

A Class of Weakly Singular Iterated Integral Inequality

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Abstract

In this paper, we establish a class of weakly singular integral inequality, which consists of iterated integral. Under several practical assumptions, the inequality is solved by adopting novel analysis techniques, such as: change of variable, amplification method, and inverse function. Explicit bounds for the unknown functions are given clearly.

Keywords: Integral inequality; Iterated integrals; Weakly singular integral kernel; Analysis technique; Estimation.

1. Introduction

In 1997 Medved [1] discuss nonlinear singular integral inequalities

$$u(t) \leq a(t) + \int_0^t (t-s)^{\beta-1} f(s)w(u(s))ds, \quad (1)$$

and the estimates of solutions are given.

In 2011, Abdeldaim et al. [2] studied a new integral inequality of Gronwall-Bellman-Pachpatte type

$$u(t) \leq u_0 + \int_{t_0}^t [f(s)u(s)[u(s) + \int_{t_0}^s h(\tau)[u(\tau) + \int_{t_0}^{\tau} g(\xi)u(\xi)d\xi]d\tau]ds. \quad (2)$$

In this paper, on the basis of [1-9], we discuss a nonlinear iterated integral inequality

$$\begin{aligned} u(t) \leq & a(t) + \int_{t_0}^t (t-s)^{\beta_1-1} f(s)w_1(u(s))[u(s) \\ & + \int_{t_0}^s (s-\tau)^{\beta_2-1} g(\tau)w_2(u(\tau))[u(\tau) \\ & + \int_{t_0}^{\tau} (\tau-\xi)^{\beta_3-1} h(\xi)w_3(u(\xi))d\xi]d\tau]ds, \end{aligned} \quad (3)$$

for all $t \in [t_0, J)$.

2. Main result

For convenience, we cite the following lemma and definition:

Definition 1. (see [1])

Let $q > 0$ be a real number and $0 < T < \infty$. We say that a function $w : R^+ \rightarrow R$ satisfies a condition (q), if

$$e^{-qt} [w(u)]^q \leq R(t) w(e^{-qt} u^q), \forall u \in R^+, t \in [0, T]. \quad (4)$$

where $R(t)$ is a continuous, nonnegative function.

Lemma 1. (see [1])

Let $\beta \in [0, 1/2]$, $0 < p < \frac{1}{1-\beta}$, then $1 + p(\beta - 1) > 0$ and

$$\int_{t_0}^t (t-s)^{p(\beta-1)} e^{ps} ds \leq \frac{e^{pt}}{p^{1+p(\beta-1)}} \Gamma(1 + p(\beta - 1)). \quad (5)$$

Lemma 2. (see [3])

Let $\alpha > 1$ is a real numbers, n is a natural number, and A_1, A_2, \dots, A_n be nonnegative real numbers. Then

$$(A_1 + A_2 + \dots + A_n)^\alpha \leq n^{\alpha-1} (A_1^\alpha + A_2^\alpha + \dots + A_n^\alpha). \quad (6)$$

Theorem 1.

Let $\beta_i \in [0, 1/2]$ ($i = 1, 2, 3$), w_1, w_2, w_3 , satisfy the condition (4) with $q_i = (1 + \beta_i) / \beta_i$, $q_1 < q_2 < q_3$, $q_2 / q_1 < 2$. If $u(t)$ satisfies (3), then

$$u(t) \leq e^t (\Omega_1^{-1} (\Omega_4^{-1} (\Omega_5^{-1} (\tilde{C}(t))))^{1/q_1}, \forall t \in [t_0, T_1], \quad (7)$$

where

$$\tilde{C}(t) = \Omega_5 [\Omega_4 (\Omega_1 (C(t)) + \int_{t_0}^t \tilde{h}(s) ds) + \int_{t_0}^t \tilde{g}(s) ds] + \int_{t_0}^t \tilde{f}(s) ds,$$

$$\Omega_1(z) = \int_{z_0}^z \frac{ds}{w_3(s)}, z_0 > 0, z \in (z_0, +\infty) \quad (8)$$

$$\Omega_2(z) = \int_{z_0}^z \frac{w_3(\Omega_1^{-1}(s)) ds}{w_2(\Omega_1^{-1}(s)) \Omega_1^{-1}(s)}, z_0 > 0, z \in (z_0, +\infty), \quad (9)$$

$$\Omega_3(z) = \int_{z_0}^z \frac{w_2(\Omega_1^{-1}(\Omega_2^{-1}(s))) ds}{w_1(\Omega_1^{-1}(\Omega_2^{-1}(s)))}, z_0 > 0, z \in (z_0, +\infty), \quad (10)$$

$$\Omega_4(z) = \int_{z_0}^z \frac{w_3(\Omega_1^{-1}(s)) ds}{w_2(\Omega_1^{-1}(s)) (\Omega_1^{-1}(s))^{2-\frac{q_2}{q_1}}}, z_0 > 0, z \in (z_0, +\infty), \quad (11)$$

$$\Omega_5(z) = \int_{z_0}^z \frac{w_2(\Omega_1^{-1}(\Omega_2^{-1}(s)))ds}{w_1(\Omega_1^{-1}(\Omega_2^{-1}(s)))(\Omega_1^{-1}(\Omega_2^{-1}(s)))^{\frac{q_2-1}{q_1}}}, z_0 > 0, z \in (z_0, +\infty), \quad (12)$$

$$\begin{aligned} C(t) &= 2^{\frac{q_3+q_2-2}{q_2} - \frac{1}{q_1}} B(t), \\ B(t) &= 2^{q_1^{-1}} a^{q_1}(t), \\ \tilde{f}(t) &= 2^{\frac{q_2-2}{q_1}} \tilde{\tilde{f}}(t), \\ \tilde{g}(t) &= \frac{q_1}{q_2} 2^{\frac{q_3+q_2-2}{q_2} - \frac{1}{q_1}} \tilde{\tilde{g}}(t), \\ \tilde{h}(t) &= 2^{\frac{q_3-1}{q_2}} \tilde{\tilde{h}}(t), \\ \tilde{\tilde{f}}(s) &= 2^{2q_1-2} K_1 f^{q_1}(s) R_1(s) e^{q_1 s}, \\ \tilde{\tilde{g}}(\tau) &= (2^{2q_2-1} K_2)^{q_2/q_1} g^{q_2}(\tau) R_2(\tau) e^{q_2 \tau}, \\ \tilde{\tilde{h}}(\xi) &= (K_3)^{q_3/q_2} h^{q_3}(\xi) R_3(\xi), \\ K_1 &= \left(\frac{\Gamma(1+p_1(\beta_1-1))}{p_1^{1+p_1(\beta_1-1)}} \right)^{q_1/p_1}, \\ K_2 &= \left(\frac{\Gamma(1+p_2(\beta_2-1))}{p_2^{1+p_2(\beta_2-1)}} \right)^{q_1/p_2}, \\ K_3 &= \left(\frac{\Gamma(1+p_3(\beta_3-1))}{p_3^{1+p_3(\beta_3-1)}} \right)^{q_2/p_3}, \end{aligned}$$

and T_1 is the largest real number such that

$$\begin{aligned} T_1 &= \text{Max}\{t \in I, \tilde{C}(t) \in \text{Dom}\Omega_3^{-1}, \Omega_3^{-1}\{\tilde{C}(t)\} \in \text{Dom}\Omega_2^{-1}, \\ &\quad \Omega_2^{-1}\{\Omega_3^{-1}\{\tilde{C}(t)\}\} \in \text{Dom}\Omega_1^{-1}\}. \end{aligned} \quad (13)$$

Proof.

Let $p_i = 1 + \beta_i$ ($i = 1, 2, 3$). Then

$$\frac{1}{p_i} + \frac{1}{q_i} = \frac{1}{1 + \beta_i} + \frac{\beta_i}{1 + \beta_i} = 1 \quad (i = 1, 2, 3)$$

and using the Holder inequality, we obtain from (3) that

$$\begin{aligned}
u(t) &\leq a(t) + \int_{t_0}^t (t-s)^{\beta_1-1} e^s f(s) e^{-s} w_1(u(s)) [u(s) \\
&\quad + \int_{t_0}^s (s-\tau)^{\beta_2-1} e^\tau g(\tau) e^{-\tau} w_2(u(\tau)) [u(\tau) \\
&\quad + \int_{t_0}^\tau (\tau-\xi)^{\beta_3-1} e^\xi h(\xi) e^{-\xi} w_3(u(\xi)) d\xi] d\tau] ds \\
&\leq a(t) + [\int_{t_0}^t (t-s)^{p_1(\beta_1-1)} e^{p_1 s} ds]^{1/p_1} [\int_{t_0}^t f^{q_1}(s) e^{-q_1 s} w_1^{q_1}(u(s)) [u(s) \\
&\quad + [\int_{t_0}^s (s-\tau)^{p_2(\beta_2-1)} e^{p_2 \tau} d\tau]^{1/p_2} [\int_{t_0}^s e^{-q_2 \tau} g^{q_2}(\tau) w_2^{q_2}(u(\tau)) [u(\tau) \\
&\quad + [\int_{t_0}^\tau (\tau-\xi)^{p_3(\beta_3-1)} e^{p_3 \xi} d\xi]^{1/p_3} \\
&\quad \times [\int_{t_0}^\tau e^{-q_3 \xi} h^{q_3}(\xi) w_3^{q_3}(u(\xi)) d\xi]^{1/q_3}]^{q_2} d\tau]^{1/q_2}]^{q_1} ds]^{1/q_1}, \forall t \in I.
\end{aligned}$$

Using (5) in Lemma 1 and (6) in Lemma 2 with $n = 2$, $\alpha = q_1$, from the above inequality we obtain that

$$\begin{aligned}
u(t) &\leq a(t) + [\int_{t_0}^t (t-s)^{p_1(\beta_1-1)} e^{p_1 s} ds]^{1/p_1} [\int_{t_0}^t f^{q_1}(s) e^{-q_1 s} w_1^{q_1}(u(s)) [2^{q_1-1} u^{q_1}(s) \\
&\quad + 2^{q_1-1} [\int_{t_0}^s (s-\tau)^{p_2(\beta_2-1)} e^{p_2 \tau} d\tau]^{q_1/p_2} \\
&\quad \times [\int_{t_0}^s e^{-q_2 \tau} g^{q_2}(\tau) w_2^{q_2}(u(\tau)) [2^{q_2-1} u^{q_2}(\tau) \\
&\quad + 2^{q_2-1} [\int_{t_0}^\tau (\tau-\xi)^{p_3(\beta_3-1)} e^{p_3 \xi} d\xi]^{q_2/p_3} \\
&\quad \times [\int_{t_0}^\tau e^{-q_3 \xi} h^{q_3}(\xi) w_3^{q_3}(u(\xi)) d\xi]^{q_2/q_3}]^{q_1/q_2}]^{q_1} ds]^{1/q_1} \\
&\leq a(t) + 2^{q_1-1} K_1^{1/q_1} e^t [\int_{t_0}^t f^{q_1}(s) e^{-q_1 s} w_1^{q_1}(u(s)) [u^{q_1}(s) \\
&\quad + 2^{q_2-1} K_2 e^{q_1 s} [\int_{t_0}^s e^{-q_2 \tau} g^{q_2}(\tau) w_2^{q_2}(u(\tau)) [u^{q_2}(\tau) \\
&\quad + K_3 e^{q_2 \tau} [\int_{t_0}^\tau e^{-q_3 \xi} h^{q_3}(\xi) w_3^{q_3}(u(\xi)) d\xi]^{q_2/q_3}]^{q_1/q_2}]^{q_1} ds]^{1/q_1}, \quad (14)
\end{aligned}$$

for all $t \in I$. Using the condition (4) in Definition 1 and the inequality (6) in Lemma 2 with $n = 2$, $\alpha = q_1$, we obtain from (14) that

$$\begin{aligned}
u^{q_1}(t) &\leq 2^{q_1-1} a^{q_1}(t) + 2^{2q_1-2} K_1 e^{q_1 t} \int_{t_0}^t f^{q_1}(s) R_1(s) e^{q_1 s} w_1(e^{-q_1 s} u^{q_1}(s)) [u^{q_1}(s) e^{-q_1 s} \\
&\quad + 2^{q_2-1} K_2 [\int_{t_0}^s g^{q_2}(\tau) R_2(\tau) e^{q_2 \tau} w_2(e^{-q_2 \tau} u^{q_2}(\tau)) [u^{q_2}(\tau) e^{-q_2 \tau}
\end{aligned}$$

$$+ K_3 \left[\int_{t_0}^{\tau} h^{q_3}(\xi) R_3(\xi) w_3(e^{-q_3 \xi} u^{q_3}(\xi)) d\xi \right]^{q_2/q_3} d\tau \right]^{q_1/q_2} ds, \forall t \in I. \quad (15)$$

Let $v(t) = u^{q_1}(t)e^{-q_1 t}$, we have from (15)

$$\begin{aligned} v(t) &\leq B(t) + \int_{t_0}^t \tilde{f}(s) w_1(v(s)) [v(s) + \int_{t_0}^s \tilde{g}(\tau) w_2(v(\tau)) [v(\tau) \\ &\quad + [\int_{t_0}^{\tau} \tilde{h}(\xi) w_3(v(\xi)) d\xi]^{q_2/q_3} d\tau]^{q_1/q_2} ds, \forall t \in I, \end{aligned} \quad (16)$$

Let $z_1(t)$ denote the function on the right-hand side of (16), which is a positive and nondecreasing function on I . From (16), we have

$$v(t) \leq z_1(t), B(t) \leq z_1(t), \forall t \in I. \quad (17)$$

Differentiating $z_1(t)$ with respect to t , using (17) we have

$$\begin{aligned} z_1'(t) &= B'(t) + \tilde{f}(t) w_1(v(t)) [v(t) + \int_{t_0}^t \tilde{g}(\tau) w_2(v(\tau)) [v(\tau) \\ &\quad + [\int_{t_0}^{\tau} \tilde{h}(\xi) w_3(v(\xi)) d\xi]^{q_2/q_3} d\tau]^{q_1/q_2} \\ &\leq B'(t) + \tilde{f}(t) w_1(z_2(t)) z_2(t), \forall t \in I, \end{aligned} \quad (18)$$

where

$$\begin{aligned} z_2(t) &= z_1(t) + [\int_{t_0}^t \tilde{g}(\tau) w_2(z_1(\tau)) [z_1(\tau) \\ &\quad + [\int_{t_0}^{\tau} \tilde{h}(\xi) w_3(z_1(\xi)) d\xi]^{q_2/q_3} d\tau]^{q_1/q_2}. \end{aligned}$$

Obviously,

$$z_2(t_0) = z_1(t_0), z_1(t) \leq z_2(t), \forall t \in I. \quad (19)$$

since $q_1 < q_2 < q_3, q_2/q_1 > 1$. Using the inequality (6) in Lemma 2 with $n = 2, \alpha = q_2/q_1$, we obtain that

$$\begin{aligned} z_2^{q_2/q_1}(t) &= 2^{\frac{q_2}{q_1}-1} z_1^{q_2/q_1}(t) + 2^{\frac{q_2}{q_1}-1} \int_{t_0}^t \tilde{g}(\tau) w_2(z_1(\tau)) [z_1(\tau) \\ &\quad + [\int_{t_0}^{\tau} \tilde{h}(\xi) w_3(z_1(\xi)) d\xi]^{q_2/q_3} d\tau. \end{aligned} \quad (20)$$

From (18) and (20), we have

$$\begin{aligned} \frac{q_2}{q_1} z_2^{\frac{q_2}{q_1}-1}(t) z_2'(t) &= \frac{q_2}{q_1} 2^{\frac{q_2}{q_1}-1} z_1^{\frac{q_2}{q_1}-1}(t) z_1'(t) + 2^{\frac{q_2}{q_1}-1} \tilde{g}(t) w_2(z_1(t)) [z_1(t) \\ &\quad + [\int_{t_0}^t \tilde{h}(\xi) w_3(z_1(\xi)) d\xi]^{q_2/q_3} \end{aligned}$$

$$\begin{aligned}
&\leq \frac{q_2}{q_1} 2^{\frac{q_2-1}{q_1}} z_2^{\frac{q_2-1}{q_1}}(t) (B'(t) + \tilde{f}(t) w_1(z_2(t)) z_2(t)) \\
&\quad + 2^{\frac{q_2-1}{q_1}} \tilde{g}(t) w_2(z_2(t)) [z_2(t) \\
&\quad + [\int_{t_0}^t \tilde{h}(\xi) w_3(z_1(\xi)) d\xi]^{q_2/q_3}] \\
&= \frac{q_2}{q_1} 2^{\frac{q_2-1}{q_1}} z_2^{\frac{q_2-1}{q_1}}(t) (B'(t) + \tilde{f}(t) w_1(z_2(t)) z_2(t)) \\
&\quad + 2^{\frac{q_2-1}{q_1}} \tilde{g}(t) w_2(z_2(t)) z_3(t), \tag{21}
\end{aligned}$$

for all $t \in I$, where $z_3(t) = z_2(t) + [\int_{t_0}^t \tilde{h}(\xi) w_3(z_2(\xi)) d\xi]^{q_2/q_3}$. Obviously, $z_2(t) \leq z_3(t)$. Using the inequality (6) with $n = 2$, $\alpha = q_3/q_2 > 1$, we obtain that

$$z_3^{q_3/q_2}(t) = 2^{\frac{q_3-1}{q_2}} z_2^{q_3/q_2}(t) + 2^{\frac{q_3-1}{q_2}} \int_{t_0}^t \tilde{h}(\xi) w_3(z_2(\xi)) d\xi, \forall t \in I. \tag{22}$$

From (21) and (22), we have

$$\begin{aligned}
\frac{q_3}{q_2} z_3^{\frac{q_3-1}{q_2}}(t) z_3'(t) &= \frac{q_3}{q_2} 2^{\frac{q_3-1}{q_2}} z_2^{\frac{q_3-1}{q_2}}(t) z_2'(t) + 2^{\frac{q_3-1}{q_2}} \tilde{h}(t) w_3(z_2(t)) \\
&\leq \frac{q_3}{q_2} 2^{\frac{q_3+q_2-2}{q_2} \frac{q_3-1}{q_1}} z_3^{\frac{q_3-1}{q_2}}(t) (B'(t) + \tilde{f}(t) w_1(z_3(t)) z_3(t)) \\
&\quad + \frac{q_1 q_3}{q_2^2} 2^{\frac{q_3+q_2-2}{q_2} \frac{q_3-1}{q_1}} z_3^{\frac{q_3-1}{q_2}}(t) \tilde{g}(t) w_2(z_3(t)) z_3(t) \\
&\quad + 2^{\frac{q_3-1}{q_2}} \tilde{h}(t) w_3(z_3(t)), \tag{23}
\end{aligned}$$

for all $t \in I$. From (23) we have

$$\begin{aligned}
z_3'(t) &\leq 2^{\frac{q_3+q_2-2}{q_2} \frac{q_3-1}{q_1}} B'(t) + 2^{\frac{q_3+q_2-2}{q_2} \frac{q_3-1}{q_1}} \tilde{f}(t) w_1(z_3(t)) z_3(t) \\
&\quad + \frac{q_1}{q_2} 2^{\frac{q_3+q_2-2}{q_2} \frac{q_3-1}{q_1}} \tilde{g}(t) w_2(z_3(t)) z_3^{\frac{2-q_2}{q_1}}(t) + 2^{\frac{q_3-1}{q_2}} \tilde{h}(t) w_3(z_3(t))
\end{aligned}$$

$$\begin{aligned}
&= C'(t) + \tilde{f}(t)w_1(z_3(t))z_3(t) + \tilde{g}(t)w_2(z_3(t))z_3^{\frac{2-q_2}{q_1}}(t) \\
&\quad + \tilde{h}(t)w_3(z_3(t)), \forall t \in I.
\end{aligned} \tag{24}$$

From (24) we have

$$\begin{aligned}
v(t) \leq & \Omega_1^{-1} \{ \Omega_4^{-1} \{ \Omega_5^{-1} [\Omega_5 [\Omega_4 (\Omega_1 (C(t)) + \int_{t_0}^t \tilde{h}(s) ds \\
&+ \int_{t_0}^t \tilde{g}(s) ds] + \int_{t_0}^t \tilde{f}(s) ds] \} \} \}, \forall t \in [t_0, T_1],
\end{aligned} \tag{25}$$

In view of $v(t) = u^{q_1}(t)e^{-q_1 t}$, we can obtain (7).

3. Summary

In this paper, we establish a class of new nonlinear weakly singular integral inequality, which consists of iterated integral, and weakly singular integral kernel be involved in each layer. Under several practical assumptions, the inequality is solved by adopting novel analysis techniques, such as: change of variable, amplification method, differential and integration, inverse function, and the dialectical relationship between constants and variables, and explicit bounds for the unknown functions are given clearly.

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