

# Precise large deviation of the surplus process in a perturbed model

Yinghua Dong

*College of Mathematics and Statistics, Nanjing University of Information Science and Technology, Nanjing, 210044, China*

*dongyinghua1@163.com*

## Abstract.

In this paper, we consider a perturbed model in which  $\{X_i, i = 1, 2, \dots\}$  are extended negatively dependent random variables with consistently varying tails,  $\{Y_k, k = 1, 2, \dots\}$  are independent, identically distributed random variables. We give precise large deviation of the surplus process .

*Keywords: Precise large deviation, the surplus process, extended negative dependence, quasi-renewal process.*

## 1. Introduction

We consider a model in which  $\{X_i, i = 1, 2, \dots\}$  form a sequence of nonnegative extended negative dependent (END) random variables with common distribution  $F$  ;  $N(t)$  denotes the appearance number of  $\{X_i, i = 1, 2, \dots\}$  in  $[0, t]$  , and  $\{N(t), t \geq 0\}$  is a general counting process. We assume  $E(N(t)) = \Lambda(t) < \infty$  for all  $t \geq 0$  , and as  $t \rightarrow \infty$  ,  $\Lambda(t) \rightarrow \infty$  . The aggregate amount up to  $t$  can be given by

$$S(t) = \sum_{i=1}^{N(t)} X_i , \quad t \geq 0.$$

$\{Y_k, k = 1, 2, \dots\}$  constitute another sequence of independent, identically distributed (i.i.d) nonnegative random variables. Suppose that their inter-arrival time  $\{\theta_i, i = 1, 2, \dots\}$  forms a sequence of identically distributed LND random

variables. Let  $T_k = \sum_{i=1}^k \theta_i$  denote the arrival time of  $Y_k$ . Then we get a quasi-renewal process  $M(t) = \sup\{n \geq 1 : T_n \leq t\}$ ,  $t \geq 0$ . Let  $E(\theta_1) = 1/\lambda_1$ . Then  $M(t)/\lambda_1 t \longrightarrow 1$ , a.e.

Let  $u > 0$  denote the initial reserve.  $\{\sigma W(t), t \geq 0\}$  denotes a perturb process, where  $\sigma$  is referred to as a diffusion coefficient, and  $\{W(t), t \geq 0\}$  is a standard Wiener process. Let  $d$  and  $-d$  denote the upper and low bounds of  $\{W(t), t \geq 0\}$ , respectively. The reserve process is presented by

$$R(t) = u + \sum_{k=1}^{M(t)} Y_k - \sum_{i=1}^{N(t)} X_i + \sigma W(t) I_{[-d, d]}(W(t)), \quad t \geq 0.$$

The surplus process can be denoted by

$$Z(t) = \sum_{i=1}^{N(t)} X_i - \sum_{k=1}^{M(t)} Y_k - \sigma W(t) I_{[-d, d]}(W(t)), \quad t \geq 0. \quad (1)$$

we assume  $\{X_i, i = 1, 2, \dots\}$ ,  $\{Y_k, k = 1, 2, \dots\}$ ,  $\{N(t), t \geq 0\}$  and  $\{M(t), t \geq 0\}$  are mutually independent. [1] showed precise large deviation of non-random sum, while [2] presented precise large deviation of  $\{S(t), t \geq 0\}$ . In the present paper, we obtain precise large deviation of the surplus process  $\{Z(t), t \geq 0\}$  in the above model.

## 2. Preliminaries

The definition of END structure was introduced by [1].

**Definition 1.** We call random variables  $\{X_i, i = 1, 2, \dots\}$  END if there is  $M > 0$  such that both

$$P\{X_1 \leq x_1, X_2 \leq x_2, \dots, X_n \leq x_n\} \leq MP\{X_1 \leq x_1\}P\{X_2 \leq x_2\} \cdots P\{X_n \leq x_n\}$$

(2)

and

$P\{X_1 > x_1, X_2 > x_2, \dots, X_n > x_n\} \leq MP\{X_1 > x_1\}P\{X_2 > x_2\} \cdots P\{X_n > x_n\}$   
hold for each  $n = 1, 2, \dots$ , and all  $x_1, \dots, x_n$ . For  $M = 1$ , if (2) holds, we call  $\{X_i, i = 1, 2, \dots\}$  LND.

In the following, we introduce some related heavy-tailed distribution class, which can be found in [3] and [4]. For convenience, denote  $\overline{F}(x) = 1 - F(x) = P(X > x)$ .

A distribution  $F$  on  $[0, \infty)$  is said to belong to the long-tailed class and write  $F \in L$ , if

$$\overline{F}(x - y) \sim \overline{F}(x), \quad \text{for any } y \in (-\infty, \infty).$$

In addition, we say that  $F$  is said to belong to the dominated variation class and written as  $F \in D$ , if

$$\overline{F}(xy) = O(1)\overline{F}(x), \text{ for all } 0 < y < 1.$$

Denote the upper Matuszewska index of  $F$  by  $J_F^+$ . If  $F \in D$ , then  $0 < J_F^+ \leq \infty$ .

The consistent variation class  $C$  is smaller than the class  $D$ . We call  $F \in C$ , if

$$\lim_{y \downarrow 1} \liminf_{x \downarrow \infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = 1.$$

It is well known that  $C$  belongs to  $D \cap L$ .

In this paper,  $N(t)$  satisfies the following assumption.

**Assumption 1:** For some  $p > J_F^+$ ,

$$EN(t)^p I(N(t) > (1 + \delta)\Lambda(t)) = O(1)\Lambda(t)$$

holds for all  $\delta > 0$ .

According to Theorem 3.1 of [2], we can get the following lemma.

**Lemma 1.** Let  $\{X_i, i = 1, 2, \dots\}$  be END. In addition to Assumption 1, suppose that  $F \in C$ ,  $E(X_1) = \mu > 0$ . Then, for any fixed  $\gamma > 0$ , it holds uniformly for all  $x \geq \gamma\Lambda(t)$  that

$$P(S(t) - \mu\Lambda(t) > x) \sim \Lambda(t)\overline{F}(x), \quad t \rightarrow \infty.$$

By the definition of the consistent variation class  $C$ , we easily get the following lemma.

**Lemma 2.** If  $F \in C$ , then

$$\lim_{x \rightarrow \infty} \frac{\overline{F}(x + o(1)x)}{\overline{F}(x)} = 1.$$

The following lemma is due to [5]

**Lemma 3.** For a distribution  $F$  on  $[0, \infty)$ , if  $F \in D$ , then for any  $p > J_F^+$ , there exists some positive number  $C_1$  and  $x_0$  such that

$$\frac{\overline{F}(xy)}{\overline{F}(x)} \leq C_1 y^{-p}, \quad \text{for all } x \geq xy \geq x_0.$$

The following is from [2].

**Lemma 4.** For a quasi-renewal process  $\{M(t), t \geq 0\}$ , the generic inter-renewal distance  $\theta$  has distribution  $G$  and expectation  $1/\lambda_1 < \infty$ . If  $G(\infty) = 1$ , then

$$\lim_{t \rightarrow \infty} \frac{\lambda_1(t)}{t} = \lambda_1, \quad \text{a.e.}$$

holds.

### 3. Main result

**Theorem.** Let  $\{X_i, i = 1, 2, \dots\}$  be END. In addition to Assumption 1, suppose that  $F \in C$ ,  $E(X_1) = \mu > 0$ . Then for any fixed  $\gamma > 0$ , it holds uniformly for all  $x \geq \gamma \Lambda(t) \geq \lambda_1 t$  satisfying  $\gamma \geq E(Y_1)$  that

$$P(Z(t) - \mu \Lambda(t) > x) \sim \Lambda(t) \bar{F}(x), \quad t \rightarrow \infty. \quad (3)$$

**Proof.** For convenience, write  $A(t) = \sum_{i=1}^{M(t)} Y_i + \sigma W(t) I_{[-d, d]}(\sigma W(t))$ .

$$\text{Since } E\left(\sum_{k=1}^{M(t)} Y_k\right) = E(Y_1) \lambda_1(t) \text{ and } E(W(t) I_{[-d, d]}(\sigma W(t))) = 0,$$

We get  $E(A(t)) = E(Y_1) \lambda_1(t)$ . According to Chen et al. (2011), we have

$$\frac{1}{E(Y_1) \lambda_1 t} \sum_{k=1}^{M(t)} Y_k = \frac{1}{M(t)} \sum_{k=1}^{M(t)} Y_k \cdot \frac{M(t)}{\lambda_1 t} \longrightarrow 1, \quad \text{a.e.}$$

In addition, it is clear that

$$\lim_{t \rightarrow \infty} \frac{W(t) I_{[-d, d]}(W(t))}{t} = 0, \quad \text{a.e.}$$

It follows from the two equalities above that

$$\frac{A(t) - E(Y_1) \lambda_1(t)}{E(Y_1) \lambda_1 t} \longrightarrow 0, \quad \text{a.e.}$$

Hence, there is a positive function  $\varepsilon(t)$  such that as  $t \rightarrow \infty$ ,  $\varepsilon(t) \rightarrow 0$  and

$$P(|A(t) - E(Y_1) \lambda_1(t)| > \varepsilon(t) E(Y_1) \lambda_1 t) = o(1).$$

Next we discuss the large deviation of the surplus process.

$$\begin{aligned} & P(Z(t) - EZ(t) > x) \\ &= P(S(t) - A(t) - ES(t) + E(Y_1) \lambda_1(t) > x) \\ &= \int_{|y - E(Y_1) \lambda_1(t)| \leq \varepsilon(t) E(Y_1) \lambda_1 t} P(S(t) - ES(t) > x - E(Y_1) \cdot \lambda_1(t) + y) P(A(t) \in dy) \end{aligned}$$

$$\begin{aligned}
& + \int_{y-E(Y_1)\lambda_1(t) < -\varepsilon(t)E(Y_1)\lambda_1 t} P(S(t) - ES(t) > x - E(Y_1) \cdot \lambda_1(t) + y) P(A(t) \in dy) \\
& + \int_{y-E(Y_1)\lambda_1(t) > \varepsilon(t)E(Y_1)\lambda_1 t} P(S(t) - ES(t) > x - E(Y_1) \cdot \lambda_1(t) + y) P(A(t) \in dy) \\
& = I_1(t) + I_2(t) + I_3(t).
\end{aligned} \tag{4}$$

First of all, we deal with  $I_1(t)$ . For  $x \geq \gamma\Lambda(t) \geq \lambda_1 t$ , we have

$$\frac{t}{x} \leq \frac{1}{\lambda_1}.$$

$$\text{When } |y - E(Y_1) \cdot \lambda_1(t)| \leq \varepsilon(t)E(Y_1)\lambda_1 t,$$

$$x - E(Y_1)\lambda_1(t) + y = x + o(1)t = x + o(1)x.$$

By Lemma 1 and Lemma 2, it holds uniformly all  $x \geq \gamma\Lambda(t)$ ,

$$\begin{aligned}
& I_1(t) \\
& = \int_{|y-E(Y_1)\lambda_1(t)| \leq \varepsilon(t)E(Y_1)\lambda_1 t} P(S(t) - ES(t) > x - E(Y_1) \cdot \lambda_1(t) + y) P(A(t) \in dy) \\
& \sim \int_{|y-E(Y_1)\lambda_1(t)| \leq \varepsilon(t)E(Y_1)\lambda_1 t} \Lambda(t) \bar{F}(x - E(Y_1) \cdot \lambda_1(t) + y) P(A(t) \in dy) \\
& = \Lambda(t) \bar{F}(x) \int_{|y-E(Y_1)\lambda_1(t)| \leq \varepsilon(t)E(Y_1)\lambda_1 t} \frac{\bar{F}(x + o(1)x)}{\bar{F}(x)} P(A(t) \in dy) \\
& \sim \Lambda(t) \bar{F}(x).
\end{aligned} \tag{5}$$

Next we discuss  $I_2(t)$ . By Lemma 3, there is some positive number

$D_2$  such that it holds uniformly for all  $x \geq \gamma\Lambda(t)$  satisfying  $\gamma > E(Y_1)$ ,

$$\begin{aligned}
I_2(t) &= \int_{y-E(Y_1)\lambda_1(t) < -\varepsilon(t)E(Y_1)\lambda_1 t} P(S(t) - ES(t) > x - E(Y_1)\lambda_1(t) + y) P(A(t) \in dy) \\
&\leq \int_{y-E(Y_1)\lambda_1(t) < -\varepsilon(t)E(Y_1)\lambda_1 t} P(S(t) - ES(t) > x - E(Y_1) \cdot \lambda_1(t)) P(A(t) \in dy) \\
&\sim \Lambda(t) \bar{F}(x) \int_{y-E(Y_1)\lambda_1(t) < -\varepsilon(t)E(Y_1)\lambda_1 t} \frac{\bar{F}(x - EY_1\lambda_1(t))}{\bar{F}(x)} P(A(t) \in dy) \\
&\leq D_2 \Lambda(t) \bar{F}(x) P(A(t) - E(Y_1) \cdot \lambda_1(t) < -\varepsilon(t)E(Y_1)\lambda_1 t) \\
&= o(1) \Lambda(t) \bar{F}(x).
\end{aligned} \tag{6}$$

Now we verify the fourth step. As  $t$  is large enough and  $x \geq \gamma\Lambda(t)$ ,

$$x - E(Y_1)\Lambda(t) = x \left( 1 - \frac{E(Y_1) \cdot \Lambda(t)}{x} \right) \geq x \left( 1 - \frac{E(Y_1)}{\gamma} \right).$$

Since  $C \subset D$ ,

$$\limsup_{x \rightarrow \infty} \frac{\bar{F}\left(x \left( 1 - \frac{E(Y_1)}{\gamma} \right)\right)}{\bar{F}(x)} < \infty.$$

Hence, as  $t$  is enough large, there is some positive number  $D_2$  such that

$$\frac{\bar{F}(x - E(Y_1)\Lambda(t))}{\bar{F}(x)} \leq D_2.$$

Finally, we deal with  $I_3(t)$ . For any fixed  $\gamma > 0$ , it holds uniformly for all  $x \geq \gamma\Lambda(t)$  that

$$\begin{aligned}
I_3(t) &= \int_{y-E(Y_1)\lambda_1(t) > \varepsilon(t)E(Y_1)\lambda_1 t} P(S(t) - ES(t) > x - E(Y_1) \cdot \lambda_1(t) + y) P(A(t) \in dy) \\
&\leq \int_{y-E(Y_1)\lambda_1(t) > \varepsilon(t)E(Y_1)\lambda_1 t} P(S(t) - ES(t) > x) P(A(t) \in dy) \\
&\sim \Lambda(t) \bar{F}(x) P(A(t) - E(Y_1)\lambda_1(t) > \varepsilon(t)E(Y_1)\lambda_1 t) \\
&= o(1) \Lambda(t) \bar{F}(x).
\end{aligned} \tag{7}$$

According to (4)-(7), we obtain (3). This ends the proof of the theorem.

## Reference:

- [1] Y. Liu, Precise large deviations for dependent random variables with heavy tails. Stat. Probab. Lett. , 47, (2004) 311-319.
- [2] Y. Chen, K. C. Yuen and K. W. Ng, Precise large deviations of random sums in the presence of negatively dependence and consistent variation. Methodol. Comput. Appl. Probab. 13 (2011) 821-833.
- [3] N. H. Bingham, C. M. Glodie, and J. L. Teugels, Regular Variation, Cambridge University Press. (1987).
- [4] D. B. H. Cline and Samorodnitsky G, Subexponentiality of the product of independent random variables, Stoch. Proc. Appl. 49, (1994) 75-98.
- [5] Q. Tang and G. Tsitsiashvili, Precise estimates for the ruin probability in finite horizon in a discrete-time model with heavy-tailed insurance and financial risks. Stoch. Proc. Appl. 108, (2003) 299-325.