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Lump Solutions for PDE's: Algorithmic Construction and Classification

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Abstract

In this paper we apply truncated Painlevé expansions to the Lax pair of a PDE to derive gauge-Bäcklund transformations of this equation. It allows us to construct an algorithmic method to derive solutions by starting from the simplest one. Actually, we use this method to obtain an infinite set of lump solutions that can be classified by means of two integer numbers N and M. Two different PDE's are used to check the method and compare the results.

1 Introduction

Real valued solutions with rationally decay or lumps have been extensively studied in recent years. For the KPI equation were found in [1] and later a study from the spectral point of view appeared in [2] (see also [3]). Lumps in DSII and three dimensional Sine-Gordon are described in [4] and [5]. In references [6], [7] and [8] nontrivial dynamics of lumps of KPI are studied. These solutions exhibit interesting scattering properties that were first noticed in [9]. The extension of these ideas and solutions to DSII equation via spectral analysis of the Dirac operator on the plane is considered in [10], while a complete study based on direct methods is performed in [11]; see also [12] for related ideas. For some interacting solutions in the Yang-Mills equation framework, see [13].

Many of the above cited papers include an expansion of the eigenfuctions in terms of poles. This fact strongly suggests a connection with Painlevé methods. This is actually the point that we would like to explore in this paper. We shall try to prove that the truncation of the Painlevé series gives us an algorithmic procedure to obtain solution. This method can be iterated. In the case of lumps two iterations can be applied in such a way that the second provides us a wave number that is the complex conjugate of the first and therefore the solution can be real. We shall apply this method to two different PDE: KPI [14] and the complex version of the Generalized dispersive long wave equation derived in [15] that we shall name GDLW [18]. As we shall see the method works exactly in the same way for both equations.

We briefly review the contents of this paper:

- Section 2 is devoted to the description of the truncation of the Painlevé series and, the subsequent iteration method, for KPI

- In Section 3, the former procedure is applied to the obtention of lumps. An infinite set of such solutions is obtained
- The same method is applied in section 4 to GDLW with similar results.

2 Truncation of the Lax pair for KPI equation

The well known KPI equation [14]

$$(u_t + u_{xxx} + 6uu_x)_x - 3u_{yy} = 0 \tag{2.1}$$

can be also written in potential form by setting

$$u = 2m_x \tag{2.2}$$

In this case equation (2.1) reads

$$\left(m_t + m_{xxx} + 6m_x^2\right)_x - 3m_{yy} = 0 \tag{2.3}$$

The Lax pair [14] can be written as:

$$I\psi_y + \psi_{xx} + 2m_x\psi = 0 \tag{2.4a}$$

$$\psi_t + 4\psi_{xxx} + 12m_x\psi_x + 6m_{xx}\psi - 6Im_y\psi = 0 \tag{2.4b}$$

or the complex conjugate

$$-I\varphi_y + \varphi_{xx} + 2m_x\varphi = 0 \tag{2.5a}$$

$$\varphi_t + 4\varphi_{xxx} + 12m_x\varphi_x + 6m_{xx}\varphi + 6Im_y\varphi = 0 \tag{2.5b}$$

where $I = \sqrt{-1}$ is the complex unit.

2.1 Truncated expansion of the Lax pair

The main idea in our method is to perform a truncated Painlevé expansion in the Lax pair that involves both the field m and the eigenfunctions ψ and φ. As it is well known the Painlevé property of a PDE requires that all its solutions can be expanded in generalized Laurent series around an arbitrary manifold depending on the initial conditions that is called the movable singularity manifold. When the Laurent series truncates at the constant level the manifold is named **singular manifold**. This is on the basis of the Weiss Singular Manifold Method [16], [17] that we shall apply in the following.

As it has been shown in many papers truncated Painlevé expansion, when applied to the Lax pair, can be consider as a binary Darboux transformation sometimes called gauge-Bäcklund transformation. To this end, let us consider a seed solution $m^{(0}$ as well as two different couples of seed eigenfuntions $\left(\psi_1^{(0}, \varphi_1^{(0)}\right)$ $\left(\psi_2^{(0}, \varphi_2^{(0)}\right)$. It means that:

$$I\psi_{i,y}^{(0} + \psi_{i,xx}^{(0} + 2m_x^{(0}\psi_i^{(0} = 0, \quad \psi_{i,t}^{(0} + 4\psi_{i,xxx}^{(0} + 12m_x^{(0}\psi_{i,x}^{(0} + 6\left(m_{xx}^{(0} - Im_y^{(0)}\right)\psi_i^{(0} = 0 \tag{2.6a}$$

$$-I\varphi_{i,y}^{(0}+\varphi_{i,xx}^{(0}+2m_x^{(0}\varphi_i^{(0}=0,\quad \varphi_{i,t}^{(0}+4\varphi_{i,xxx}^{(0}+12m_x^{(0}\varphi_{i,x}^{(0}+6\left(m_{xx}^{(0}+Im_y^{(0}\right)\varphi_i^{(0}=0 \tag{2.6b}$$

$$i=1,2$$

Truncated Painlevé expansion of (2.4)-(2.5) can therefore be written as:

$$m^{(1}=m^{(0}+\frac{\phi_{1,x}^{(0}}{\phi_1^{(0}} \tag{2.7a}$$

$$\psi_2^{(1}=\psi_2^{(0}-\psi_1^{(0}\frac{\Omega_{1,2}^{(0}}{\phi_1^{(0}},\quad \varphi_2^{(1}=\varphi_2^{(0}-\varphi_1^{(0}\frac{\Omega_{2,1}^{(0}}{\phi_1^{(0}} \tag{2.7b}$$

where $\phi_1^{(0}$ is the singular manifold. Substitution of (2.7) in (2.4)-(2.5) provides several polynomials in $\phi_1^{(0}$ whose coefficients should be 0. We have manage the different equations with MAPLE and the result is that we can define a matrix $\Omega_{i,j}^{(0}$ whose exact derivative is given by:

$$d\Omega_{i,j}^{(0}=\varphi_i^{(0}\psi_j^{(0}dx+I\left[\varphi_i^{(0}\psi_{j,x}^{(0}-\psi_j^{(0}\varphi_{i,x}^{(0}\right]dy+$$

$$+\left[\varphi_i^{(0}\left(3I\psi_{j,y}^{(0}-\psi_{j,x,x}^{(0}\right)-\psi_j^{(0}\left(3I\varphi_{i,y}^{(0}+\varphi_{j,xx}^{(0}\right)+4\,\varphi_{i,x}^{(0}\psi_{j,x}^{(0}\right]dt \tag{2.8}$$

such that

$$\phi_1^{(0}=\Omega_{1,1}^{(0},\qquad \phi_2^{(0}=\Omega_{2,2}^{(0} \tag{2.9}$$

Therefore, the knowledge of two seed solutions $\left(\psi_i^{(0},\varphi_i^{(0)}\right)$, $i=1,2$ of the Lax pair allows us to compute the matrix elements $\Omega_{i,j}^{(0}$ given by (2.8) and yields the transformation (2.7) that can be understood as a Darboux transformation in the sense that preserves the Lax pair by transforming the seed functions $\left(\psi_i^{(0},\varphi_i^{(0},m^{(0)}\right)$ into new solutions $\left(\psi_i^{(1},\varphi_i^{(1},m^{(1)}\right)$.

2.2 Iteration

It is a trivial exercise (It only requires a lot of calculations easily managed by MAPLE) to prove that the matrix $\Omega_{i,j}^{(0}$ defined in (2.8) can be also expanded in truncated Painlevé series of the form

$$\Omega_{i,j}^{(1}=\Omega_{i,j}^{(0}-\frac{\Omega_{i,1}^{(0}\Omega_{1,j}^{(0}}{\phi_1^{(0}} \tag{2.10}$$

In particular, for the diagonal elements

$$\phi_2^{(1}=\phi_2^{(0}-\frac{\Omega_{2,1}^{(0}\Omega_{1,2}^{(0}}{\phi_1^{(0}} \tag{2.11}$$

We have therefore, that $\psi_i^{(1}$, $\varphi_i^{(1}$, $i=1,2$ defined by (2.7b) and $\phi_2^{(1}$ defined in (2.11) are respectively eigenfunctions and singular manifold for m^1. It allows us to take m^1, $\psi_i^{(1}$, $\varphi_i^{(1}$ as the new seed solutions of the Lax pair and expand it again in truncated Painlevé expansion by taking $\phi_2^{(1}$ as singular manifold. It just means that:

$$m^{(2} = m^{(1} + \frac{\left(\phi_2^{(1}\right)_x}{\phi_2^{(1}} \tag{2.12a}$$

$$\psi_j^{(2} = \psi_j^{(1} - \psi_2^{(1} \frac{\Omega_{2,j}^{(1}}{\phi_2^{(1}}, \quad \varphi_j^{(2} = \varphi_j^{(1} - \varphi_2^{(1} \frac{\Omega_{j,2}^{(1}}{\phi_2^{(1}} \tag{2.12b}$$

By combining (2.12a) with (2.7a) the second iteration can be written as:

$$m^{(2} = m^{(0} + \frac{(\tau_{1,2})_x}{\tau_{1,2}} \tag{2.13}$$

where $\tau_{1,2} = \phi_2^{(1}\phi_1^{(0}$. By using (2.11) and (2.9), we get:

$$\tau_{1,2} = \mid \Omega_{i,j}^{(0} \mid .\, i, j = 1,2 \tag{2.14}$$

Therefore, all that we need to obtain the first and second iteration of $m^{(0}$ is to compute the matrix $\Omega_{i,j}^{(0}$ given in (2.8).

3 Lump solutions of KPI

Now, we shall apply the above described method to obtain lump solutions of KPI. Let us start with the trivial seed solution

$$m^{(0} = 0$$

Solutions of Lax pair (2.6) are in this case

$$\psi_j^{(0} = e^{k_j Q(x,y,t,k_j)} Z^{[N]}(x,y,t,k_j) \tag{3.1a}$$

$$\varphi_j^{(0} = e^{-n_j Q(x,y,t,n_j)} Z^{[N]}(x,-y,t,-n_j) \tag{3.1b}$$

where

$$Q(x,y,t,k_j) = x + Ik_j y - 4k_j^2 t \tag{3.2}$$

and $Z^{[N]}(x,y,t,k_j)$ are polynomials of order N in x that can be written as:

$$Z^{[N]}(x,y,t,k_j) = \sum_{h=0}^{N} \frac{N!}{h!(N-h)!} \varepsilon_h(y,t,k_j) x^{N-h},\ \varepsilon_0 = 1 \tag{3.3}$$

Functions $\varepsilon_h(y,t,k_j)$ can be obtained through the recursion relation

$$\frac{\partial \varepsilon_{h+1}}{\partial y} = I(h+1)\left[2k_j \varepsilon_h + h\varepsilon_{h-1}\right] \tag{3.4a}$$

$$\frac{\partial \varepsilon_{h+1}}{\partial t} = -4(h+1)\left[3k_j^2\varepsilon_h + 3k_j h\varepsilon_{h-1} + h(h-1)\varepsilon_{h-2}\right] \tag{3.4b}$$

The first three elements of this expansion are:

$$\varepsilon_1(y,t,k_j) = 2Ik_jy - 12k_j^2t \tag{3.5a}$$

$$\varepsilon_2(y,t,k_j) = \varepsilon_1^2 + \delta_2 \quad \Longrightarrow Z^{[2]} = \left(Z^{[1]}\right)^2 + \delta_2 \tag{3.5b}$$

$$\varepsilon_3(y,t,k_j) = \varepsilon_1^3 + 3\delta_2\varepsilon_1 + \delta_3 \quad \Longrightarrow Z^{[3]}(y,t,k_j) = \left(Z^{[1]}\right)^3 + 3\delta_2 Z^{[1]} + \delta_3 \tag{3.5c}$$

where

$$\delta_2(y,t,k_j) = 2Iy - 24k_jt, \quad \delta_3(y,t,k_j) = -24t \tag{3.6}$$

In order to have polynomial solutions for the singular manifolds defined in (2.8)-(2.9), it is necessary to have $n_j = k_j$ to suppress the exponentials in ϕ_1^0. On the other hand from (3.2), it is easy to check that $Q(x,y,t,k_j)^* = Q(x,y,t,-k_j^*)$. It suggests us that if we take k_j for the first iteration, we should select $k_2 = -k_1^*$ for the second one in order to have a real expression for (2.13). Furthermore it would be necessary to have $\phi_2^0 = \left(\phi_1^0\right)^*$.

With the above requirements, the form of the seed eigenfuntions in which we are interested, whould be:

$$\psi_1^{(0} = e^{k_1Q_1}Z^{[N]}(k_1), \qquad \varphi_1^{(0} = e^{-k_1Q_1}\left(Z^{[M]}(-k_1^*)\right)^* \tag{3.7a}$$

$$\psi_2^{(0} = e^{-k_1^*Q_1^*}Z^{[M]}(-k_1^*), \qquad \varphi_2^{(0} = e^{k_1^*Q_1^*}\left(Z^{[N]}(k_1)\right)^* \tag{3.7b}$$

where we have defined

$$Q_1 = Q(x,y,t,k_1) = x + Ik_1y - 4k_1^2t \tag{3.8a}$$

$$Z^{[N]}(k_1) = Z^{[N]}(x,y,t,k_1), \quad Z^{[M]}(-k_1^*) = Z^{[M]}(x,y,t,-k_1^*) \tag{3.8b}$$

and we have used the obvious relations (check (3.3))

$$Z^{[N]}(x,-y,t,k_1^*) = \left(Z^{[N]}(x,y,t,k_1)\right)^*, \quad Z^{[M]}(x,y,t,-k_1^*) = \left(Z^{[M]}(x,-y,t,-k_1)\right)^* \tag{3.9}$$

One important property of the above defined polynomials (this property is very useful because it allows us to perform integration by parts) is that:

$$\frac{\partial Z^{[N]}}{\partial x} = NZ^{[N-1]} \tag{3.10}$$

Actually, we have an infinite set of solutions that can be classified in terms of two integer numbers N and M. Let us compute the first three cases

3.1 $N = M = 0$

In this case, we have:

$$Z^{[0]}(k_1) = 1, \quad Z^{[0]}(-k_1^*) = 1 \tag{3.11}$$

and the eigenfunctions (3.1) are:

$$\psi_1^{(0} = e^{k_1 Q_1}, \qquad \varphi_1^{(0} = e^{-k_1 Q_1}, \qquad \psi_2^{(0} = e^{-k_1^* Q_1^*}, \qquad \varphi_2^{(0} = e^{k_1^* Q_1^*} \tag{3.12}$$

Integration of (2.8) gives trivially

$$\Omega_{i,j}^{(0)} = \begin{pmatrix} Z^{[1]}(k_1) & -\frac{e^{-[k_1 Q_1 + k_1^* Q_1^*]}}{k_1 + k_1^*} \\ \frac{e^{[k_1 Q_1 + k_1^* Q_1^*]}}{k_1 + k_1^*} & \left(Z^{[1]}(k_1)\right)^* \end{pmatrix} \tag{3.13}$$

According to (3.3) and (3.5)

$$Z^{[1]}(k_1) = x + \varepsilon_1(y,t,k_1) = x + 2Ik_1 y - 12k_1^2 t \tag{3.14}$$

Therefore, (2.14) gives us the following real positive defined expression

$$\tau_{1,2} = Z^{[1]}(k_1)\left(Z^{[1]}(k_1)\right)^* + \left(\frac{1}{k_1 + k_1^*}\right)^2 \tag{3.15}$$

that can be explicitly written as:

$$\tau_{1,2} = X_1^2 + Y_1^2 + \left(\frac{1}{2a_1}\right)^2 \tag{3.16}$$

where

$$k_1 = a_1 + Ib_1, \qquad Z^{[1]}(k_1) = X_1 + IY_1 \tag{3.17a}$$

$$X_1 = x - 2b_1 y - 12t(a_1^2 - b_1^2), \quad Y_1 = 2a_1(y - 12b_1 t) \tag{3.17b}$$

3.2 $N = 1, M = 0$

In this case, we need to use:

$$Z^{[2]}(k_1) = (Z^{[1]}(k_1))^2 + \delta(y,t,k_1), \quad Z^{[1]}(k_1) = x + \varepsilon_1(y,t,k_1), \quad Z^{[0]}(-k_1^*) = 1 \tag{3.18}$$

and the eigenfunctions (3.1) are:

$$\psi_1^{(0} = e^{k_1 Q_1} Z^{[1]}(k_1), \qquad \varphi_1^{(0} = e^{-k_1 Q_1}, \qquad \psi_2^{(0} = \left(\varphi_1^{(0)}\right)^*, \qquad \varphi_2^{(0} = \left(\psi_1^{(0)}\right)^* \tag{3.19}$$

Integration of (2.8) yields

$$\Omega_{i,j}^{(0)} = \begin{pmatrix} \frac{Z^{[2]}}{2} & -\frac{e^{-[k_1 Q_1 + k_1^* Q_1^*]}}{k_1 + k_1^*} \\ \frac{e^{[k_1 Q_1 + k_1^* Q_1^*]}}{k_1 + k_1^*}\left(\left(Z^{[1]} - \frac{1}{k_1 + k_1^*}\right)\left((Z^{[1]})^* - \frac{1}{k_1 + k_1^*}\right) + \left(\frac{1}{k_1 + k_1^*}\right)^2\right) & \frac{\left(Z^{[2]}\right)^*}{2} \end{pmatrix} \tag{3.20}$$

Therefore

$$\tau_{1,2}=\frac{Z^{[2]}(k_1)\left(Z^{[2]}(k_1)\right)^*}{4}+$$
$$+\left(\frac{1}{k_1+k_1^*}\right)^2\left\{\left(Z^{[1]}(k_1)-\frac{1}{k_1+k_1^*}\right)\left(\left(Z^{[1]}(k_1)\right)^*-\frac{1}{k_1+k_1^*}\right)+\left(\frac{1}{k_1+k_1^*}\right)^2\right\} \quad (3.21)$$

and finally, from (3.3) and (3.5), we have for $\tau_{1,2}$ the real positive defined expression:

$$\tau_{1,2}=\frac{(X_1^2-Y_1^2+X_2)^2+(2X_1Y_1+Y_2)^2}{4}+\left(\frac{1}{2a_1}\right)^2\left\{\left(X_1-\frac{1}{2a_1}\right)^2+Y_1^2+\left(\frac{1}{2a_1}\right)^2\right\} \quad (3.22)$$

where we have used (3.6) to define

$$\delta_2(y,t,k_1)=X_2+IY_2,\quad X_2=-24a_1t,\quad Y_2=2y-24b_1t \quad (3.23)$$

3.3 $N=1, M=1$

From (3.5a) it is easy to see that $Z^{[1]}(-k_1^*)=\left(Z^{[1]}(k_1)\right)^*$ Therefore, the eigenfunctions (3.1) are:

$$\psi_1^{(0}=e^{k_1Q_1}Z^{[1]}(k_1),\quad \varphi_1^{(0}=e^{-k_1Q_1}Z^{[1]}(k_1),\quad \psi_2^{(0}=\left(\varphi_1^{(0)}\right)^*,\quad \varphi_2^{(0}=\left(\psi_1^{(0)}\right)^* \quad (3.24)$$

Integration of (2.8) with these eigenfuncions yields

$$\Omega_{1,1}^{(0}=\frac{Z^{[1]}Z^{[2]}}{2}-\frac{Z^{[3]}}{6},\qquad \Omega_{2,2}^{(0}=\left(\Omega_{1,1}^{(0}\right)^* \quad (3.25a)$$

$$\Omega_{1,2}^{(0}=-\frac{e^{-[k_1Q_1+k_1^*Q_1^*]}}{k_1+k_1^*}\left(\left(Z^{[1]}+\frac{1}{k_1+k_1^*}\right)\left(\left(Z^{[1]}\right)^*+\frac{1}{k_1+k_1^*}\right)+\left(\frac{1}{k_1+k_1^*}\right)^2\right) \quad (3.25b)$$

$$\Omega_{2,1}^{(0}=\frac{e^{[k_1Q_1+k_1^*Q_1^*]}}{k_1+k_1^*}\left(\left(Z^{[1]}-\frac{1}{k_1+k_1^*}\right)\left(\left(Z^{[1]}\right)^*-\frac{1}{k_1+k_1^*}\right)+\left(\frac{1}{k_1+k_1^*}\right)^2\right) \quad (3.25c)$$

Therefore $\tau_{1,2}$ is the real positive defined expression:

$$\tau_{1,2}=\left(\frac{Z^{[1]}Z^{[2]}}{2}-\frac{Z^{[3]}}{6}\right)\left(\frac{Z^{[1]}Z^{[2]}}{2}-\frac{Z^{[3]}}{6}\right)^*+$$
$$+\left(\frac{1}{k_1+k_1^*}\right)^2\left\{\left(Z^{[1]}+\frac{1}{k_1+k_1^*}\right)\left(\left(Z^{[1]}\right)^*+\frac{1}{k_1+k_1^*}\right)+\left(\frac{1}{k_1+k_1^*}\right)^2\right\} \quad (3.26)$$
$$\left\{\left(Z^{[1]}-\frac{1}{k_1+k_1^*}\right)\left(\left(Z^{[1]}\right)^*-\frac{1}{k_1+k_1^*}\right)+\left(\frac{1}{k_1+k_1^*}\right)^2\right\}$$

that with the aid of (3.3) and (3.5) can be explicitly written as:

$$\tau_{1,2} = \left(\frac{X_1^3 - 3X_1Y_1^2 - \frac{X_3}{2}}{9}\right)^2 + \left(\frac{3X_1^2Y_1 - Y_1^3 - \frac{Y_3}{2}}{9}\right)^2 + \tag{3.27}$$
$$+\left(\frac{1}{2a_1}\right)^2 \left\{\left(X_1 + \frac{1}{2a_1}\right)^2 + Y_1^2 + \left(\frac{1}{2a_1}\right)^2\right\}\left\{\left(X_1 - \frac{1}{2a_1}\right)^2 + Y_1^2 + \left(\frac{1}{2a_1}\right)^2\right\}$$

where we have:

$$\delta_3 = X_3 + IY_3 \Longrightarrow X_3 = -24t, \quad Y_3 = 0 \tag{3.28}$$

4 GDLW equation

A different case in which the above described method can be tested is the following Lax pair

$$I\psi_t + \psi_{xx} + 2m_x\psi = 0 \tag{4.1a}$$

$$2m_y\psi_{xy} + \left(I\int m_{xy}dt - m_{xy}\right)\psi_y + 2m_y^2\psi = 0 \tag{4.1b}$$

and its complex conjugate

$$-I\varphi_t + \varphi_{xx} + 2m_x\varphi = 0 \tag{4.2a}$$

$$2m_y\varphi_{xy} - \left(I\int m_{xy}dt + m_{xy}\right)\varphi_y + 2m_y^2\varphi = 0 \tag{4.2b}$$

This spectral pair appears in [18] and yields to the equation

$$m_y^2\left(n_{yt} + m_{xxxy}\right) + m_{xy}\left(n_y^2 + m_{xy}^2\right) - m_y\left(n_y^2 + m_{xy}^2\right)_x + 4m_y^3m_{xx} \tag{4.3}$$

It was proved in [18], that this equation is related through Miura transformations to the dispersive wave equation proposed by Boiti et al in [15], as well as to the system proposed by Fokas in [19].

4.1 Truncation of the Lax pair

Let be $m^{(0}$ a seed solution of (4.3) and $(\psi_i^{(0}, \varphi_i^{(0})$ eigenfuntions of the Lax pair. Truncation of the Lax pair can be understood as the gauge-Bäcklund transformation

$$m^{(1} = m^{(0} + \frac{\phi_{1,x}^{(0}}{\phi_1^{(0}}, \quad \psi_2^{(1} = \psi_2^{(0} - \psi_1^{(0}\frac{\Omega_{1,2}^{(0}}{\phi_1^{(0}}, \quad \varphi_2^{(1} = \varphi_2^{(0} - \varphi_1^{(0}\frac{\Omega_{2,1}^{(0}}{\phi_1^{(0}} \tag{4.4}$$

Straightforward calculation yields:

$$d\Omega_{i,j}^{(0} = \varphi_i^{(0}\psi_j^{(0}dx - \left[\frac{\left(\varphi_i^{(0)}\right)_y\left(\psi_j^{(0)}\right)_y}{m_y^{(0}}\right]dy + \left[\varphi_i^{(0}\left(\psi_j^{(0)}\right)_y - \psi_j^{(0}\left(\varphi_i^{(0)}\right)_y\right]dt \tag{4.5a}$$

$$\phi_1^{(0} = \Omega_{1,1}^{(0}, \quad \phi_2^{(0} = \Omega_{2,2}^{(0} \tag{4.5b}$$

4.2 Lump solutions

As it is easy to see in (4.5a), the matrix $\Omega_{i,j}^{(0}$ is not defined for $m_y^{(0} = 0$. Therefore, we shall use as seed solution

$$m_y^{(0} = -1$$

In this case the Lax pair gives us:

$$I\left(\psi_j^{(0}\right)_t + \left(\psi_j^{(0}\right)_{xx} = 0, \quad -\left(\psi_j^{(0}\right)_{xy} + \psi_j^{(0} = 0 \tag{4.6a}$$

$$-I\left(\varphi_j^{(0}\right)_t + \left(\varphi_j^{(0}\right)_{xx} = 0, \quad -\left(\varphi_j^{(0}\right)_{xy} + \varphi_j^{(0} = 0 \tag{4.6b}$$

The solutions of (4.6) can be written as:

$$\psi_j^{(0} = e^{k_j Q(x,y,t,k_j)} Z^{[N]}(x,y,t,k_j), \quad \varphi_j^{(0} = e^{-n_j Q(x,y,t,n_j)} Z^{[N]}(x,y,-t,-n_j) \tag{4.7}$$

where

$$Q(x,y,t,k_j) = x + \frac{y}{k_j^2} + Ik_j t \tag{4.8}$$

and $Z^{[N]}(x,y,t,k_j)$ can be written as the expansion (3.3) whose coefficients obey the recursion relations

$$\frac{\partial \varepsilon_{h+1}}{\partial y} = -\frac{(h+1)}{k_j^2}\left[\varepsilon_h + k_j \frac{\partial \varepsilon_h}{\partial y}\right] \tag{4.9a}$$

$$\frac{\partial \varepsilon_{h+1}}{\partial t} = I(h+1)\left[2k_j \varepsilon_h + j\varepsilon_{h-1}\right] \tag{4.9b}$$

The first three elements of this expansion are:

$$\varepsilon_1(y,t,k_j) = -\frac{y}{k_j^2} + 2Ik_j t \tag{4.10a}$$

$$\varepsilon_2(y,t,k_j) = \varepsilon_1^2 + \delta_2 \quad \Longrightarrow Z^{[2]} = \left(Z^{[1]}\right)^2 + \delta_2 \tag{4.10b}$$

$$\varepsilon_3(y,t,k_j) = \varepsilon_1^3 + 3\delta_2\varepsilon_1 + \delta_3 \quad \Longrightarrow Z^{[3]} = \left(Z^{[1]}\right)^3 + 3\delta_2 Z^{[1]} + \delta_3 \tag{4.10c}$$

where

$$\delta_2 = \frac{2y}{k_j^3} + 2It, \qquad \delta_3 = \frac{-6y}{k_j^4} \tag{4.11}$$

Therefore the $\tau_{1,2}$-functions are given by the expressions (3.16), (3.22) and (3.27) where we have:

$$X_1 = x - \frac{a_1^2 - b_1^2}{(a_1^2 + b_1^2)^2} y - 2b_1 t, \qquad Y_1 = \frac{2a_1 b_1}{(a_1^2 + b_1^2)^2} y + 2a_1 t \tag{4.12a}$$

$$X_2 = 2a_1 \frac{a_1^2 - 3b_1^2}{(a_1^2 + b_1^2)^3} y, \qquad Y_2 = 2b_1 \frac{b_1^2 - 3a_1^2}{(a_1^2 + b_1^2)^3} y + 2t \tag{4.12b}$$

$$X_3 = -6 \frac{a_1^4 + b_1^4 - 6a_1^2 b_1^2}{(a_1^2 + b_1^2)^4} y, \qquad Y_3 = -24a_1 b_1 \frac{b_1^2 - a_1^2}{(a_1^2 + b_1^2)^4} y \tag{4.12c}$$

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