Countable compactness in generalized L-topological spaces

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Abstract.

In this paper, the generalized countable L-compact sets and generalized Lindelöf sets are introduced in generalized L-topological spaces, based on the notion of generalized L-compactness. They are described by cover form and finite intersection property. They are preserved under generalized L-continuous mapping, inherited for L-closed subsets, and finitely additive. And an L-subset is generalized L-compact if and only if it is generalized Lindelöf and generalized countably L-compact.

Keywords: generalized L-topology, generalized countable L-compactness, generalized Lindelöf set

1. Introduction and Preliminaries

In [2], Bai introduced the concept of generalized L-topological spaces, and studied the basic concepts and basic properties in generalized L-topological spaces. Following the lines of [2], in [3], Bai introduced generalized L-compactness. In this paper, our aim is to continue the research of generalized countable L-compact sets and generalized Lindelöf sets in generalized L-topological spaces.

Throughout this paper, $(L, \lor, \land, ')$ is a completely distributive De Morgan algebra and $X$ is a nonempty set. $L^X$ is the set of all L-fuzzy sets on $X$. The smallest element and the largest element of $L^X$ will be denoted by $0$ and $1$ respectively. The set of non-unit prime elements[4] in $L$ is denoted by $pr(L)$. The set of nonzero co-prime elements[4] in $L$ and $L^X$ is denoted by
\[ M(L) \] and \[ M^*(L) \] respectively. Clearly, \( r \in pr(L) \) if and only if \( r' \in M(L) \). The greatest minimal family of \( a \) in \( L \) is denoted by \( \beta(a) \). The greatest maximal family of \( a \) in \( L \) is denoted by \( \alpha(a) \) [5,8]. Moreover for \( a \) in \( L \), define \( \beta^*(a) = \beta(a) \cap M(L) \) and \( \alpha^*(a) = \alpha(a) \cap pr(L) \). For each \( \psi \subseteq L \), we define \( \psi' = \{ A' : A \in \psi \} \). For \( r \in L \), \( \mathcal{E}_r(A) = \{ x \in X : A(x) \geq r \} \).

Definition 1.1.[2]. Let \( L \) be a completely distributive De Morgan algebra, \( X \) be a nonempty set and \( \delta \) be a collection of subsets of \( L^X \). Then \( \delta \) is called a generalized L-topology (briefly GL-t) on \( X \) if \( \emptyset \in \delta \) and \( G_i \in \delta \) for \( i \in I \neq \emptyset \) implies \( G = \bigvee_{i \in I} G_i \in \delta \). We call the pair \( (L^X, \delta) \) a generalized L-topological space (briefly GL-ts) on \( X \). The element of \( \delta \) are called generalized L-open sets (briefly GL-open sets) and the complements are called generalized L-closed sets (briefly GL-closed sets). We say \( \delta \) is strong if \( 1 \in \delta \).

Definition 1.2.[6]. Each mapping \( f : X \to Y \) induces a mapping \( f_L^{-} : X \to Y \) (called an L-valued Zadeh function or an L-fuzzy mapping or an L-forward power set operator), which is defined by \( f_L^{-}(A) = \bigvee \{ A(x) \mid f(x) = y \} \quad (\forall A \in L^X, y \in Y) \). The right adjoint to \( f_L^{-} \) (called L-backward power set operator) is denoted \( f_L^{+} \) and given by \( f_L^{+}(B) = \bigvee \{ A \in L^X \mid f_L^{-}(A) \leq B \} = B \circ f \quad (\forall B \in L^X) \).

Definition 1.3.[2]. Let \( (L^X, \delta) \) and \( (L^Y, \tau) \) be two GL-ts’s and \( f^{-} : L^X \to L^Y \) an L-fuzzy mapping. \( f^{-} \) is called a generalized L-continuous mapping (briefly GL-continuous mapping) if \( f^{-}(B) \in \delta \) for each \( B \in \tau \).

Definition 1.4.[2]. Let \( (L^X, \delta) \) be a GL-ts and \( x_A \in M^*(L^X) \). \( A \in \delta' \) is
called a generalized $L$-closed remote-neighborhood (briefly GLC-RN) of $x_\lambda$, if $x_\lambda \leq A$. $B \in L^X$ is called a generalized $L$-remote-neighborhood (briefly GL-RN) of $x_\lambda$ if there is a GLC-RN $A$ of $x_\lambda$ such that $B \leq A$. The set of all GLC-RNs(GL-RNs) of $x_\lambda$ is denoted by $\eta^{-}(x_\lambda)$ ($\eta(x_\lambda)$).

Definition 1.5[2]. Let $(\delta, L^X)$ be a GL-ts, $A \in L^X$ and $\alpha \in M(L)$. $\phi \subset \delta'$ is called an $\alpha$-closed -remote neighborhood family of $A$ (briefly $\alpha$-C-RF of $A$) if for each $x_\alpha$ in $A$, there exists a $P \in \phi$ such that $P \in \eta(x_\alpha)$. $\phi$ is called an $\alpha^{-}$-C-RF of $A$.

Definition 1.6[2]. Let $(\delta, L^X)$ be a GL-ts and $A \in L^X$. $A$ is called generalized $L$-compact (briefly GL-compact) if every $\alpha$-C-RF $\phi$ of $A$ has a finite subfamily which is an $\alpha^{-}$-C-RF of $A$. $(\delta, L^X)$ is called GL-compact if $1_X$ is GL-compact.

2. Generalized countable $L$-compactness

Definition 2.1. Let $(L^X, \delta)$ be a GL-ts and $A \in L^X$. $A$ is called generalized countably $L$-compact if every countable $\alpha$-C-RF $\phi$ of $A$ has a finite subfamily which is an $\alpha^{-}$-C-RF of $A$. $(\alpha \in M(L))$. $(L^X, \delta)$ is called generalized countably $L$-compact if $1_X$ is generalized countably $L$-compact.

From the Definitions 2.1 and 1.6 we immediately obtain the following results.

Corollary 2.2. Every generalized $L$-compact set is generalized countably $L$-compact.

Definition 2.3. Let $(L^X, \delta)$ be a GL-ts, $A \in L^X$ and $r \in pr(L)$. $\mu \subset \delta$ is called an $r$-cover of $A$ if for each $x \in \varepsilon_r(A)$, there exists an $U \in \mu$ such that $U(x) \leq r$. $\mu$ is called an $r^{-}$-cover of $A$ if there exists a $t \in \alpha^*(r)$ such that $\mu$ is a $t$-cover of $A$.

Theorem 2.4. Let $(L^X, \delta)$ be a GL-ts and $r \in pr(L)$. $A \in L^X$ is generalized countably $L$-compact if and only if every countable $r$-cover $\mu$ of $A$ has a finite subfamily $\nu$ which is an $r^{-}$-cover of $A$. 

1625
Proof. Let $A$ be generalized countably $L$-compact, $\mu$ a countable $r$-cover of $A$ and $r \in pr(L)$. Put $\phi = \mu'$, then $\phi \subset \delta'$ and for each $x \in \varepsilon_r(A)$ there exists a $Q = U' \in \phi$ such that $U(x) \leq r$, i.e., $r' \leq Q(x)$.

Since $r \in pr(L)$, $r' \in M(L)$. By $x_r \leq Q \not\equiv \emptyset$ have $Q \in \eta(x_r)$, hence $\phi$ is a countable $r'$-C-RF of $A$. Since $A$ is generalized countably $L$-compact, there is a finite subfamily $\nu$ of $\mu$ such that $\nu = \nu'$ is an $(r')^-$-C-RF of $A$, i.e., for some $t \in \beta^*(r')$ and each $x \in \varepsilon_r(A)$, there is a $\nu'(x) \in \nu$ such that $t \leq \nu'(x)$, equivalently, for some $t' \in \alpha'(r)$ and each $x \not\in \varepsilon_r(A)$, there is a $\nu'(x) \in \nu$ such that $V(x) \leq t'$. Thus $\mu$ has a finite subfamily $\nu$ which is an $r^+$-cover of $A$.

Conversely, suppose every countable $r$-cover $\mu$ of $A$ has a finite subfamily is an $r^+$-cover of $A$. Let $\phi$ be a countable $\alpha$-C-RF of $A$, $\mu = \phi'$ and $r = \alpha'$. Since $\alpha \in M(L)$, $r \in pr(L)$. With the method of dual above, it is easily to prove that $\mu$ is a countable $r$-cover of $A$. Suppose $\psi$ is a finite subfamily of $\mu$ such that $\psi$ is an $r^+$-cover of $A$. Put $\psi = \nu'$, then $\psi$ is an $\alpha^-$-C-RF of $A$. Thus $A$ is generalized countably $L$-compact.

Definition 2.5. Let $(L^X, \delta)$ be a GL-ts, $A \in L^X$, $r \in pr(L)$ and $\mu \subset L^X$. If for every finite subfamily $\nu$ of $\mu$ and for each $t \in \alpha^*(r)$, there is an $x \in \varepsilon_r(A)$ such that $\nu(x) \geq t'$, then we say that $\mu$ has an $r^+$-finite intersection property in $A$.

Theorem 2.6. Let $(L^X, \delta)$ be a GL-ts and $r \in pr(L)$. $A \in L^X$ is generalized countably $L$-compact if and only if every countable subfamily of GL-closed sets $\mu$ has an $r^+$-finite intersection property in $A$ and there is an $x \in \varepsilon_r(A)$ such that $(\wedge \nu)(x) \geq r'$.

Proof. Let $A$ be generalized countably $L$-compact. Suppose there is a prime element $e \in pr(L)$ and some countable subfamily of GL-closed sets $\mu$ has an $e^+$-finite intersection property in $A$ for each $x \in \varepsilon_r(A)$ such that $(\wedge \mu)(x) \geq e'$. Then there exists a $B \in \mu$ such that $B(x) \geq e'$, i.e., $B'(x) \leq e$. This
shows $\mu'$ is a countable $e$-cover of $A$. By the Theorem 2.4, there is a finite subfamily $\nu = \{B_1, \cdots, B_n\}$ of $\mu$ such that $\nu'$ is an $e^+$-cover of $A$. Hence for some $t \in \alpha^+(e)$ and each $x \in \mathcal{C}_e(A)$, there is a $B_i \in \nu$

such that $B_i'(x) \leq t$. And so $(\vee_{i=1}^n B_i')(x) \leq t$, i.e. $(\forall \nu)(x) = (\vee_{i=1}^n B_i)(x) \geq t'$, which contradicts that $\mu$ has an $e^+$-finite intersection property in $A$.

Conversely, let $\mu$ be a countable $r$-cover of $A$ and $r \in pr(L)$. If none of the finite subfamily $\nu$ of $\mu$ is $r^+$-cover of $A$, then every $t \in \alpha^+(r)$ there is an $x \in \mathcal{C}_e(A)$ such that $C(x) \leq t$ for each $C \in \nu$. And so $(\forall \nu)(x) \leq t$, equivalently, $(\forall \nu')(x) \geq t'$. This shows that subfamily of GL-closed sets $\mu'$ having an $r^+$-finite intersection property in $A$. Hence there is an $x \in \mathcal{C}_e(A)$ such that $(\vee \mu)(x) \geq r'$, i.e. $(\forall \mu)(x) \leq r$. This implies that $\mu$ is not a countable $r$-cover of $A$, a contradiction. By the Theorem 2.4, $A$ is generalized countably L-compact.

Theorem 2.7. Let $(L^X, \delta)$ be a GL-ts and $A, B \in L^X$. If $A$ is generalized countably L-compact and $B \in \delta'$, then $A \land B$ is generalized countably L-compact.

Proof. Let $\phi \subset \delta'$ be a countable $\alpha$-C-RF of $A \land B$ ($\alpha \in M(L)$). Then $\phi_1 = \phi \cup \{B\}$ is a countable $\alpha$-C-RF of $A$. In fact, for each $x_\alpha \in B$ then $x_\alpha \in A \land B$. Hence, there is $P \in \phi \subset \phi_1$ such that $P \in \eta(x_\alpha)$. If $x_\alpha \notin B$, then $B \in \phi$ and $B \in \eta(x_\alpha)$. Thus, $\phi_1$ is indeed a countable $\alpha$-C-RF of $A$. Since $A$ is generalized countably L-compact, there exists an $r \in f^B(\alpha)$ and finite subfamily $\psi_1$ of $\phi_1$ such that $\psi_1$ is an $r$-C-RF of $A$. Let $\delta = \psi_1 - \{B\}$, then $\delta$ is a finite subfamily of $\phi$ and $\psi$ is an $r$-C-RF of $A \land B$. In fact, $x_\alpha \in A \land B$, then $x_\alpha \in A$, from the definition of $\psi_1$, there is $P \in \psi_1$, with $P \in \eta_1(x_\alpha)$. But $x_\alpha \in B$, so $P \neq B$, and thus $P \in \psi_1 - \{B\} = \psi$. Hence, $A \land B$ is generalized countably L-compact.

Theorem 2.8. If $A$ and $B$ are generalized countably L-compact in GL-ts
Theorem 2.9. Let \( (L^X, \delta) \) and \( (L^Y, \tau) \) be two GL-ts's, \( f : L^X \to L^Y \) a GL-continuous mapping and \( A \) a generalized countable L-compact set in \( (L^X, \delta) \). Then \( f^{-\top}(A) \) is generalized countable L-compact in \( (L^Y, \tau) \).

Proof. Let \( \phi \subseteq \tau' \) be a countable \( \alpha \)-C-RF of \( f^{-\top}(A) \) and \( x_\alpha \in A(\alpha \in M(L)) \). To begin with, let us show that \( f^{\top-}(\phi) = \{f^{\top-}(P) : P \in \phi\} \) a countable \( \alpha \)-C-RF of \( A \). Since \( f^{\top-} \) is GL-continuous and \( x_\alpha \in A \), \( f^{\top-}(\phi) \subseteq \delta' \) and \( f^{\top-}(x_\alpha) = (f^{\top-}(x))_\alpha \leq f^{\top-}(A) \). By \( \phi \) is a countable \( \alpha \)-C-RF of \( f^{\top-}(A) \), there is a \( P \in \phi \) with \( P \in \eta((f^{\top-}(x))_\alpha) \), i.e. \( (f^{\top-}(x))_\alpha \leq P \) or, equivalently, \( P(f^{\top-}(x)) \geq \alpha \).

By the definition of inverse mapping, \( f^{\top-}(P)(x) = P(f^{\top-}(x)) \geq \alpha \) hence \( x_\alpha \notin f^{\top-}(P) \), i.e. \( f^{\top-}(P) \in \eta(x_\alpha) \).

Therefore \( f^{\top-}(\phi) \) is a countable \( \alpha \)-C-RF of \( A \).

Since \( A \) is generalized countably L-compact, there exists an \( r \in \beta^*(\alpha) \) and a finite subfamily \( \psi \) of \( \phi \) such that \( f^{\top-}(\psi) \) is an \( r \)-C-RF of \( A \). Again, by the generalized countable L-compactness of \( A \), there exists an \( r_1 \in \beta^*(r) \) and a finite subset \( \psi_1 \) of \( f^{\top-}(\psi) \) such that \( \psi_1 \) is an \( r_1 \)-C-RF of \( A \). Obviously we can take \( \psi_1 = f^{\top-}(\psi) \).

Now we will show that \( \psi \) is an \( r \)-C-RF of \( f^{\top-}(A) \). Let \( y_r \leq f^{\top-}(A) \), by the Lemma 4.10 [8], \( r = \sup \{\lambda \in L : \exists x \in f^{\top-}(y), A(x) \geq \lambda \text{ and } \lambda \leq r\} \). Since \( r_1 \in \beta^*(r) \), we have \( r_1 \in \beta(r) \) and hence there is an \( \lambda \in L \) and \( x \leq f^{\top-}(y) \) with \( A(x) \geq \lambda \), \( \lambda \leq r \), and \( \lambda \geq r_1 \); thus \( x_\lambda \leq A \). It follows from \( f^{\top-}(\psi) \) is an \( r \)-C-RF of \( A \) that there is a \( P \in \psi \) with \( f^{\top-}(P) \in \eta(x_\lambda) \), i.e. \( f^{\top-}(P)(x) \geq r_1 \). Hence \( P(y) = P(f^{\top-}(x)) \geq r_1 \) and therefore certainly \( P(y) \geq r \), i.e. \( P \in \eta(y_r) \). Thus \( f^{\top-}(A) \) is generalized countably GL-compact.
Corollary 2.10. Let \((L^X, \delta)\) be a countable L-compact space and \(f : (L^X, \delta) \rightarrow (L^Y, \tau)\) a surjective GL-continuous mapping. Then \((L^Y, \tau)\) is generalized countably L-compact.

3. Generalized Lindelöf sets

Definition 3.1. Let \((L^X, \delta)\) be a GL-ts and \(A \in L^X\). \(A\) is called generalized Lindelöf sets if every \(\alpha\)-C-RF \(\phi\) of \(A\) has a countable subfamily which is an \(\alpha^-\)-C-RF \(\phi\) of \(A\) \((\alpha \in M(L))\). \((L^X, \delta)\) is called generalized Lindelöf space if \(1_X\) is generalized Lindelöf.

From the Definitions 3.1 and 2.6 we immediately obtain the following results.

Theorem 3.2. Let \((L^X, \delta)\) be a GL-ts and \(A \in L^X\). Then \(A\) is generalized L-compact set if and only if \(A\) is generalized Lindelöf and generalized countably L-compact.

Analogous to generalized countable L-compactness, we have the following results.

Theorem 3.3. Let \((L^X, \delta)\) be a GL-ts and \(r \in pr(L)\). Then \(A\) is generalized Lindelöf set if and only if every \(r\)-cover \(\mu\) of \(A\) has a countable subfamily \(\nu\) which is an \(r^-\)-cover of \(A\).

Theorem 3.4. Let \((L^X, \delta)\) be a GL-ts and \(A, B \in L^X\). If \(A\) is generalized Lindelöf set and \(B \in \delta'\), then \(A \land B\) is generalized Lindelöf.

Theorem 3.5. If \(A\) and \(B\) is generalized Lindelöf sets in GL-ts \((L^X, \delta)\), then \(A \lor B\) is generalized Lindelöf.

Theorem 3.6. Let \((L^X, \delta)\) and \((L^Y, \tau)\) be two GL-ts’s, \(f : L^X \rightarrow L^Y\) a GL-continuous mapping and \(A\) a generalized Lindelöf set in \((L^X, \delta)\). Then \(f^{-1}(A)\) is generalized Lindelöf in \((L^Y, \tau)\).
Corollary 3.7. Let \((L^X, \delta)\) and \((L^Y, \tau)\) be a generalized Lindelöf set and \(f : (L^X, \delta) \to (L^Y, \tau)\) a surjective GL-continuous mapping. Then \((L^Y, \tau)\) is generalized Lindelöf.

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References


