A New Theorem on Bargaining Sets in TU Games

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Abstract. In this paper, we provide a new existence theorem by proving that Mas-Colell bargaining sets exist for all TU games.

Introduction

Let \( N = \{1, 2, \ldots, n\} \) be the set of \( n \) players. Any subset of \( N \) is called a coalition.

Definition 1.1. A cooperative game (or a TU game) in characteristic function form with player set \( N \) is a map \( \nu : 2^N \rightarrow \mathbb{R} \) with the property \( \nu(\emptyset) = 0 \).

A payoff vector \( x \in \mathbb{R}^n \) is said to be individual rational if \( x_i \geq \nu(\{i\}) \) for each \( i \in N \).

Definition 1.2. The imputation set \( I(\nu) \) of a cooperative game \( \nu \) is the set

\[
I(\nu) = \left\{ x \in \mathbb{R}^n \mid \sum_{i \in N} x_i = \nu(N), x_i \geq \nu(\{i\}) \text{ for each } i \in N \right\}.
\]

Cooperative games have been studied extensively in the literature. A central question in cooperative games is to study solution concepts and their relationships, those well-known solution concepts include cores, stable sets, Shapley values, bargaining sets, and so on.

To state Vohra’s result formally, let us recall some necessary concepts from [4].

A non-transferable utility game (NTU game) in characteristic function form is defined as a pair \((N, V)\), where \( V : 2^N \rightarrow \mathbb{R}^N \) is a correspondence satisfying

(i) for all non-empty \( S \in 2^N \), \( V(S) \) is non-empty, closed, and comprehensive,
(ii) for all \( i \in N \), \( V(\{i\}) = \left\{ x \in \mathbb{R}^N \mid x_i \leq 0 \right\} \),
(iii) for all \( S \in 2^N \), \( V(S)_j \cap \mathbb{R}_j \) is bounded.

A TU game \( \nu \) in characteristic function form is equivalent to an NTU game \((N, V)\) such that for every non-empty \( S \in 2^N \),

\[
V(S) = \left\{ x \in \mathbb{R}^N \mid \sum_{i \in S} x_i \leq \nu(S) \right\}. \quad (1.1)
\]

In fact, Condition (ii) in the definition above by Vohra also requires \( \nu(\{i\}) = 0 \) for all \( i \in N \), which can be achieved by zero normalization.

Weak Superadditivity (version 1): For any \( S \in 2^N \) and \( i \notin S \), if \( x \in V(S) \), then \( y \in V(S \cup \{i\}) \), where \( y_i = 0 \) and \( y_j = x_j \) for \( j \neq i \).

This has the following equivalent form given in [2].

Weak Superadditivity (version 2): An NTU game \((N, V)\) is weakly superadditive if for every \( i \in N \) and every \( S \subseteq N \setminus \{i\} \) satisfying \( S \neq \emptyset \), \( V(S) \times V(\{i\}) \subseteq V(S \cup \{i\}) \).

Clearly, for TU games, the weak superadditivity is equivalent to the following according to version 2 and (1.1).
Weak Superadditivity for TU games: \( \nu(S) + \nu(\{i\}) \leq \nu(S \cup \{i\}) \) for each \( S \subseteq N \) and each \( i \in N \setminus S \).


**Theorem 1.3** (Vohra, 1991). If \( \nu \) is a weakly superadditive TU game, then the Mas-Colell bargaining set \( MB(\nu) \) of \( \nu \) is non-empty.

In this paper, we prove the following stronger existence theorem for Mas-Colell bargaining sets in TU games.

**Theorem 1.4.** If \( \nu \) is a TU game such that \( \nu(S) \leq \nu(N) \) for each \( S \subseteq N \), then the Mas-Colell bargaining set \( MB(\nu) \) of \( \nu \) is non-empty.

**Lemma 1.7.** Let \( \nu \) be a TU game and let \( \nu_0 \) be the zero-normalized game of \( \nu \). Then \( x \in MB(\nu) \) if and only if \( x' \in MB(\nu_0) \), where \( x'_i = x_i - \nu(i) \) for each \( i \in N \).

**Proof of Theorem 1.4**

In this section, we will give a proof for Theorem 1.4 by proving the following Theorem 2.2 which implies Theorem 1.4. Our proof is motivated in part by the ideas from [4] and [5]. Let \( \nu \) be a TU game. For an imputation \( x \in I(\nu) \) and a coalition \( S \subseteq N \), the excess of \( S \) at \( x \) is

\[
e(S,x) = \nu(S) - \sum_{i \in S} x_i
\]

Clearly, we have following remark from the definitions.

**Remark 2.1.** An objection \((S,y)\) at \( x \) exists if and only if \( e(S,x) > 0 \).

Next, for the purpose of overcoming difficulties in our proof for Theorem 1.4, we introduce strong counterobjection as follows, where the special conditions imposed on strong counterobjection is just a technical device.

**Strong Counterobjection:** Given an objection \((S,y)\) at \( x \in I(\nu) \), a strong counterobjection to \((S,y)\) at \( x \) is a pair \((T,z)\), where \( T \) is a coalition such that \( T \setminus S \neq \phi \) and there exists \( h \in S \setminus T \) satisfying \( y_h - x_h = \max \{ y_i - x_i | i \in S \} > 0 \), and \( z \) is a vector in \( R^{|T|} \) satisfying that

\[
z(T) = \sum_{i \in T} z_i = e(T), z_i \geq y_i \text{ for each } i \in S \setminus T, \text{ and } z_i \geq x_i + \sum_{j \in S \setminus T} \frac{y_j - x_j}{|T \setminus S|} \text{ for each } i \in T \setminus S.
\]

An imputation \( x \in I(\nu) \) is said to belong to strong Mas-Colell bargaining set \( MB_s(\nu) \) if for any objection \((S,y)\) at \( x \), there exists a strong counterobjection to it at \( x \).

**Theorem 2.2.** If \( \nu \) is a TU game such that \( \nu(S) \leq \nu(N) \) for each \( S \subseteq N \), then the strong Mas-Colell bargaining set \( MB_s(\nu) \) of \( \nu \) is non-empty.

**Lemma 2.3.** Given an objection \((S,y)\) at \( x \) and a non-empty coalition \( T \) such that \( T \setminus S \neq \phi \) and there exists \( h \in S \setminus T \) satisfying \( y_h - y_h = \max \{ y_i - x_i | i \in S \} > 0 \), then a strong counterobjection \((T,z)\) to \((S,y)\) at \( x \) exists if and only if \( e(T,x) \geq e(S,x) \).

Next we introduce the concept of balanced collection and a result from [11] which is needed in our proof.

Let \( \Delta^N \) be the standard simplex:

\[
\Delta^N = \left\{ x \in R^N \left| x_i \geq 0 \text{ for each } i \in N \text{ and } \sum_{i=1}^{n} x_i = 1 \right. \right\}.
\]

Its \( i \)-th face is \( \Delta^N[i] = \{ x \in \Delta^N | x_i = 0 \} \). For each \( S \subseteq N \), denote \( e^S \) the n-dimensional vector
with \( e^S_i = 1 \) if \( i \in S \) and \( e^S_i = 0 \) if \( i \notin S \).

**Definition 2.4.** A collection \( B \) of non-empty subsets (coalitions) of \( N \) is balanced if there exist positive numbers \( \lambda_S \) for \( S \in B \) such that

\[
\sum_{S \in B} \lambda_S e^S = e^N.
\]  

(2.1)

The numbers \( \lambda_S \) are called balancing coefficients.

Clearly, the condition in (2.1) for a balanced collection \( B \) is equivalent to the following.

\[
\sum_{S \in B \cap i \in S} \lambda_S = 1 \quad \text{for each } \ i \in N.
\]

(2.2)

The next theorem is proved by Zhou.

**Theorem 2.5** (Zhou, 1994). If \( \{ O_S \}_{S \in N} \) is a family of open sets of \( \Delta^N \) that satisfy

(1) \( \Delta^{N,i} \subseteq O_{\{ i \}} \) for each \( i \in N \) and

(2) \( \bigcup_{S \in N} O_S = \Delta^N \),

then there is a balanced collection \( B \) of non-empty subsets (coalitions) of \( N \) such that \( \bigcap_{S \in B} O_S \neq \phi \).

Let \( \nu \) be a TU game. Note that the core \( C(\nu) \) of \( \nu \) consists of all \( x \in I(\nu) \) such that \( e(S, x) \leq 0 \) for all \( S \subseteq N \). It follows from Remark 2.1 that the core \( C(\nu) \) is a subset of Mas-Colell bargaining set \( MB(\nu) \). Thus, whenever \( \nu \) has a non-empty core, \( MB(\nu) \) is non-empty. This means that, when we deal with the existence of \( MB(\nu) \), we may assume that \( C(\nu) = \phi \), that is, for any \( x \in I(\nu) \), there exists \( S \subseteq N \) such that \( e(S, x) > 0 \). For each \( x \in I(\nu) \), let \( \epsilon_x = \min \{ e(S, x) \mid S \subseteq N \text{ with } e(S, x) > 0 \} \) and set

\[
\epsilon_x = \min \left\{ \frac{1}{n} e_x, \frac{1}{n} \nu(N) \right\}
\]

(2.3)

Then, under the assumption that \( \nu(N) > 0 \) and \( C(\nu) = \phi \), \( \epsilon_x > 0 \) for each \( x \in I(\nu) \).

Let \( \nu \) be a TU game and \( x \in I(\nu) \). We say an objection \( (S, y) \) at \( x \) is strongly justified if there is no strong counterobjection to \( (S, y) \) at \( x \). For each non-empty \( S \subseteq N \), define \( O_S \) as follows:

\[
O_{\{ i \}} = \{ x \in I(\nu) \mid x_i < \epsilon_x \} \quad \text{for each } \ i \in N,
\]

\[
O_S = \{ x \in I(\nu) \mid \text{there exists a strongly justified objection } (S, y) \text{ at } x \} \quad \text{if } |S| \geq 2.
\]

The following fact follows from the definition immediately.

**Fact 2.6.** Let \( \nu \) be a TU game with empty core and \( \nu(N) > 0 \). For each \( i \in N \), \( \Delta^{N,i} \subseteq O_{\{ i \}} \).

**Lemma 2.7.** Let \( \nu \) be a TU game with \( \nu(N) > 0 \). Then, for each non-empty \( S \subseteq N \), \( O_S \) is open.

**Lemma 2.8.** Let \( \nu \) be a TU game such that \( \nu(N) > 0 \) and \( \nu(S) \leq \nu(N) \) for each \( S \subseteq N \). Then for any balanced collection \( B \) of coalitions, \( \bigcap_{S \in B} O_S = \phi \).

The next lemma allows us to assume \( \nu(N) > 0 \) when dealing with the non-emptiness of strong Mas-Colell bargaining sets.

**Lemma 2.9.** Let \( \nu \) be a TU game and let \( b > 0 \) be such that \( \nu(N) + b > 0 \). Define \( \nu' \) to be the game such that \( \nu'(S) = \nu(S) + \frac{|S|}{n} b \) for each \( S \subseteq N \). Then \( x \in MB_s(\nu) \) if and only if
We now prove Theorem 1.4 by proving Theorem 2.2.

Proof of Theorem 2.2. Let \( v \) be a TU game such that \( v(S) \leq v(N) \) for all \( S \subseteq N \). In view of Lemma 2.9, we may assume \( v(N) > 0 \). In fact, if \( v(N) \leq 0 \), then let \( b > 0 \) be such that \( v(N) + b > 0 \) and define \( v' \) to be the game such that \( v'(S) = v(S) + \frac{|S|}{n} b \) for each \( S \subseteq N \). Then \( v'(N) = v(N) + b > 0 \) and \( v'(S) \leq v'(N) \) for each \( S \subseteq N \). By Lemma 2.9, \( MB_v \) is non-empty if and only if \( MB_{v'} \) is non-empty. Thus, we may assume \( v(N) > 0 \). If the core \( C(v) \) is non-empty, then we have the strong Mas-Colell bargaining set \( MB_v \) is non-empty. Thus, we may assume that the core \( C(v) \) is empty.

Recall that for each \( x \in I(v) \), \( \sum_{i=1}^{n} x_i = v(N) > 0 \). We map \( Q = I(v) \) onto the standard simplex \( \Delta^N \) by \( f: \)

\[
f: x \rightarrow \frac{x}{\sum_{i=1}^{n} x_i}
\]

Suppose, to the contrary, that the strong Mas-Colell bargaining set \( MB_v \) is empty. Then we have \( Q \setminus U_0 \neq \bigcup_{S \subseteq N} O_S \). This means that \( \Delta^N = f(Q) = U_0 \neq \bigcup_{S \subseteq N} f(O_S) \). By Fact 2.6, \( \Delta^N \subseteq f(O_i) \) for each \( i \in N \). It follows from Theorem 2.5 that there is a balanced collection \( B \) of coalitions such that \( \bigcap_{S \in B} f(O_s) \neq \phi \). But, by Lemma 2.8, we have \( \bigcap_{S \in B} O_S = \phi \). It follows that \( \bigcap_{S \in B} f(O_s) = \phi \), a contradiction. Thus, the theorem holds.

Conclusion

In this paper, we proofed a stronger existence theorem by proving that Mas-Colell bargaining sets exist for all TU games.

References


