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Nonlinear-Integral-Equation Construction of Orthogonal Polynomials

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Abstract

The nonlinear integral equation $P(x) = \int_\alpha^\beta dy\, w(y)\, P(y)\, P(x+y)$ is investigated. It is shown that for a given function $w(x)$ the equation admits an infinite set of polynomial solutions $P_n(x)$. For polynomial solutions, this nonlinear integral equation reduces to a finite set of coupled linear algebraic equations for the coefficients of the polynomials. Interestingly, the set of polynomial solutions is orthogonal with respect to the measure $x\,w(x)$. The nonlinear integral equation can be used to specify all orthogonal polynomials in a simple and compact way. This integral equation provides a natural vehicle for extending the theory of orthogonal polynomials into the complex domain. Generalizations of the integral equation are discussed. Finally, it is observed that since the integral equation is independent of the degree of the polynomials it may possibly be a useful tool in determining and studying the asymptotic behaviors of polynomials.

1 Introduction

The work reported here is a review (in Secs. 1 and 2) of the research reported in Ref. [1], where it is shown that any class of orthogonal polynomials can be constructed by using a nonlinear integral equation. Section 3 describes new work in which we show how to use this integral equation to examine the asymptotic properties of the polynomials.

Let us begin by recalling that there are many ways to specify a set of orthogonal polynomials. We mention two completely general methods below:

• We can specify the domain (α,β) and the measure $g(x)$ with respect to which the polynomials are orthogonal. We then can use the well known Gram-Schmidt orthogonalization procedure to determine the polynomials sequentially. For example, on the domain $(-1,1)$ and for the measure $g(x) = (1-x^2)^{-1/2}$, the Gram-Schmidt procedure yields the Chebyshev polynomials $T_n(x)$:

$$T_0(x) = 1,$$

$$\begin{aligned} T_1(x) &= x, \\ T_2(x) &= 2x^2-1, \\ T_3(x) &= 4x^3-3x, \end{aligned}$$

and so on.

• We can use a recursion relation. For example, if we take

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$

along with the initial conditions $T_0(x) = 1$ and $T_1(x) = x$, we again obtain the Chebyshev polynomials $T_n(x)$.

Other methods to construct or specify a set of orthogonal polynomials can also be used. The following two methods do not have a general applicability:

• The classical orthogonal polynomials [2, 3, 4] obey a differential-equation eigenvalue problem (a Sturm-Liouville problem). For example, the differential equation

$$(1-x^2)y''(x) - xy'(x) + n^2y(x) = 0$$

along with appropriate boundary conditions at $x = -1$ and $x = 1$ leads to the Chebyshev polynomials $T_n(x)$ once again. Of course, it is unusual to find that a set of polynomials obey a differential equation. For example, the Krawtchouk, Charlier, Meixner, and Hahn polynomials satisfy difference equations but not differential equations [5].

• We can introduce a generating function $G(x,t)$ for polynomials. If the generating function is expanded as a Taylor series in powers of t, the coefficients are polynomials in x. For example, the generating function

$$G(x,t) = \frac{1-xt}{1-2xt+t^2} = \sum_0^\infty T_n(x)t^n$$

reproduces the Chebyshev polynomials $T_n(x)$. Of course, this method is again not general because there is no procedure for predicting when a given generating function will in fact produce orthogonal polynomials.

The new method described in this paper for constructing orthogonal polynomials is general and makes use of the following nonlinear integral equation:

$$P(x) = \int_\alpha^\beta dy\, w(y)\, P(y)\, P(x+y), \tag{1.1}$$

where the function $w(x)$ satisfies the constraint

$$\int_\alpha^\beta dx\, w(x) = 1. \tag{1.2}$$

We find that the nth degree polynomial solutions $P_n(x)$ to this integral equation are orthogonal with respect to the measure

$$g(x) = x\, w(x). \tag{1.3}$$

The proof of orthogonality following (1.8) requires that $w(x)$ satisfy some rather weak conditions: The moments of $w(x)$ must all exist and the determinant of the matrix B_n in (1.9) must not vanish. Note that it is sufficient that $xw(x)$ be integrable and positive almost everywhere.

To find the polynomial solutions to this integral equation, we simply substitute a polynomial of degree n,

$$P_n(x) = \sum_{k=0}^{n} a_{n,k} x^k, \tag{1.4}$$

in which the coefficients $a_{n,k}$ are arbitrary, into the integral equation (1.1). We then solve the resulting algebraic equations for the coefficients. Note that the polynomial of degree $n = 0$, $P_0(x) = 1$, already solves the integral equation by virtue of the constraint (1.2).

To demonstrate the procedure we introduce the following notation:

$$\langle f \rangle \equiv \int_{\alpha}^{\beta} dy\, w(y)\, f(y). \tag{1.5}$$

Let us now see how to solve for the orthogonal polynomials for some low-degree cases.
Example: Substitute $P_1(x)$, where $P_1(x) = a_{1,0} + a_{1,1}x$, into the integral equation (1.1). We obtain the equation

$$a_{1,0} + a_{1,1}x = \langle P_1(y)\, [a_{1,0} + a_{1,1}y + a_{1,1}x] \rangle.$$

Then, matching coefficients of powers of x gives a system of two linear equations for the coefficients P_1:

$$\begin{aligned} \langle P_1(x) \rangle &= 1, \\ \langle x P_1(x) \rangle &= 0. \end{aligned} \tag{1.6}$$

Example: Substitute $P_2(x)$, where $P_2(x) = a_{2,0} + a_{2,1}x + a_{2,2}x^2$, into the integral equation (1.1). The result is

$$a_{2,0} + a_{2,1}x + a_{2,2}x^2 = \langle P_2(y) \left[a_{2,0} + a_{2,1}(x+y) + a_{2,2}(x^2 + 2xy + y^2) \right] \rangle.$$

Matching coefficients of powers of x gives a system of three linear equations for the coefficients of P_2:

$$\begin{aligned} \langle P_2(x) \rangle &= 1, \\ \langle x P_2(x) \rangle &= 0, \\ \langle x^2 P_2(x) \rangle &= 0. \end{aligned} \tag{1.7}$$

The general pattern for the case of nth degree polynomials is now obvious. If we substitute $P_n(x)$ in (1.4) into the integral equation (1.1) and match coefficients of powers of x, we obtain the following system of $n+1$ linear equations:

$$\langle x^k P_n(x) \rangle = \delta_{k,0} \quad (k = 0, 1, \ldots, n). \tag{1.8}$$

Equations (1.6) and (1.7) are special cases of this equation for $n = 1$ and $n = 2$.

Cramer's rule tells us that the linear inhomogeneous algebraic system (1.8) has a unique solution. Thus, while the nonlinear integral equation (1.1) admits an infinite set of polynomial solutions, the set of polynomials is unique. Thus, for each $w(x)$ there is only one polynomial solution of degree n.

The system of algebraic equations (1.8) can be used to construct a quick proof of orthogonality:

$$\begin{aligned}\langle xP_nP_m\rangle &= \sum_{k=0}^{m} a_{m,k}\langle x^{k+1}P_n(x)\rangle\\ &= \sum_{k=1}^{m+1} a_{m,k-1}\langle x^k P_n(x)\rangle\\ &= 0,\end{aligned}$$

where we have used (1.8) explicitly for $k=0,1,2,\ldots,n-1$.

Let us now display the explicit form of the polynomial solutions $P_n(x)$. We define the moment notation $m_n \equiv \langle x^n\rangle$. Then, for all n we have

$$P_n(x) = \frac{\det A_n}{\det B_n}, \tag{1.9}$$

where the matrices A_n and B_n are given by

$$A_n = \begin{pmatrix} 1 & x & \cdots & x^n \\ m_1 & m_2 & \cdots & m_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ m_n & m_{m+1} & \cdots & m_{2n} \end{pmatrix} \tag{1.10}$$

and

$$B_n = \begin{pmatrix} m_0 & m_1 & \cdots & m_n \\ m_1 & m_2 & \cdots & m_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ m_n & m_{m+1} & \cdots & m_{2n} \end{pmatrix}. \tag{1.11}$$

The first three polynomials are given explicitly as

$$\begin{aligned}P_0(x) &= 1,\\ P_1(x) &= \frac{m_2 - xm_1}{m_2 - m_1^2},\\ P_2(x) &= \frac{(m_2m_4 - m_3^2) + (m_2m_3 - m_1m_4)x + (m_1m_3 - m_2^2)x^2}{m_4(m_2 - m_1^2) - m_3^2 + 2m_1m_2m_3 - m_2^3}.\end{aligned}$$

Note that the polynomial formula in (1.9) is precisely what one obtains if one is given the measure $g(x)$ and one then uses the Gram-Schmidt orthogonalization procedure. This shows that the nonlinear integral equation (1.1) is equivalent to the Gram-Schmidt procedure. The noteworthy feature of the nonlinear integral equation is that it is independent of the degree n of the polynomial; *all* polynomials, regardless of their degree, satisfy the integral equation. Technically speaking, the Gram-Schmidt procedure is an iterative technique in which one calculates the polynomials in order of increasing degree P_0, P_1, P_2, and so on, where each new polynomial is orthogonal to all previously found polynomials. When one uses the nonlinear integral equation (1.1), one simply specifies the degree of the required polynomial and then determines it directly without knowing the other polynomials. However, since the final formula for the polynomials is unique, one must regard the integral equation as being equivalent to the Gram-Schmidt procedure.

Example: To illustrate the construction of a set of orthogonal polynomials, let us choose $\alpha = 0$ and $\beta = 1$ and take the function $w(x)$ to have the general form $x^a + x^b$. Then the condition in (1.2) implies that $ab = 1$. For these choices we obtain a complicated-looking one-parameter family of orthogonal polynomials whose coefficients are palindromic:

$$\begin{aligned} P_0(x) &= 1, \\ P_1(x) &= \frac{(a+2)(2a+1)}{a^6+10a^5+20a^4+10a^3+20a^2+10a+1}[2a^4+17a^3 \\ &\quad +34a^2+17a+2-x(3a^4+22a^3+46a^2+22a+3)]. \end{aligned}$$

An interesting question to ask is, What happens if $w(x) = g(x)/x$ is singular at the origin $x = 0$? This at first appears to be a serious problem, but in fact the solution is elementary. We simply choose the path of integration in the integral equation (1.1) to avoid the origin!

Example: Consider the case of Legendre polynomials. For these polynomials $\alpha = -1$, $\beta = 1$, and $g(x)$ is a constant. There are infinitely many topologically distinct integration paths connecting -1 to 1, where the paths are characterized by their winding numbers. For definiteness, choose a path that runs from -1 to 1 in the positive (counterclockwise) direction and does not encircle the origin. On this path $\int dx/x = i\pi$. Thus, to maintain the normalization condition $\int_\alpha^\beta dx\, w(x) = 1$ we use $w(x) = 1/(i\pi x)$. The moments $m_n = \langle x^n \rangle$ are $m_0 = 1$, $m_1 = 2/(i\pi)$, $m_2 = 0$, $m_3 = 2/(3i\pi)$. The first four polynomials are

$$\begin{aligned} P_0(x) &= 1, \\ P_1(x) &= \frac{i\pi}{2}x, \\ P_2(x) &= 1-3x^2, \\ P_3(x) &= \frac{3i\pi}{8}(3x-5x^3). \end{aligned}$$

These are precisely the Legendre polynomials with an unusual overall multiplicative normalization factor — every other polynomial contains a multiplicative factor of i. Note that these polynomials are symmetric under the combined reflections

$$\begin{aligned} x &\to -x, \\ i &\to -i. \end{aligned} \tag{1.12}$$

This symmetry has been heavily studied by mathematical physicists and is known as $\mathscr{P}\mathscr{T}$ symmetry [6, 7]. Of course, the notion of polynomials on complex contours is not new and has been examined in the past by many mathematicians; see, for example, Refs. [8, 9].

2 Other nonlinear integral equations having polynomial solutions

There are many other kinds of nonlinear integral equations having polynomial solutions. For example, one can have a *multiplicative* rather than an additive argument; that is, one can replace

$P(x+y)$ in the nonlinear integral equation (1.1) by $P(xy)$:

$$P(x) = \int_\alpha^\beta dy\, w(y)\, P(y)\, P(xy). \tag{2.1}$$

If we now substitute $P_n(x) = \sum_{k=0}^n a_{n,k}\, x^k$, we get

$$a_{n,k}\langle x^k P_n(x)\rangle = a_{n,k} \quad (k = 0, 1, \ldots, n).$$

Unlike the previous case, these equations are *quadratic* and there are now 2^{n-1} solutions because each coefficient $a_{n,k}$ can be either zero or nonzero for $k = 0, \ldots, n-1$.

We find that for one special class of solutions all of the coefficients are nonzero and the polynomials are orthogonal with respect to the measure

$$g(x) = (1-x)\, w(x).$$

Here are some additional examples of nonlinear integral equations that have a particularly rich and interesting structure:

Example: Replace $P(x+y)$ in the nonlinear integral equation (1.1) by $P(x+a+by)$:

$$P(x) = \int_\alpha^\beta dy\, w(y)\, P(y)\, P(x+a+by).$$

Example: Replace $P(x+y)$ in the nonlinear integral equation (1.1) by $P[x+f(y)]$, where f is an arbitrary function:

$$P(x) = \int_\alpha^\beta dy\, w(y)\, P(y)\, P[x+f(y)].$$

Example: Replace $P(y)$ in (1.1) by $f[P(y)]$, where f is an arbitrary function:

$$P(x) = \int_\alpha^\beta dy\, w(y)\, f[P(y)]\, P(x+y).$$

3 Application: asymptotic behavior of polynomials

Because the nonlinear integral equation (1.1) does not contain the degree of the polynomial solutions explicitly, it is easy to use this integral equation to study the asymptotic behavior of classes of polynomials. We illustrate this by deriving asymptotic properties of some classical polynomials.

Example: The Laguerre polynomials $L_n^\gamma(x)$ are orthogonal on interval $(0,\infty)$ with respect to measure

$$g(x) = x^\gamma e^{-x}/\Gamma(\gamma). \tag{3.1}$$

Thus, from (1.1) we know that the nth Laguerre polynomial satisfies the following nonlinear integral equation:

$$L_n^\gamma(x) = \frac{1}{\Gamma(\gamma)} \int_0^\infty dy\, y^{\gamma-1} e^{-y} L_n^\gamma(y) L_n^\gamma(x+y). \tag{3.2}$$

It is well known that the asymptotic behavior of the Laguerre polynomials for large degree is given by [10]

$$\lim_{n\to\infty} n^{-\gamma} L_n(x/n) = x^{-\gamma/2} J_\gamma(2\sqrt{x}). \tag{3.3}$$

It is easy to verify this asymptotic behavior directly by using the integral equation (3.2). We simply scale all the arguments in (3.2) by n and multiply both sides by $n^{-\gamma}$:

$$\frac{1}{n^\gamma} L_n^\gamma\left(\frac{x}{n}\right) = \frac{1}{\Gamma(\gamma)} \int_0^\infty dy\, y^{\gamma-1} e^{-y/n} \frac{1}{n^\gamma} L_n^\gamma\left(\frac{y}{n}\right) \frac{1}{n^\gamma} L_n^\gamma\left(\frac{x+y}{n}\right).$$

Next, we take the limit $n \to \infty$ and substitute the limiting behavior of the Laguerre polynomial given in (3.3). We obtain the following integral identity involving Bessel functions:

$$x^{-\gamma/2} J_\gamma(2\sqrt{x}) \quad = \quad \frac{1}{\Gamma(\gamma)} \int_0^\infty dy\, y^{\gamma/2-1} J_\gamma(2\sqrt{y})(x+y)^{-\gamma/2} J_\gamma(2\sqrt{x+y}).$$

This identity transforms to a standard identity that can be found in Gradsteyn and Ryzhik [10]:

$$2^\gamma \frac{J_\gamma(z)}{2z^\gamma} = \frac{1}{\Gamma(\gamma)} \int_0^\infty \frac{dw}{w} w^\gamma J_\gamma(w) \frac{J_\gamma(\sqrt{w^2+z^2})}{\left(\sqrt{w^2+z^2}\right)^\gamma}. \tag{3.4}$$

Here is a second example involving Jacobi polynomials:

Example: The Jacobi polynomials $P_n^{\alpha,\beta}(x)$ are orthogonal on the interval $(-1,1)$ with respect to measure

$$g(x) = \frac{(1-x)^\alpha (1+x)^\beta}{2^{\alpha+\beta} B(\alpha, \beta+1)}. \tag{3.5}$$

The nth Jacobi polynomial satisfies the *multiplicative* integral equation (2.1):

$$P_n^{\alpha,\beta}(x) = \frac{2^{-\alpha-\beta}}{B(\alpha, \beta+n+1)} \int_{-1}^1 dy\, (1-y)^{\alpha-1} (1+y)^\beta P_n^{\alpha,\beta}(y) P_n^{\alpha,\beta}(xy). \tag{3.6}$$

It is known that the asymptotic behavior of the Jacobi polynomials for large β is given by [10]

$$\lim_{\beta\to\infty} P_n^{\alpha,\beta}\left(1 - \frac{2x}{\beta}\right) = L_n^\alpha(x). \tag{3.7}$$

Let us perform the asymptotic limit of the integral equation (3.6). To do so, we make the change of variables $x \to 1 - 2x/\beta$ and $y \to 1 - 2y/\beta$:

$$\begin{aligned} P_n^{\alpha,\beta}\left(1 - \frac{2x}{\beta}\right) \quad &= \quad \beta^{-\alpha} \frac{\Gamma(\alpha+\beta+n+1)}{\Gamma(\alpha)\Gamma(\beta+n+1)} \\ &\times \int_{-\beta/2}^{\beta/2} dy\, y^{\alpha-1} \left(1 - \frac{y}{\beta}\right)^\beta P_n^{\alpha,\beta}\left(1 - \frac{2x}{\beta}\right) P_n^{\alpha,\beta}\left[\left(1 - \frac{2x}{\beta}\right)\left(1 - \frac{2y}{\beta}\right)\right]. \end{aligned}$$

Next, we take the limit $\beta \to \infty$:

$$L_n^\alpha(x) = \frac{1}{\Gamma(\alpha)} \int_0^\infty dy\, y^{\alpha-1} e^{-y} L_n^\alpha(y) L_n^\alpha(x+y). \tag{3.8}$$

Observe that this is precisely the nonlinear integral equation (3.2) satisfied by the Laguerre polynomials!

These examples show that the nonlinear integral equation (1.1) is extremely useful in elucidating the properties of known orthogonal polynomials. We hope that it can also be used to discover new and interesting classes of orthogonal polynomials.

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