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Lift of Invariant to Non-Invariant Solutions of Complex Monge-Ampère Equations

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Abstract

We show how partner symmetries of the elliptic and hyperbolic complex Monge-Ampère equations (*CMA* and *HCMA*) provide a lift of non-invariant solutions of three- and two-dimensional reduced equations, i.e., a lift of invariant solutions of the original *CMA* and *HCMA* equations, to non-invariant solutions of the latter four-dimensional equations. The lift is applied to non-invariant solutions of the two-dimensional Helmholtz equation to yield non-invariant solutions of *CMA*, and to non-invariant solutions of three-dimensional wave equation and three-dimensional hyperbolic Boyer-Finley equation to yield non-invariant solutions of *HCMA*. By using these solutions as metric potentials, it may be possible to construct four-dimensional Ricci-flat metrics of Euclidean and ultra-hyperbolic signatures that have non-zero curvature tensors and no Killing vectors.

1 Introduction

Solutions of Plebañski's first and second heavenly equations yield a potential that determines Ricci-flat (anti-)self-dual metrics on 4-dimensional complex manifolds [13]. In other words, these "heavenly" metrics satisfy complex vacuum Einstein equations. In the case of the first heavenly equation, physically important ones are two real cross sections of these complex metrics, Kähler metrics with Euclidean or ultra-hyperbolic signature, when the first heavenly equation coincides with the elliptic (*CMA*) and hyperbolic (*HCMA*) complex Monge-Ampère equation respectively. In particular, among the solutions $u(z_1,\bar{z_1},z_2,\bar{z_2})$ of the elliptic *CMA*

$$u_{1\bar{1}}u_{2\bar{2}}-u_{1\bar{2}}u_{2\bar{1}}=1 \tag{1.1}$$

there are gravitational instantons, the most important of which is $K3$, the Kummer surface [1]. The explicit construction of the $K3$ metric is still an unsolved challenging problem. The main difficulty is that this metric should have no Killing vectors and therefore the corresponding solution of *CMA* should be a non-invariant solution (with no point symmetries). That was a basic motive

for us to develop methods for finding non-invariant solutions of nonlinear partial differential equations (PDEs). In the context of Lie's theory of symmetries of differential equations, the standard method for solving nonlinear PDEs is symmetry reduction that yields only invariant solutions and therefore cannot produce Kähler potential of $K3$. Recently, we have developed method of partner symmetries that yields non-invariant solutions to the elliptic and hyperbolic *CMA* and to the second heavenly equation. Using them as metric potentials, we have obtained some heavenly metrics with no Killing vectors [6, 7, 8] (nonexistence of conformal Killing vectors was not proved).

Here we develop further our method of partner symmetries so, that we are able now to obtain non-invariant solutions of the four-dimensional *CMA* and *HCMA* starting from invariant solutions of these equations that satisfy the corresponding reduced equations of lower (three and two) dimensions and these "seed" solutions should be non-invariant solutions of the reduced equations. We have called this procedure *"lift"*. In particular, we have obtained non-invariant solutions of elliptic *CMA* (1.1) by the lift from solutions of two-dimensional Helmholtz equation and non-invariant solutions of hyperbolic *HCMA*

$$u_{1\bar{1}}u_{2\bar{2}} - u_{1\bar{2}}u_{2\bar{1}} = -1 \tag{1.2}$$

by the lift from solutions of three-dimensional wave equation and of the three-dimensional hyperbolic Boyer-Finley equation [2]:

$$\psi_{z\bar{z}} = \mathrm{e}^{\psi_x}\psi_{xx}. \tag{1.3}$$

In section 2 we discuss partner symmetries of the elliptic and hyperbolic complex Monge-Ampère equations.

In section 3 we obtain non-invariant solutions of the elliptic *CMA* by the lift from two-dimensional Helmholtz equation.

In section 4 we make the lift of solutions of three-dimensional wave equation to non-invariant solutions of the hyperbolic *HCMA*.

In section 5 we use Legendre transformation of *HCMA* and equations for rotational partner symmetries to obtain hyperbolic Boyer-Finley equation (*BF*) and Bäcklund transformations for *BF* that we discovered earlier [5].

In section 6, using results of the previous section, we obtain non-invariant solutions of *HCMA* by the lift from our non-invariant solutions to hyperbolic *BF* (1.3) that we obtained earlier [10]. Noninvariant solutions to the elliptic *BF* were obtained first by D. Calderbank and P. Tod [3]. A little later, we had independently obtained these solutions to the elliptic *BF* and also non-invariant solutions to the hyperbolic *BF* [10] by our version of the method of group foliation [12]. We had also proved non-invariance of all these solutions.

By using non-invariant solutions of *HCMA* as metric potentials, it may be possible to construct metrics with ultra-hyperbolic signature that have no Killing vectors [9]. A comprehensive survey of results on four-dimensional anti-self-dual metrics with the ultra-hyperbolic (neutral) signature was given by M. Dunajski and S. West in [4].

We are working now on a modification of this method in order to obtain non-invariant solutions of the elliptic *CMA* (1.1) by the lift from solutions of the three-dimensional elliptic *BF* [2]

$$\psi_{z\bar{z}} + \mathrm{e}^{\psi_x}\psi_{xx} = 0 \tag{1.4}$$

to non-invariant solutions of the elliptic *CMA* (1.1) with the final goal to obtain Ricci-flat heavenly metrics with Euclidean signature that admit no Killing vectors.

2 Partner symmetries of complex Monge-Ampère equations

The hyperbolic and elliptic *CMA* have the same set of symmetries whose characteristics φ satisfy the condition

$$u_{2\bar{2}}\varphi_{1\bar{1}} + u_{1\bar{1}}\varphi_{2\bar{2}} - u_{2\bar{1}}\varphi_{1\bar{2}} - u_{1\bar{2}}\varphi_{2\bar{1}} = 0. \tag{2.1}$$

Define the operators

$$L_1 = \lambda(u_{1\bar{2}}D_{\bar{1}} - u_{1\bar{1}}D_{\bar{2}}), \quad L_2 = \lambda(u_{2\bar{2}}D_{\bar{1}} - u_{2\bar{1}}D_{\bar{2}}), \tag{2.2}$$

where $D_i, D_{\bar{i}}$ are operators of total derivatives with respect to $z_i, \bar{z_i}$ and λ is a complex constant. Then the symmetry condition (2.1) can be expressed as a total divergence

$$D_1 L_2 \varphi = D_2 L_1 \varphi \tag{2.3}$$

so that locally there exists a symmetry potential ψ defined by the differential equations

$$\psi_1 = L_1\varphi, \quad \psi_2 = L_2\varphi. \tag{2.4}$$

It is easy to see that if φ satisfies (2.3), then ψ also satisfies (2.3) and so the potential ψ of a symmetry φ is itself a symmetry of *CMA* [6, 8]. These φ and ψ are called *partner symmetries*.

Differential equations (2.4) are recursion relations for symmetries

$$\psi = R_1\varphi, \quad \psi = R_2\varphi \tag{2.5}$$

with the recursion operators

$$R_1 = D_1^{-1}L_1, \quad R_2 = D_2^{-1}L_2. \tag{2.6}$$

The transformation inverse to (2.4) is obtained by taking complex conjugates of equations (2.4), solving them algebraically with respect to φ_1 and φ_2, and using *CMA*:

$$\varphi_1 = \mp\bar{\lambda}^{-1}(u_{1\bar{2}}\psi_{\bar{1}} - u_{1\bar{1}}\psi_{\bar{2}}), \quad \varphi_2 = \mp\bar{\lambda}^{-1}(u_{2\bar{2}}\psi_{\bar{1}} - u_{2\bar{1}}\psi_{\bar{2}}) \tag{2.7}$$

where the minus and plus signs correspond to the elliptic and hyperbolic *CMA* respectively. Note that if $|\lambda| = 1$ the inverse transformation (2.7) reads

$$\varphi = \mp R_1\psi, \quad \varphi = \mp R_2\psi, \tag{2.8}$$

and then for *HCMA* there is a simplifying choice $\psi = \varphi$, when the transformation (2.4) coincides with its inverse (2.7) and becomes

$$\varphi_1 = \lambda(u_{1\bar{2}}\varphi_{\bar{1}} - u_{1\bar{1}}\varphi_{\bar{2}}), \quad \varphi_2 = \lambda(u_{2\bar{2}}\varphi_{\bar{1}} - u_{2\bar{1}}\varphi_{\bar{2}}). \tag{2.9}$$

The two operators in (2.9) form the Lax pair of Mason and Newman [11] and hence all integrability conditions of system (2.9) and the complex conjugate equations

$$\varphi_{\bar{1}} = \lambda^{-1}(u_{2\bar{1}}\varphi_1 - u_{1\bar{1}}\varphi_2), \quad \varphi_{\bar{2}} = \lambda^{-1}(u_{2\bar{2}}\varphi_1 - u_{1\bar{2}}\varphi_2) \tag{2.10}$$

are identically satisfied as a consequence of these equations and *HCMA*. The same is true for equations (2.7) together with their complex conjugates for elliptic *CMA* since they can be written in the form (2.9), (2.10) for the complex potential $\Phi = \varphi + i\psi$.

If we choose for φ a characteristic of any Lie point symmetry of *HCMA*, then (2.9) and (2.10) become differential constraints, joined to *HCMA*, that select some particular solutions of *HCMA*. Equations (2.9) and (2.10) are not independent: any three equations out of the four ones imply the fourth equation together with *HCMA* itself as their algebraic consequences. Alternatively, any two of these four equations together with *HCMA* imply the remaining two constraints. In sections 4 and 5 we shall use *HCMA* together with the first equations in (2.9) and (2.10)

$$\varphi_1 = \lambda(u_{1\bar{2}}\varphi_{\bar{1}} - u_{1\bar{1}}\varphi_{\bar{2}}), \quad \varphi_{\bar{1}} = \lambda^{-1}(u_{2\bar{1}}\varphi_1 - u_{1\bar{1}}\varphi_2) \tag{2.11}$$

as basic independent equations: the original PDE and two constraints for one unknown u.

3 Lift of non-invariant solutions of elliptic CMA from two-dimensional Helmholtz equation

For the elliptic *CMA* the ansatz $\psi = \varphi$ implies the trivial solution $\varphi = \psi = 0$, so in this section we use the more general equations (2.7) with $\varphi \neq \psi$ and $|\lambda| = 1$.

Introducing real variables x, y, z, t by the relations $z_1 = (x+iy)/2$, $z_2 = (t+iz)/2$, we consider the elliptic *CMA* (1.1) in a real form

$$(u_{xx}+u_{yy})(u_{zz}+u_{tt}) - (u_{xt}+u_{yz})^2 - (u_{yt}-u_{xz})^2 = 1. \tag{3.1}$$

First, consider solutions that are invariant under translations in x, selected by the condition $u_1 + u_{\bar{1}} \equiv 2u_x = 0$, for which (3.1) reduces to

$$u_{yy}(u_{zz}+u_{tt}) - u_{yz}^2 - u_{yt}^2 = 1 \iff u_{yy}u_{2\bar{2}} - u_{y2}u_{y\bar{2}} = 1. \tag{3.2}$$

Applying to this the Legendre transformation $v = u - yu_y$, $p = u_y$, we obtain the three-dimensional Laplace equation

$$v_{2\bar{2}} + v_{pp} = 0. \tag{3.3}$$

We consider now solutions of (3.3) that are invariant under the symmetry generator $X = v\partial_v + \partial_p$ due to the condition $v_p - v = 0$ and thus imply further reduction of (3.3). Then $v = e^p\theta(z_2, \bar{z}_2)$, where θ satisfies the Helmholtz equation

$$\theta_{2\bar{2}} + \theta = 0. \tag{3.4}$$

In order to make a lift from solutions of these low-dimensional reduced linear equations to non-invariant solutions of four-dimensional elliptic *CMA*, we need to arrive at these equations without symmetry reduction. For this purpose we choose the symmetries of translations and dilations

$$\varphi = u_1 + u_{\bar{1}}, \qquad \psi = u - z_1u_1 - \bar{z}_1u_{\bar{1}}, \tag{3.5}$$

respectively, for characteristics of partner symmetries in the equations (2.7) (with the minus sign) and their complex conjugates.

After the Legendre transformation

$$v = u - z_1 u_1 - \bar{z}_1 u_{\bar{1}}, \quad p = u_1, \quad \bar{p} = u_{\bar{1}} \tag{3.6}$$

in the new variables $p, \bar{p}, v$ formulas (3.5) become $\varphi = p + \bar{p}, \quad \psi = v$ and elliptic *CMA* (1.1) takes the form

$$v_{p\bar{p}} v_{2\bar{2}} - v_{p\bar{2}} v_{\bar{p}2} = v_{p\bar{p}} v_{2\bar{2}} - v_{p\bar{p}}^2 \tag{3.7}$$

where $v = v(p, \bar{p}, z_2, \bar{z}_2)$. The Legendre transformation (3.6) maps the equations (2.7) with the choice of symmetries (3.5), their complex conjugates and the transformed *CMA* (3.7) to the following system of five independent equations (for more details see [6])

$$v_{pp} = A v_{p\bar{p}}, \quad v_{p\bar{2}} = C v_{p\bar{p}}, \quad v_{2\bar{2}} = B v_{p\bar{p}} \tag{3.8}$$

together with their complex conjugates. Here the coefficients are defined as

$$A = \frac{1 + v_p^2 + i v_2}{\Delta}, \quad B = \frac{v_2 v_{\bar{2}} + i(v_2 - v_{\bar{2}})}{\Delta}, \quad C = \frac{v_p v_{\bar{2}} + i(v_p - v_{\bar{p}})}{\Delta}$$

where $\Delta = 1 + v_p v_{\bar{p}}$. Equations (3.8) imply one more equation [6]

$$v_{p\bar{p}} = 1 + v_p v_{\bar{p}} \tag{3.9}$$

as their differential consequence. We note once again that the transformed *CMA* (3.7) is satisfied automatically on solutions of (3.8) and (3.9).

The logarithmic substitution $v = -\ln w$ linearizes equations (3.8) and (3.9) in the form

$$\begin{aligned} &w_{p\bar{p}} + w = 0, && w_{pp} + w - i w_2 = 0, \\ &w_{p\bar{2}} - i(w_p - w_{\bar{p}}) = 0, && w_{2\bar{2}} - i(w_2 - w_{\bar{2}}) = 0 \end{aligned} \tag{3.10}$$

plus two complex conjugate equations. System (3.10) implies

$$w_{2\bar{2}} - (w_{pp} + w_{\overline{p\bar{p}}} - 2 w_{p\bar{p}}) = 0. \tag{3.11}$$

If $p = \alpha + i\beta$, $\bar{p} = \alpha - i\beta$, then in real variables α and β (3.11) becomes the three-dimensional Laplace equation

$$w_{2\bar{2}} + w_{\beta\beta} = 0 \tag{3.12}$$

which coincides with (3.3) up to a change in variables, while the first equation (3.10) coincides with the two-dimensional Helmholtz equation (3.4) in the new variables.

Thus, if we know a solution of one of these low-dimensional equations, depending on arbitrary constants, we consider them as arbitrary functions of all other variables that do not appear explicitly in this equation. Then all the other equations (3.10) determine a dependence of these functions on these parameters and so we obtain a non-invariant solution of the four-dimensional equation (3.7). We call this a lift of invariant solutions of *CMA*, satisfying the reduced equation (3.3) or (3.4), to noninvariant solutions of *CMA*.

For example, let us consider a lift from Helmholtz equation (3.4), that coincides in new variables with the first equation (3.10), starting from its solution of the form

$$w=\sum_j A_j(z_2,\bar{z}_2)\left\{e^{2s_j\mathrm{Re}(\alpha_j p)}\mathrm{Re}\left(F_j e^{2i\mathrm{Im}(\alpha_j p)}\right)+e^{-2s_j\mathrm{Re}(\alpha_j p)}\mathrm{Re}\left(G_j e^{2i\mathrm{Im}(\alpha_j p)}\right)\right\}. \tag{3.13}$$

Then other equations (3.10) imply the following restrictions on solution (3.13)

$$A_j=\exp\left\{2\,\mathrm{Im}\Big(\big(\alpha_j^2(s_j^2+1)+1\big)z_2\Big)\right\},\qquad s_j=\sqrt{1-1/|\alpha_j|^2},$$

while α_j, F_j, G_j are arbitrary complex constants. This is a non-invariant solution of the Legendre-transformed *CMA* [6].

4 Lift of non-invariant solutions of hyperbolic CMA from three-dimensional wave equation

For the translational symmetry reduction we will need the real form of *HCMA* obtained by the change of variables $z_1=(x+iy)/2$, $\bar{z}_1=(x-iy)/2$, $z_2=(t+iz)/2$, $z_2=(t-iz)/2$:

$$(u_{xx}+u_{yy})(u_{zz}+u_{tt})-(u_{xt}+u_{yz})^2-(u_{yt}-u_{xz})^2=-1. \tag{4.1}$$

We consider solutions of (4.1), invariant under translations in x, that are selected by the condition $u_x=0$. Then (4.1) reduces to

$$u_{yy}(u_{zz}+u_{tt})-u_{yz}^2-u_{yt}^2=-1. \tag{4.2}$$

Applying to (4.2) the Legendre transformation

$$v=u-yu_y,\quad q=u_y \tag{4.3}$$

we end up with the three-dimensional wave equation

$$v_{qq}=v_{tt}+v_{zz} \tag{4.4}$$

for the new unknown $v=v(q,t,z)$.

Now we will not perform any symmetry reduction but use equations (2.9) and (2.10) for partner symmetries with φ equal to the characteristic of the symmetry of translations in x: $\varphi=u_1+u_{\bar{1}}$, and $\lambda=i$. With these choices, after the Legendre transformation similar to (4.3)

$$v=u-z_1u_{z1}-\bar{z}_1u_{\bar{z}_1},\quad p=u_x,\quad q=u_y \tag{4.5}$$

the real form (4.1) of *HCMA* becomes

$$(v_{pp}+v_{qq})(v_{tt}+v_{zz})-(v_{pt}-v_{qz})^2-(v_{pz}+v_{qt})^2=v_{pp}v_{qq}-v_{pq}^2, \tag{4.6}$$

while (2.9) and (2.10) yield only three independent equations [8]

$$v_{qq}=v_{pz}+v_{qt}, \tag{4.7a}$$

$$v_{pq} = v_{qz} - v_{pt}, \tag{4.7b}$$

$$v_{qq} = v_{tt} + v_{zz}. \tag{4.7c}$$

Equation (4.7c) formally coincides with the three-dimensional wave equation (4.4) that determines solutions of (4.6), invariant under translations in x. However, the unknown v in this equation depends also on the fourth variable p, so that (4.7c) depends on an extra parameter p and therefore it is actually not the reduced equation (4.4). Note that the transformed non-reduced *HCMA* (4.6) is a consequence of the system (4.7a–4.7c) and so the latter equations determine some partial solutions of (4.6).

The solution set of (4.7c) can be written as the double Fourier integral

$$\begin{aligned} v = \int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} \Big(& a(p,\alpha,\beta)\exp\left\{-i\left(\alpha t+\beta z+\sqrt{\alpha^2+\beta^2}\,q\right)\right\} \\ & + b(p,\alpha,\beta)\exp\left\{-i\left(\alpha t+\beta z-\sqrt{\alpha^2+\beta^2}\,q\right)\right\}\Big)\, d\alpha\, d\beta. \end{aligned} \tag{4.8}$$

Imposing the remaining equations (4.7a) and (4.7b) on solution (4.8), we finally obtain the general solution of the system (4.7a)–(4.7c)

$$\begin{aligned} v = \int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} \Bigg(& a(\alpha,\beta)\exp\left\{-i\sqrt{\alpha^2+\beta^2}\left(\frac{\sqrt{\alpha^2+\beta^2+\alpha}}{\beta}p+q\right)\right\} \\ & + b(\alpha,\beta)\exp\left\{-i\sqrt{\alpha^2+\beta^2}\left(\frac{\sqrt{\alpha^2+\beta^2-\alpha}}{\beta}p-q\right)\right\}\Bigg) e^{-i(\alpha t+\beta z)}\, d\alpha\, d\beta. \end{aligned} \tag{4.9}$$

which is a partial solution of the Legendre-transformed *HCMA* (4.6).

Thus, using equations (4.7a)–(4.7c), implied by our choice of partner symmetries, we have made a lift of solutions of the reduced equation (4.4) to a set of particular solutions of four-dimensional Legendre-transformed *HCMA* (4.6).

5 Rotational partner symmetries, Legendre transformation and Boyer-Finley equation

Boyer-Finley equation usually arises from rotational symmetry reduction of *HCMA*, subjected to the combination of the point and Legendre transformation in the first pair of variables $z_1, \bar{z}_1$

$$z_1 = e^{\zeta_1},\ \bar{z}_1 = e^{\bar{\zeta}_1},\ \zeta_1 = \psi_q,\ \bar{\zeta}_1 = \psi_{\bar{q}},\ u = q\psi_q + \bar{q}\psi_{\bar{q}} - \psi,\ u_{\zeta_1} = q,\ u_{\bar{\zeta}_1} = \bar{q}. \tag{5.1}$$

Here we do not perform any symmetry reduction but still apply the same transformation (5.1) to *HCMA* and the two independent constraints (2.11), rename $z_2 = z$ and choose φ as the rotational symmetry characteristic $\varphi = i(z_1u_1 - \bar{z}_1u_1^-) = i(q-\bar{q})$. Then *HCMA* becomes

$$\psi_{q\bar{q}}\psi_{z\bar{z}} - \psi_{q\bar{z}}\psi_{\bar{q}z} + e^{\psi_q+\psi_{\bar{q}}}(\psi_{qq}\psi_{\bar{q}\bar{q}} - \psi^2_{q\bar{q}}) = 0, \tag{5.2}$$

where $\psi(q,\bar{q},z,\bar{z})$ is the new unknown, and the constraints (2.11) take the form

$$e^{\psi_{\bar{q}}}(\psi_{\bar{q}\bar{q}} + \psi_{q\bar{q}}) = \lambda\,\psi_{q\bar{z}}, \quad \lambda e^{\psi_q}(\psi_{qq} + \psi_{q\bar{q}}) = \psi_{\bar{q}z}. \tag{5.3}$$

Now, we express $\psi_{q\bar{z}}$ and $\psi_{\bar{q}z}$ from (5.3) and substitute them into *HCMA* (5.2) with the result

$$\psi_{z\bar{z}} = e^{\psi_q+\psi_{\bar{q}}}(\psi_{qq} + 2\psi_{q\bar{q}} + \psi_{\bar{q}\bar{q}}). \tag{5.4}$$

In the real coordinates x,y in the complex q-plane ($q = x + iy$, $\bar{q} = x - iy$), (5.4) becomes the (hyperbolic) Boyer-Finley equation

$$\psi_{z\bar{z}} = e^{\psi_x}\psi_{xx}. \tag{5.5}$$

The constraints (5.3) take the form

$$\psi_{zx} + i\psi_{zy} = 2\lambda\left[e^{(\psi_x - i\psi_y)/2}\right]_x, \quad \psi_{\bar{z}x} - i\psi_{\bar{z}y} = 2\lambda^{-1}\left[e^{(\psi_x + i\psi_y)/2}\right]_x. \tag{5.6}$$

The variable y does not appear explicitly in the Boyer-Finley equation (5.5), and so it can be regarded as a parameter of a symmetry group of this equation: a change of y does not affect the equation. Let ω be any symmetry characteristic of the Boyer-Finley equation in the form

$$\tilde{\psi}_{z\bar{z}} = \exp(\tilde{\psi}_{xx}), \tag{5.7}$$

related to (5.5) by the substitution $\psi = \tilde{\psi}_x$. Then a symmetry characteristic of (5.5) is $i\omega_x$ (where the factor i is introduced for convenience) and the Lie equation for the group with the parameter y reads

$$\psi_y = i\omega_x. \tag{5.8}$$

Eliminating ψ_y in the constraints (5.6) with the aid of (5.8) and then integrating the result with respect to x, we obtain

$$\omega_z = \psi_z - 2\lambda e^{(\psi_x+\omega_x)/2}, \quad \omega_{\bar{z}} = -\psi_{\bar{z}} + 2\lambda^{-1}e^{(\psi_x-\omega_x)/2}. \tag{5.9}$$

These are Bäcklund transformations for the Boyer-Finley equation that we discovered earlier [5]. The differential compatibility condition $(\omega_z)_{\bar{z}} = (\omega_{\bar{z}})_z$ of the system (5.9) reproduces the Boyer-Finley equation (5.5), while the compatibility condition in the form $(\psi_z)_{\bar{z}} = (\psi_{\bar{z}})_z$ yields the equation for symmetry characteristics of the Boyer-Finley equation (5.7):

$$\omega_{z\bar{z}} - e^{\psi_x}\omega_{xx} = 0. \tag{5.10}$$

Thus, without any symmetry reduction, the Boyer-Finley equation arises as a linear combination of the Legendre-transformed *HCMA* and differential constraints (5.3) implied by our choice of rotational symmetry for both partner symmetries. Furthermore, the differential constraints themselves turn out to be Bäcklund transformations for the Boyer-Finley equation in a new disguise.

6 Lift of solutions of Boyer-Finley equation to non-invariant solutions of *HCMA*

Now consider a reverse procedure. We start with the three-dimensional Boyer-Finley equation together with its Bäcklund transformations, differentiated with respect to x, and consider a symmetry group parameter as the fourth coordinate y in these equations, according to (5.8). Then we arrive

at 4–dimensional Legendre-transformed *HCMA* as a linear combination of these three equations. As a consequence, the partner symmetries lift three-dimensional non-invariant solutions of *BF*, that are still invariant solutions of *HCMA*, to four-dimensional non-invariant solutions of *HCMA*.

We start with non-invariant solutions to the hyperbolic *BF*

$$v_{z\bar{z}} = (e^v)_{xx} \tag{6.1}$$

that we had obtained earlier in [10] by our version of the method of group foliation. Those solutions involve a couple of arbitrary holomorphic and anti-holomorphic functions $b(z)$ and $\bar{b}(\bar{z})$ that arise as "constants" of integrations. In our construction, *BF* equation (5.5) and its solutions depend also on the fourth variable, the parameter y, and hence the "constants" of integration, b and $\bar{b}$, should also depend on y:

$$v(x,y,z,\bar{z}) = \ln[x+b(z,y)] + \ln[x+\bar{b}(\bar{z},y)] - 2\ln(z+\bar{z}). \tag{6.2}$$

BF equations (5.5) and (6.1) are related to each other by the substitution $v = \psi_x$ and hence solutions of (5.5), $\psi = \int v dx$, are obtained by integrating the formula (6.2) with respect to x with the "constant" of integration $F(z,\bar{z},y)$:

$$\begin{aligned}\psi &= [x+b(z,y)]\ln[x+b(z,y)] + [x+\bar{b}(\bar{z},y)]\ln[x+\bar{b}(\bar{z},y)] \\ &- 2x[\ln(z+\bar{z})+1] + F(z,\bar{z},y).\end{aligned} \tag{6.3}$$

For arbitrary functions b, $\bar{b}$, and F, this is a solution to *BF* (5.5). The unknown y-dependence in (6.3) is determined by the requirement that ψ should also satisfy the Legendre-transformed *HCMA* (5.2).

We substitute the expression (6.3) for ψ in *HCMA* (5.2) and, since all the x-dependence in (6.3) is known explicitly, (5.2) splits into several equations, corresponding to groups of terms with a different dependence on x. We were able to solve these equations and make a complete analysis of all possible solutions. They have the form

$$\begin{aligned}\psi &= [q+b(z)]\ln[q+b(z)] + [\bar{q}+\bar{b}(\bar{z})]\ln[\bar{q}+\bar{b}(\bar{z})] \\ &-(q+\bar{q})[\ln(z+\bar{z})+1] + \int\int \frac{b(z)+\bar{b}(\bar{z})}{(z+\bar{z})^2}\, dz d\bar{z} + r(y),\end{aligned} \tag{6.4a}$$

$$\begin{aligned}\psi &= [q+b(z)]\ln[q+b(z)] + [\bar{q}+\bar{b}(\bar{z})]\ln[\bar{q}+\bar{b}(\bar{z})] \\ &-(q+\bar{q})[\ln(z+\bar{z})+1] + \int\int \frac{b(z)+\bar{b}(\bar{z})}{(z+\bar{z})^2}\, dz d\bar{z} \\ &+ 2iy\ln\left(\frac{\bar{z}}{z}\right) + r(y),\end{aligned} \tag{6.4b}$$

$$\begin{aligned}\psi &= [q+b(z)]\ln[q+b(z)] + [\bar{q}+\bar{b}(\bar{z})]\ln[\bar{q}+\bar{b}(\bar{z})] \\ &-(q+\bar{q})[\ln(z+\bar{z})+1] + \int\int \frac{b(z)+\bar{b}(\bar{z})}{(z+\bar{z})^2}\, dz d\bar{z} \\ &+2i\int \ln\left[\frac{\bar{z}+2ik(y)}{z-2ik(y)}\right] dy + r(y).\end{aligned} \tag{6.4c}$$

Here $r(y)$ and $k(y)$ are arbitrary smooth real-valued functions, while $b(z)$ and $\overline{b}(\overline{z})$ are arbitrary holomorphic and anti-holomorphic functions of one complex variable that arise when the y-dependence of $b(z,y)$ and $\overline{b}(\overline{z},y)$ is completely determined. Solution (6.4b) is a particular simple case of the more general solution (6.4c) when $k(y)=0$.

Note that, by construction, we have obtained the solutions of *HCMA* that satisfy only one additional differential constraint, the Boyer-Finley equation, though we have two constraint equations produced by partner symmetries. If we require that both constraints should be satisfied, we obtain a subset of solutions that are invariant with respect to non-local symmetries of *HCMA*, though this does not mean invariant solutions in the usual sense [6, 8]. For solutions with such special property we have

$$r(y)=2(\alpha-\pi)y+r_0 \tag{6.5}$$

$$r(y)=2\alpha y+r_0 \tag{6.6}$$

in (6.4a) and $(6.4b, 6.4c)$ respectively. Here r_0 is an arbitrary real constant and $\lambda=e^{i\alpha}$.

It can be proved that if the functions $b(z)$, $\overline{b}(\overline{z})$ are not constants, the formulas (6.4a)–(6.4c) yield non-invariant solutions of (5.2). As a consequence, by the reasoning similar to [8], the ultra-hyperbolic metrics governed by the potentials ψ in (6.4a)–(6.4c) have no Killing vectors [9] (though nonexistence of conformal Killing vectors was not proved).

7 Conclusions

We are interested in obtaining non-invariant solutions of four-dimensional heavenly equations because they may yield gravitational metrics with no Killing vectors. This is a characteristic property of the famous gravitational instanton $K3$ where the metric potential should be a non-invariant solution of the elliptic complex Monge-Ampère equation. Constructing an explicit metric on $K3$ is our final goal. In this paper we have used a new approach for solving such a problem which we call "lift". We use partner symmetries for lifting invariant solutions of elliptic and hyperbolic *CMA*, that satisfy equations of lower dimensions, to non-invariant solutions of *CMA*.

A symmetry reduction of a partial differential equation reduces by one the number of independent variables in the original equation, so that the reduced equation is easier to solve. Its solutions are solutions of the original PDE that are invariant under the symmetry that was used in the reduction. Even if we found non-invariant solutions of the reduced equation, it would only mean that no further symmetry reduction was made and they would still be invariant solutions of the original equation.

For complex Monge-Ampère equations, we have shown that partner symmetries provide a procedure reverse to the symmetry reduction: a lift of invariant solutions of *CMA* to non-invariant solutions of *CMA*. This means holographic property of the symmetry used for the reduction, i.e. the information on solutions is not completely lost under the reduction but can be reconstructed for a certain class of non-invariant solutions. We have performed such a procedure for the elliptic and hyperbolic *CMA* and obtained non-invariant solutions of these equations. Using these solutions as metric potentials, it may be possible to obtain gravitational metrics of Euclidean and ultra-hyperbolic signatures that have no Killing vectors [9]. We are now developing a modified lifting procedure from non-invariant solutions of the elliptic Boyer-Finley equation to non-invariant solutions of the elliptic *CMA*.

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