\]

\[
\epsilon_1 = \epsilon_2 = \epsilon_6 = \epsilon_7 = \epsilon_{11} = \epsilon_{12} = \epsilon_9 = \epsilon_{16} = \epsilon_{17} = \epsilon_{19} = \epsilon_{20} = 0.
\]

Also the conditions (2.5a,b) and (2.10) are identically satisfied for the above parametric restrictions which in turn leads to the existence of second integral \( I_2 \). Furthermore equation (2.21) can be rewritten as

\[ w_{n+3} = -w_n - \frac{2\lambda_1 w_{n+1} w_{n+2} + \lambda_2 (w_{n+1} + w_{n+2}) + \lambda_3}{\lambda_4 w_{n+1} w_{n+2} + \lambda_1 (w_{n+1} + w_{n+2}) + \lambda_5}, \tag{2.22} \]

where

\[
\tilde{\beta}_2 = \frac{\lambda_1}{2\alpha_4}, \quad \tilde{\gamma}_8 = \frac{\lambda_2}{\alpha_4}, \quad \tilde{\gamma}_9 = \frac{\lambda_3}{\alpha_4}, \quad \tilde{\beta}_1 = \frac{\lambda_4}{2\alpha_4}, \quad \epsilon_9 = \frac{\alpha_4 \gamma_8 - \lambda_5}{2}.
\]
and so the integral $I_2$ is

$$I_2 = [\lambda_4 w_{n+1} + \lambda_1 w_n + \lambda_1 w_{n+1} + \lambda_5] w_{n+2}^2 + [\lambda_4 w_{n+1} + \lambda_5] w_n + \lambda_1 w_{n+1}^2 - \lambda_4 w_{n+1} w_n$$

$$+ 2\lambda_1 w_n w_{n+1} + \lambda_3 w_n + \lambda_5 w_n - \lambda_1 w_{n+1}^2 + (\lambda_2 - \lambda_5) w_{n+1} + \lambda_4 w_{n+1} w_n$$

$$+ \lambda_5 w_n - \lambda_1 w_{n+1} w_n + (\lambda_2 - \lambda_5) w_{n+1} w_n + \lambda_3 w_n + (\lambda_5 - \lambda_2) w_n^2. \quad (2.23)$$

It is easy to see that the third order $O\Delta E$ (2.22) admitting the integral $I_1$ assumes exactly the same form as (2.22) by choosing the parameters $\beta_1, \beta_3, \gamma_2$ and $\gamma_4$ as

$$\alpha_4 = \frac{\lambda_2 - \lambda_5}{\lambda_1}, \quad \beta_1 = \lambda_4, \quad \beta_3 = \lambda_1, \quad \gamma_2 = \lambda_2, \quad \gamma_4 = \lambda_3.$$

and the integral $I_1$ becomes

$$I_1 = (\lambda_4 w_{n+1} + \lambda_1 w_n + \lambda_1 w_{n+1} + \lambda_2 - \lambda_5) (\lambda_4 w_{n+1} + \lambda_1 w_n + \lambda_1 w_{n+1} + \lambda_5) w_{n+2}$$

$$+ 2\lambda_1 w_n w_{n+1} + \lambda_3 w_n + \lambda_2 w_{n+1} + \lambda_3 w_{n+2} + (\lambda_1 w_{n+1} + \lambda_2 - \lambda_5) \lambda_1 w_{n+1} + \lambda_5 w_n$$

$$+ \lambda_2 w_{n+1} + \lambda_3 w_n + (\lambda_5 w_{n+1} + \lambda_3) (\lambda_2 - \lambda_5) w_{n+1}. \quad (2.24)$$

Thus we conclude that the third order $O\Delta E$ (2.22) admits 2 independent integrals $I_1$ and $I_2$.

Proceeding along the similar lines described above one can identify more than one third order $O\Delta E$ possessing 2 independent integrals since the factorisation of $F$ and $G$ explained earlier is not unique. For example $F$ and $G$ can also be factored as

$$F = 2(\beta_3 w_1 + \beta_3 w_2 + \beta_4) [\beta_3 (w_1 + w_2)^2 + \gamma_7 (w_1 + w_2) + \gamma_9]$$

$$G = (\beta_3 w_1 + \beta_3 w_2 + \beta_4)^2$$

provided

$$\delta_3 = \beta_3^2, \quad \delta_2 = 2\beta_3, \quad \delta_3 = 2\beta_3, \quad \delta_3 = \beta_3^2,$$

$$\epsilon_3 = 4\delta_3, \quad \epsilon_4 = 2\beta_3 (\delta_4 + \gamma_7), \quad \epsilon_5 = \beta_3 \gamma_9 + 2\delta_4^2, \quad \epsilon_7 = 2\delta_3 \gamma_7, \quad \epsilon_8 = 2\beta_3 \gamma_7,$$

$$\epsilon_9 = 2(\beta_3 \gamma_7 + \beta_3 \gamma_9) - \epsilon_5, \quad \epsilon_{10} = 2\beta_3 \gamma_9, \quad \epsilon_{12} = 2\beta_3 \gamma_7, \quad \epsilon_{13} = 2\beta_3 \gamma_7,$$

$$\epsilon_{14} = 2(\beta_3 \gamma_7 + \beta_3 \gamma_9) - \epsilon_5, \quad \epsilon_{15} = \beta_3 \gamma_9, \quad \epsilon_{16} = \beta_3^2, \quad \epsilon_{17} = 2\beta_3 (\gamma_7 - \beta_4),$$

$$\epsilon_{18} = -2\beta_3 \gamma_7 + \beta_3 \gamma_9 + \gamma_7^2 + \beta_4^2, \quad \epsilon_{19} = \gamma_9 (\gamma_7 - \beta_4),$$

$$\delta_1 = \delta_2 = \epsilon_1 = \epsilon_2 = \epsilon_6 = \epsilon_{11} = \epsilon_{20} = 0.$$

Here again the conditions (2.5ab) and (2.10) are identically satisfied for the above parametric restrictions. In this case equation (2.20) becomes

$$w_{n+3} = -w_n - \frac{\lambda_1 (w_{n+1} + w_{n+2})^2 + \lambda_2 (w_{n+1} + w_{n+2}) + \lambda_3}{\lambda_1 (w_{n+1} + w_{n+2}) + \lambda_4}, \quad (2.25)$$

where

$$\beta_3 = \lambda_1, \quad \gamma_7 = \lambda_2, \quad \gamma_9 = \lambda_3, \quad \beta_4 = \lambda_4.$$
and admits second integral \( I_2 \) as

\[
I_2 = (\lambda_1 w_n + \lambda_1 w_{n+1} + \lambda_4 w_{n+2} + \lambda_2 w_n w_{n+2} + \lambda_4 w_n w_{n+1} + \lambda_1 w_{n+1} w_{n+2} + \lambda_3 w_{n+1} w_{n+2})
\]

\[
+ (\lambda_2 - \lambda_4) w_{n+1} + \lambda_1 w_{n+1} w_{n+2} + (\lambda_2 - \lambda_4) w_{n+1} + \lambda_3] .
\]

(2.26)

It is easy to see that equation (2.14) possessing an integral \( I_1 \) takes exactly the same form as (2.25) for the parametric restrictions

\[
c_2 = b_3 = \lambda_1, \quad e_2 = 2\lambda_1, \quad e_3 = \lambda_2, \quad f_3 = \lambda_3, \quad c_3 = \lambda_4, \quad a_1 = a_2 = b_2 = a_3 = 0
\]

and the integral \( I_1 \) becomes

\[
I_1 = (\lambda_1 w_n + \lambda_1 w_{n+1} + \lambda_4) w_{n+2} + (\lambda_1 w_n + 2\lambda_1 w_n w_{n+1} + \lambda_2 w_n + \lambda_1 w_{n+1} + \lambda_2 w_{n+1} + \lambda_3 w_{n+1} + \lambda_2 w_{n+1} w_{n+2} + \lambda_4 w_{n+1} w_{n+2} + \lambda_3 w_{n+1} w_{n+2})
\]

\[
+ \lambda_3] w_{n+2} + \lambda_1 w_{n+1} w_n^2 + \lambda_4 w_n^2 + \lambda_1 w_{n+1} w_n + \lambda_2 w_{n+1} w_n + \lambda_3 w_{n+1} + \lambda_4 w_{n+1} + \lambda_3 w_{n+1} .
\]

(2.27)

A detailed calculation shows that there exists no other third order difference equation possessing 2 independent polynomial integrals. We wish to mention that the identified third order difference equations (2.22) and (2.25) admitting 2 integrals are also measure preserving with measure 1 and hence they are integrable. Equations (2.22) and (2.25) were also obtained by Iatrou (2003a) using a different method. However, our analysis shows that there exists a third order difference equation involving at least 10 parameters possessing one cyclic integral \( I_1 \).

3 Construction of integrals for fourth order autonomous ordinary difference equation

Consider an autonomous fourth order \( O\Delta E \) having the form

\[
w_4 = F(\omega_0, w_1, w_2, w_3).
\]

(3.1)

Assume that equation (3.1) admits an integral \( I(w_0, w_1, w_2, w_3) \) having the form

\[
I(w_0, w_1, w_2, w_3) = \sum_{j=1}^{3} [A_{1j}(w_1, w_2) w_3^2 + A_{2j}(w_1, w_2) w_3 + A_{3j}(w_1, w_2)] w_0^{3-j}.
\]

(3.2)

Then the integrability condition \( I(n+1) - I(n) = 0 \) leads to a quadratic equation in \( w_4 \)

\[
F_1(w_1, w_2, w_3) w_4^2 + F_2(w_1, w_2, w_3) w_4
\]

\[
- [F_1(w_1, w_2, w_3) w_0^3 + F_2(w_1, w_2, w_3) w_0 + (F_3(w_1, w_2, w_3) - F_3(w_1, w_2, w_3))] = 0,
\]

(3.3)

where

\[
F_1(w_1, w_2, w_3) = \sum_{j=1}^{3} A_{ij}(w_2, w_3) w_1^{3-j}, i = 1, 2, 3,
\]

\[
F_{k+3}(w_1, w_2, w_3) = \sum_{j=1}^{3} A_{jk}(w_1, w_2) w_3^{3-j}, k = 1, 2, 3.
\]
Hereafter we denote \( F_i(w_1, w_2, w_3) \) as \( F_i \) for the remaining section unless otherwise specified. Equation (3.3) can be factorised as

\[
\left( w_4 + w_0 + \frac{F_5 + F_2}{2F_1} \right) \left( w_4 - w_0 + \frac{F_2 - F_5}{2F_1} \right) = 0 \tag{3.4}
\]

provided

\[
A_{11}(w_2, w_3)w_1^2 + A_{12}(w_2, w_3)w_1 + A_{13}(w_2, w_3) = A_{11}(w_1, w_2)w_3^2 + A_{21}(w_1, w_2)w_3 + A_{31}(w_1, w_2) \tag{3.5a}
\]

and

\[
(F_5 + F_2)(F_5 - F_2) = 4(F_6 - F_3)F_1. \tag{3.5b}
\]

Thus we obtain,

\[
w_4 = -w_0 - \frac{F_2 + F_5}{2F_1}, \tag{3.6}
\]

\[
w_4 = w_0 - \frac{F_2 - F_5}{2F_1}. \tag{3.7}
\]

It is easy to check that equation (3.5a) is satisfied with

\[
A_{11}(w_1, w_2) = (a_1 w_1^2 + a_2 w_1 + a_3)w_2^2 + (a_2 w_1^2 + a_5 w_1 + a_6)w_2 + a_3 w_1^2 + a_8 w_1 + a_9 \tag{3.8a}
\]

\[
A_{12}(w_1, w_2) = (a_2 w_1^2 + a_5 w_1 + a_8)w_2^3 + A_{122}(w_1)w_2 + A_{123}(w_1) \tag{3.8b}
\]

\[
A_{13}(w_1, w_2) = (a_3 w_1^2 + a_6 w_1 + a_9)w_2^3 + A_{132}(w_1)w_2 + A_{133}(w_1) \tag{3.8c}
\]

\[
A_{21}(w_1, w_2) = (a_2 w_2^2 + a_5 w_2 + a_6)w_1^2 + A_{122}(w_2)w_1 + A_{132}(w_2) \tag{3.8d}
\]

\[
A_{31}(w_1, w_2) = (a_3 w_2^2 + a_5 w_2 + a_9)w_1^2 + A_{123}(w_2)w_1 + A_{133}(w_2), \tag{3.8e}
\]

where \( a_i \)'s are constants while \( A_{122}, A_{123}, A_{132} \) and \( A_{133} \) are arbitrary functions. As pointed out for third order difference equations, equation (3.5b) can be solved for two distinct possibilities:

\[
(i) \quad F_5 = F_2 \quad \text{and} \quad F_6 = F_3, \tag{3.9}
\]

\[
(ii) \quad F_5 \neq F_2 \quad \text{and} \quad F_6 \neq F_3. \tag{3.10}
\]

Considering the condition (3.9) we have

\[
A_{11}(w_1, w_2) = (a_1 w_1^2 + a_2 w_1 + a_3)w_2^2 + (a_2 w_1^2 + a_5 w_1 + a_6)w_2 + a_3 w_1^2 + a_8 w_1 + a_9 \tag{3.11a}
\]

\[
A_{12}(w_1, w_2) = (a_2 w_1^2 + a_5 w_1 + a_8)w_2^3 + (b_1 w_1^2 + b_5 w_1 + b_6)w_2 + b_7 w_1^2 + b_8 w_1 + b_9 \tag{3.11b}
\]

\[
A_{13}(w_1, w_2) = (a_3 w_1^2 + a_6 w_1 + a_9)w_2^3 + (b_7 w_1^2 + c_5 w_1 + c_6)w_2 + c_7 w_1^2 + c_8 w_1 + c_9 \tag{3.11c}
\]

\[
A_{21}(w_1, w_2) = (a_2 w_1^2 + b_4 w_1 + b_7)w_2^3 + (a_5 w_1^2 + b_5 w_1 + c_5)w_2 + a_6 w_1^2 + b_6 w_1 + c_6 \tag{3.11d}
\]
Polynomial integrals for third- and fourth-order ordinary difference equations

\[ A_{22}(w_1, w_2) = (a_5 w_1^2 + b_5 w_1 + b_8) w_2^2 + (b_5 w_1^2 + e_5 w_1 + e_6) w_2 + c_5 w_2^2 + e_6 w_1 + e_9 \]  
\[ A_{23}(w_1, w_2) = (a_8 w_1^2 + b_6 w_1 + b_9) w_2^2 + (b_8 w_1^2 + e_6 w_1 + e_9) w_2 + c_8 w_2^2 + f_8 w_1 + f_9 \]  
\[ A_{31}(w_1, w_2) = (a_3 w_1^2 + b_7 w_1 + c_7) w_2^2 + (a_8 w_1^2 + b_8 w_1 + c_8) w_2 + a_9 w_1^2 + b_9 w_1 + c_9 \]  
\[ A_{32}(w_1, w_2) = (a_6 w_1^2 + c_5 w_1 + c_8) w_2^2 + (b_6 w_1^2 + e_6 w_1 + f_8) w_2 + c_6 w_2^2 + e_9 w_1 + f_9 \]  
\[ A_{33}(w_1, w_2) = (a_9 w_1^2 + c_6 w_1 + c_9) w_2^2 + (b_3 w_1^2 + e_9 w_1 + f_9) w_2 + c_9 w_2^2 + f_9 w_1 + j_9 \]

where \(a_i's, b_i's, c_i's, e_i's, f_i's\) and \(j_9\) are arbitrary constants. Thus we obtain a fourth order difference equation

\[ w_4 = -w_0 - \frac{F_2 + F_5}{2F_1} = -w_0 - \frac{F}{G} \]  
\[ I_1 = \left[ (a_1 w_1^2 + a_2 w_1 + a_3) w_2^2 + (a_2 w_1^2 + a_3 w_1 + a_6) w_2 + a_3 w_2^2 + a_8 w_1 + a_9 \right] G_0 \]

\[ I_1 = \left[ (a_2 w_1^2 + a_3 w_1 + a_6) w_2^2 + (a_3 w_1^2 + a_6 w_1 + a_9) w_2^2 + a_8 w_1 + a_9 \right] w_0 \]

Proceeding further along the lines described for third order \(O\Delta E\) we find that there exist six fourth order \(O\Delta E\) possessing 2 independent integrals. They are

\[ w_{n+4} = -w_n - \frac{\lambda_1(w_{n+2}^2 + w_{n+2}(w_{n+1} + w_{n+3}) + \lambda_3(w_{n+1} + w_{n+3}) + \lambda_2 w_{n+2} + \lambda_4)}{\lambda_1 w_{n+2} + \lambda_2} \]
\[ I_1 = (\lambda_1 w_{n+1} + \lambda_3) w_{n+3}^2 + ((2\lambda_1 w_{n+1} + 2\lambda_3) w_{n+2} + \lambda_1 w_{n+1}^2 + \lambda_2 w_{n+1} + \lambda_4) w_{n+3} + (\lambda_1 w_{n+1} + 2\lambda_3) w_{n+2}^2 + (\lambda_1 w_{n+1} + (\lambda_2 + 2\lambda_3) w_{n+1} + 2\lambda_4) w_{n+2} + 2\lambda_3 w_{n+1} + 2\lambda_4 w_{n+1} + (\lambda_1 w_{n+2} + \lambda_3) w_n^2 + ((2\lambda_1 w_{n+2} + 2\lambda_3) w_{n+1} + \lambda_1 w_{n+2}^2 + \lambda_2 w_{n+2} + \lambda_4) w_n. \]  

\[ (ii) \quad w_{n+4} = -w_n - \frac{\lambda_1(w_{n+1} + w_{n+2} + w_{n+3})^2 + \lambda_2(w_{n+1} + w_{n+2} + w_{n+3}) + \lambda_3}{\lambda_1(w_{n+1} + w_{n+2} + w_{n+3}) + \lambda_4}. \]  

\[ I_1 = (\lambda_1 w_n + \lambda_1 w_{n+1} + \lambda_4) w_{n+3}^2 + (\lambda_1 w_n + (2\lambda_1 w_2 + 2\lambda_1 w_{n+1} + \lambda_2) w_n + \lambda_1 w_{n+1}^2 + \lambda_2 w_{n+1} + \lambda_3) w_{n+3} + (\lambda_1 w_n + \lambda_2 + \lambda_3) w_{n+2}^2 + (\lambda_1 w_{n+1} + \lambda_2 w_{n+1} + \lambda_3) w_{n+2} + \lambda_2 w_{n+1} + \lambda_3 w_{n+1} + (\lambda_1 w_{n+2} + \lambda_4) w_{n+3} + (\lambda_1 w_{n+1} + \lambda_2 + \lambda_3) w_{n+2} + \lambda_2 w_{n+2} + \lambda_3 w_{n+2} + \lambda_4 w_{n+2} + \lambda_3 w_{n}. \]  

\[ (iii) \quad w_{n+4} = -w_n - \frac{\lambda_1(w_{n+1} + w_{n+2} + w_{n+3})^2 + \lambda_2(w_{n+1} + w_{n+2} + w_{n+3}) + \lambda_3}{\lambda_1(w_{n+1} + w_{n+2} + w_{n+3}) + \lambda_4}. \]  

\[ I_1 = (\lambda_1 w_n + \lambda_2 - \lambda_4)(\lambda_1 w_{n+2} + \lambda_2 + \lambda_4)(\lambda_1 w_{n+1} + \lambda_2 - \lambda_4)[(\lambda_1 w_n + 1 + \lambda_2 w_{n+2} + \lambda_4) w_{n+3} + (\lambda_1 w_n + \lambda_2 + \lambda_3) w_{n+3} + (\lambda_1 w_n + \lambda_2 + \lambda_3) w_{n+3} + \lambda_1 w_{n+1}^2 + \lambda_2 w_{n+1} + \lambda_3 w_{n+1} + \lambda_4 w_{n+1} + \lambda_3 w_{n+1} + \lambda_4 w_{n+1} + \lambda_3 w_{n+1} + \lambda_4 w_{n+1} + \lambda_3 w_{n+1} + \lambda_4 w_{n+1} + \lambda_3 w_{n+1} + \lambda_4 w_{n+1} + \lambda_3 w_{n+1} + \lambda_4 w_{n+1}]. \]  

\[ (iv) \quad w_{n+4} = -w_n - \frac{\lambda_1(w_{n+1} + 2\lambda_3)(w_{n+1} + w_{n+3}) + \lambda_1(w_{n+2} + w_{n+1} + w_{n+3}) + \lambda_2 w_{n+2} + \lambda_4}{\lambda_1 w_{n+2} + \lambda_3}. \]  

\[ I_1 = (\lambda_1 w_{n+1} + \lambda_3) w_{n+3}^2 + ((\lambda_1 w_{n+2} + \lambda_1 w_{n+1} + 2\lambda_3) w_n + (\lambda_1 w_{n+1} + 2\lambda_3) w_{n+2} + \lambda_1 w_{n+1}^2 + \lambda_2 w_{n+1} + \lambda_4) w_{n+3} + (\lambda_1 w_{n+1} + \lambda_2 + \lambda_3) w_{n+2}^2 + (\lambda_1 w_{n+1} + \lambda_2 w_{n+1} + \lambda_3) w_{n+2} + \lambda_2 w_{n+1} + \lambda_3 w_{n+1} + (\lambda_1 w_{n+2} + \lambda_4) w_{n+3} + (\lambda_1 w_{n+1} + \lambda_2 + \lambda_3) w_{n+2} + \lambda_2 w_{n+2} + \lambda_3 w_{n+2} + \lambda_4 w_{n+2} + \lambda_3 w_{n}. \]  

\[ (v) \quad w_{n+4} = -w_n - \frac{\lambda_1(w_{n+1} + 2w_{n+2} + w_{n+3}^2 + (w_{n+1} + w_{n+2} + w_{n+3})^3 + w_{n+1} w_{n+3} + w_{n+2} + w_{n+3})}{\lambda_1 w_{n+1} + \lambda_1 w_{n+3} + 2\lambda_1 w_{n+2} + \lambda_3}, \]  

\[ (3.14) \]  

\[ (3.15) \]  

\[ (3.16) \]  

\[ (3.17) \]  

\[ (3.18) \]
Appendix A.

We have identified two distinct third order equations (2.22) and (2.25) each of them admits 2 independent integrals. It is appropriate to mention here that the integrals belong to this category and they are expected to be integrable. The integrability of the above difference equations can be established (hopefully) by other means, for example through a criterion developed for discrete equations (Halburd, 2005). The effectiveness of the method is also discussed for N-th order $O\Delta E$ in the Appendix.

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Appendix A

From the analysis we observe that $A_{21}(w_{n+1}, w_{n+2}) = A_{12}(w_{n+2}, w_{n+1})$, $A_{32}(w_{n+1}, w_{n+2}) = A_{23}(w_{n+2}, w_{n+1})$, $A_{31}(w_{n+1}, w_{n+2}) = A_{13}(w_{n+2}, w_{n+1})$ in all the six fourth order difference equations and therefore we presented only the forms of $A_{11}(w_{n+1}, w_{n+2})$, $A_{12}(w_{n+1}, w_{n+2})$, $A_{13}(w_{n+1}, w_{n+2})$, $A_{22}(w_{n+1}, w_{n+2})$, $A_{23}(w_{n+1}, w_{n+2})$, $A_{33}(w_{n+1}, w_{n+2})$.

Explicit forms for $A_{ij}(w_{n+1}, w_{n+2})$ associated with fourth order difference equation (3.14):

\[ A_{11} = 0, \]

\[ A_{12} = \lambda_1(\lambda_1 w_{n+1} + \lambda_3)(\lambda_1 w_{n+2} + \lambda_3), \]

\[ A_{13} = (\lambda_1 w_{n+1} + \lambda_3)((\lambda_1 \lambda_3 w_{n+2} + \lambda_1 \lambda_4) + 2\lambda_3^2 - \lambda_2 \lambda_3), \]

\[ A_{22} = (2\lambda_3^2 - \lambda_2 \lambda_3 + \lambda_1 \lambda_3 w_{n+2} + \lambda_1 \lambda_4)(\lambda_1 w_{n+1} + \lambda_3) + \lambda_1(\lambda_1 w_{n+2} + \lambda_3) \]
\[ \times (\lambda_4 + \lambda_1 w_{n+1}^2 + \lambda_1 w_{n+1} w_{n+2} + \lambda_2 w_{n+1} + \lambda_3 w_{n+2}), \]

\[ A_{23} = (\lambda_1 \lambda_3 w_{n+2} + \lambda_1 \lambda_4 + 2\lambda_3^2 - \lambda_3 \lambda_2)(\lambda_4 + \lambda_1 w_{n+1}^2 + \lambda_1 w_{n+1} w_{n+2} + \lambda_2 w_{n+1} + \lambda_3 w_{n+2}), \]

\[ A_{33} = (2\lambda_3^2 - \lambda_3 \lambda_2 + \lambda_1 \lambda_4)(\lambda_3 w_{n+2} + \lambda_4 w_{n+2} + \lambda_4 w_{n+1}) + \lambda_1 \lambda_3(\lambda_3 w_{n+2} + \lambda_3 w_{n+1} + \lambda_1) w_{n+1} w_{n+2} + \lambda_3(2\lambda_3^2 - \lambda_3 \lambda_2 + \lambda_1 \lambda_4)(w_{n+2} + w_{n+1}) w_{n+1}. \]

Explicit forms for $A_{ij}(w_{n+1}, w_{n+2})$ associated with fourth order difference equation (3.15):

\[ A_{11} = \lambda_1^2(\lambda_1 w_{n+2} + \lambda_1 w_{n+1} - 2\lambda_4 + \lambda_2), \]

\[ A_{12} = \lambda_1(\lambda_1 w_{n+1} + \lambda_1 w_{n+2} + \lambda_4)(\lambda_1 w_{n+2} + \lambda_1 w_{n+1} - 2\lambda_4 + \lambda_2) \]
\[ + \lambda_1^2((2\lambda_2 - 4\lambda_4) w_{n+1} + \lambda_1 w_{n+1} w_{n+2} + \lambda_1 w_{n+1}^2 + \lambda_2 w_{n+2} - 2\lambda_4 w_{n+2}), \]

\[ A_{13} = \lambda_1(\lambda_1 w_{n+1} + \lambda_1 w_{n+2} + \lambda_4)(2\lambda_2 w_{n+1} - 4\lambda_4 w_{n+1} + \lambda_1 w_{n+1} w_{n+2} + \lambda_1 w_{n+1}^2 \]
\[ + \lambda_2 w_{n+2} - 2\lambda_4 w_{n+2}), \]

\[ A_{22} = 2\lambda_3^3 w_{n+2}^3 + \lambda_1(\lambda_1(\lambda_3 - 16\lambda_4 w_{n+1} + 7\lambda_1 w_{n+1}^2 + 11\lambda_2 w_{n+1}) + 2\lambda_2(\lambda_2 - 2\lambda_4)) w_{n+2} \]
\[ + \lambda_1^2(5\lambda_2 - 7\lambda_4 + 7\lambda_1 w_{n+1}) w_{n+2} + 2\lambda_1^2 w_{n+1}^2 + \lambda_1^2(5\lambda_2 - 7\lambda_4) w_{n+1}^2 + \lambda_1(\lambda_1 \lambda_3 \]
\[ + 2\lambda_2^2 - 4\lambda_2 \lambda_4) w_{n+1} + \lambda_1 \lambda_2 \lambda_3 + \lambda_4(4\lambda_2 \lambda_4 - \lambda_2^2 - 4\lambda_4 - 2\lambda_1 \lambda_3), \]

\[ A_{23} = 2\lambda_3^3(\lambda_1 w_{n+1} + \lambda_2 - 2\lambda_4) w_{n+1}^3 \]
\[ + (\lambda_3^2(5\lambda_1 w_{n+1} + 8\lambda_2 - 13\lambda_4) w_{n+1}) + \lambda_1(2\lambda_2^2 \]
\[ + 2\lambda_2 - 5\lambda_2 \lambda_4)) w_{n+1}^2 + (\lambda_1(\lambda_1(4\lambda_1 w_{n+1} + 9\lambda_2 - 13\lambda_4) w_{n+1}) + 4\lambda_2^2 - 8\lambda_2 \lambda_4 \]
\[ + \lambda_1 \lambda_3) w_{n+1} + \lambda_1(4\lambda_2^2 - 4\lambda_2 \lambda_4 - 2\lambda_1 \lambda_3 + \lambda_2^2 + \lambda_1 \lambda_2 \lambda_3) w_{n+2} + \lambda_1^2 w_{n+1}^2 \]
\[ + \lambda_1^2(3\lambda_2 - 4\lambda_4) w_{n+1}^3 + \lambda_1(\lambda_1 \lambda_3 + 2\lambda_2^2 - 4\lambda_2 \lambda_4) w_{n+1}^2 + 2\lambda_1 \lambda_3(\lambda_2 - 2\lambda_4) w_{n+1}, \]
Explicit forms for $A_{ij}(w_{n+1}, w_{n+2})$ associated with fourth order difference equation (3.16):

\[ A_{11} = \lambda_1^2(\lambda_1 w_{n+1} + \lambda_2 - 2\lambda_4)(w_{n+1} + w_{n+2}), \]
\[ A_{12} = \lambda_1^2(\lambda_1 w_{n+1} + \lambda_2 + \lambda_1 w_{n+1} + \lambda_2 - \lambda_4)(w_{n+1} + w_{n+2}) + \lambda_1(\lambda_2 - \lambda_4)(\lambda_2^2 w_{n+1} + \lambda_2 - \lambda_4)(\lambda_1 w_{n+1} + \lambda_2 - \lambda_4), \]
\[ A_{13} = (\lambda_1 w_{n+1} + \lambda_2 - \lambda_4)(\lambda_2^2 w_{n+1} + \lambda_2 - \lambda_4)(\lambda_1 w_{n+1} + \lambda_2 - \lambda_4), \]
\[ A_{22} = \lambda_1^2(\lambda_1 w_{n+1} + \lambda_2)(\lambda_2 - \lambda_4 + \lambda_1 w_{n+1}) w_{n+1}^2 + \lambda_1^2(\lambda_1(2\lambda_2 - \lambda_4) w_{n+1}^2 + (\lambda_1 + 2\lambda_2 - \lambda_4 - \lambda_1^2) w_{n+1} + \lambda_3(\lambda_2 - \lambda_4) w_{n+2}) + (\lambda_2 - \lambda_4)(\lambda_2^2(\lambda_2 w_{n+1} + \lambda_3) w_{n+1} + (\lambda_2 - \lambda_4)(\lambda_2^2 - \lambda_2^2)), \]
\[ A_{23} = \lambda_1(\lambda_2 - \lambda_4)(\lambda_2^2 w_{n+1} + \lambda_1(2\lambda_2 - \lambda_4) w_{n+1} + (\lambda_2 - \lambda_4) w_{n+2}) + (\lambda_2 - \lambda_4)(\lambda_2^2(\lambda_2 w_{n+1} + \lambda_3) w_{n+1} - (\lambda_2 - \lambda_4)((\lambda_2 - \lambda_4)^2 - 2\lambda_1\lambda_4 w_{n+1})) w_{n+2} + (\lambda_2 - \lambda_4)^2(\lambda_4 w_{n+1} + \lambda_3)(\lambda_1 w_{n+1} + \lambda_2 - \lambda_4), \]
\[ A_{33} = (\lambda_2 - \lambda_4)^2((\lambda_2^2 w_{n+1} + \lambda_1\lambda_2 w_{n+1} + (\lambda_2 - \lambda_4)^2) w_{n+2} + (\lambda_1\lambda_2 w_{n+1} - (\lambda_2^2 - 3\lambda_2\lambda_4 + 2\lambda_4^2 - \lambda_1\lambda_3) w_{n+1}) w_{n+2} + (\lambda_2 - \lambda_4)^2 w_{n+1}). \]

Explicit forms for $A_{ij}(w_{n+1}, w_{n+2})$ associated with fourth order difference equation (3.17):

\[ A_{11} = 0, \]
\[ A_{12} = a_{11} \lambda_1(\lambda_1 w_{n+1} + \lambda_3), \]
\[ A_{13} = \lambda_1(\lambda_1 w_{n+1} + \lambda_3)(\alpha_{11}(w_{n+1} + w_{n+2}) + \alpha_{12}), \]
\[ A_{22} = \alpha_{12} \lambda_1(\lambda_1 w_{n+1} + w_{n+2}) + 2\lambda_3) + \alpha_{11}(\lambda_1(\lambda_1 w_{n+1} + w_{n+2}) + (3\lambda_1 w_{n+1} + 4\lambda_3 w_{n+2} + 4\lambda_1 w_{n+1} + \lambda_4) + \lambda_3(3\lambda_3 - \lambda_2)), \]
Explicit forms for $A_{ij}(w_{n+1}, w_{n+2})$ associated with fourth order difference equation (3.18):

$$A_{11} = \alpha_1^2,$$

$$A_{12} = \lambda_1 (\lambda_2 w_{n+1} + 2 \lambda_3 w_{n+2} + \lambda_4),$$

$$A_{13} = \lambda_2 (\lambda_1 w_{n+1} + \lambda_2 w_{n+2} + \lambda_3),$$

$$A_{22} = \alpha_1 (\lambda_2 w_{n+1} + 2 \lambda_3 w_{n+2} + \lambda_4),$$

$$A_{23} = \alpha_2 (\lambda_2 w_{n+1} + \lambda_2 w_{n+2} + \lambda_3).$$

Explicit forms for $A_{ij}(w_{n+1}, w_{n+2})$ associated with fourth order difference equation (3.19):

$$A_{11} = A_{12} = 0,$$

$$A_{13} = (\lambda_1 w_{n+1} + \lambda_4 + \lambda_5)(\lambda_4 + \lambda_1 w_{n+1} - \lambda_5),$$

$$A_{22} = (\lambda_1 w_{n+1} + \lambda_4 + \lambda_5)(\lambda_1 w_{n+2} + \lambda_4 + \lambda_5),$$

$$A_{23} = (\lambda_1 w_{n+1} + \lambda_4 + \lambda_5)(\lambda_1 w_{n+1} + \lambda_5 w_{n+2} + \lambda_4 w_{n+2} + \lambda_3).$$
Consider an Nth-order autonomous $O\Delta E$ having the form

$$w_{n+N} = F(w_n, ... w_{n+N-1}). \quad (B1)$$

Let us assume that equation (B1) admits a polynomial integral $I(n)$ having the form

$$I(n) = \sum_{j=1}^{3} [A_{ij}(n)w_{n+1}^2w_{n+1} + A_{2j}(n)w_{n+N-1} + A_{3j}(n)]w_{n}^{3-j}, \quad (B2)$$

where

$$A_{ij}(n) = A_{ij}(w_{n+1}, w_{n+2}, ..., w_{n+2}), \quad i, j = 1, 2, 3$$

are unknown functions. Then the integrability condition $I(n+1) - I(n) = 0$ leads to a quadratic equation in $w_{n+N}$

$$F_1(n)w_{n+N}^2 + F_2(n)w_{n+N} + F_3(n) - (F_4(n)w_{n}^2 + F_5(n)w_{n} + F_6(n)) = 0, \quad (B3)$$

where

$$F_1(n) = \sum_{j=1}^{3} A_{ij}(n+1)w_{n+1}^{3-j}, \quad i = 1, 2, 3, \quad F_k(n) = \sum_{j=1}^{3} A_{jk}(n)w_{n+N-1}^{3-j}, \quad k = 1, 2, 3.$$  

It is straightforward to check that equation (B3) can be factorised as

$$\left( w_{n+N} + w_n + \frac{F_2(n) + F_5(n)}{2F_1(n)} \right) \left( w_{n+N} - w_n + \frac{F_2(n) - F_5(n)}{2F_1(n)} \right) = 0 \quad (B4)$$

provided

$$A_{11}(n+1)w_{n+1}^2 - A_{11}(n)w_{n+N-1}^2 + A_{12}(n+1)w_{n+1} - A_{21}(n)w_{n+N-1} = A_{31}(n) - A_{13}(n+1)(B5a)$$

and

$$[F_3(n) + F_2(n)][F_5(n) - F_2(n)] = 4[F_6(n) - F_3(n)]F_1(n). \quad (B5b)$$

Thus we obtain,

$$w_{n+N} = -w_n - \frac{F_2(n) + F_5(n)}{2F_1(n)}, \quad (B6)$$

$$w_{n+N} = w_n - \frac{F_2(n) - F_5(n)}{2F_1(n)}. \quad (B7)$$

Obviously (B5b) is satisfied for distinct possibilities

1. $F_5 = F_2$ and $F_6 = F_3,$
(2) $F_5 \neq F_2$ and $F_6 \neq F_3$.

Demanding that one of the integrals of the given $O\Delta E$ (B1), say, $I_1(n)$ is cyclic invariant, that is

$$I_1(w_n, w_{n+1}, \cdots, w_{n+N-2}, w_{n+N-1}) = I_1(w_{n+1}, w_{n+2}, \cdots, w_{n+N-1}, w_n)$$

$$= I_1(w_{n+2}, w_{n+3}, \cdots, w_n, w_{n+1}) = \cdots = I_1(w_{n+N-1}, w_n, \cdots, w_{n+N-3}, w_{n+N-2})$$

it is easy to verify that both the conditions given in equation (B5a) as well as $F_5 = F_2$ and $F_6 = F_3$ satisfy. In this case the integral $I_1$ will take the form

$$I_1(n) = \sum_{0 \leq i_1 \leq i_2 \leq \cdots \leq i_n \leq 2} a_{i_1i_2\cdots i_n} w_{n+1}^{i_1} w_{n+2}^{i_2} \cdots w_{n+N+1}^{i_n},$$

where

$$a_{i_1i_2\cdots i_n} = a_{i_2i_3\cdots i_ni_1} = \cdots = a_{i_ni_1\cdots i_{n-2}i_{n-1}}, \text{ provided } i_1 \neq i_2 \neq \cdots \neq i_n.$$  

Remaining integrals $I_2(n), I_3(n) \cdots I_{N-1}(n)$ may be constructed by demanding them not to be cyclic invariant. We find that the integrals (2.24) of the third order difference equation (2.22) and (2.27) of (2.25) are cyclic invariant.

References


Polynomial integrals for third- and fourth-order ordinary difference equations


