Journal of Nonlinear Mathematical Physics Volume 15, Supplement 3 (2008), 1–12 ARTICLE

# Quantum Integrability of the Dynamics on a Group Manifold

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**Abstract**

We study the dynamics of a particle moving on the $SU(2)$ group manifold. An exact quantization of this system is accomplished by finding the unitary and irreducible representations of a finite-dimensional Lie subalgebra of the whole Poisson algebra in phase space. In fact, the basic position and momentum operators, as well as the Hamiltonian, are found in the enveloping algebra of the anti-de Sitter group $SO(3,2)$. The present algorithm mimics the one previously used in Ref. [1]. Our construction can be extended to more general semi-simple Lie groups. This framework would allow us to achieve the quantization of the geodesic motion in a symmetric pseudo-Riemannian manifold.

## 1 Introduction

Since the pioneering work by Wigner [2] a huge amount of papers have been devoted to the analysis of solvable quantum systems through their symmetries. They constitute a powerful tool in the explicit construction of eigenstates and eigenvalues of a given symmetrical Hamiltonian. But symmetry can be taken beyond this technical ability and considered as the basic bricks for (fundamental) physical systems in the sense that all objects related to it, such as space-time, solution manifold, wave functions, quantum operators, etc., can be explicitly constructed from canonical structures on a particular Lie group. This viewpoint has been demonstrated in many finite- and infinite-dimensional cases by applying a Group Approach to Quantization (GAQ) developed since the original paper [3], where the quantum free Galilean particle and the harmonic oscillator were derived. Then, this algorithm has been applied to less elementary groups as those associated with relativistic particles, in particular the relativistic harmonic oscillator [4, 5, 6], field theories in curved space-times, non-linear $\sigma$-models, the Virasoro group and others concerning conformal symmetry and quantum gravity (see, for instance [7, 8, 9]).

In applying this GAQ it appears the necessity of finding with complete accuracy the structure of a given Lie group, considered as the basic symmetry group of the corresponding physical system, from which any referent to this problem can be derived. However, in many practical cases, the fully understanding of such a symmetry is nearly tantamount to solve the classical equations of motion (something that GAQ is intended to avoid). In fact, the classical Hamiltonian, $H$, of a particular physical problem scarcely closes a Poisson subalgebra with the standard coordinate, $q$, and momentum, $p$. In this situation one has to resort to a possibly infinite-dimensional Poisson subalgebra containing $\langle H,q,p\rangle$ or, alternatively, as we shall do in the present contribution, to look for a slightly different, finite-dimensional Poisson subalgebra in the enveloping algebra of which the original functions $\langle H,q,p\rangle$ can be found, and then quantized. After all, in the GAQ scheme, not only the generators of the original group $G$ can be quantized, but also the entire universal enveloping algebra. This procedure has been explicitly achieved in dealing with the quantum dynamics of a particle in a (modified) Pöschl-Teller potential [1], where the "first-order" group $G$ used was $SL(2,R)$.

The present paper is organized as follows. In the next section we briefly introduce a group-theoretical framework to achieve a natural *non-canonical quantization* where symmetry plays an essential role and a notion of *integrability* also naturally arises. In Sec. 3 we sketch the example of the free non-relativistic particle as a prototype of (integrable) quantum linear system. In Sec. 4 the free relativistic particle constitutes an example of an intermediary step between linear and non-linear systems; a physical system the equations of which are linear but the Lagrangian is not. Finally in Sec. 5. the case of the dynamics on the $SU(2)$ manifold is explicitly presented to exemplify the scheme of treatment of a class of physical problems whose symmetries can be found in the enveloping algebra of a finite-dimensional Lie group.

## 2 Group Approach to Quantization

The essential idea underlying this group framework for quantization consists in selecting a given subalgebra $\tilde{\mathscr{G}}$ of the classical Poisson algebra including $\langle H,\, p_i,\, x^j,\, 1\rangle$ and finding its *unitary irreducible representations* (unirreps), which constitute the possible *quantizations*. Although the actual procedure for finding unirreps might not be what really matters from the physical point of view we proceed along a well-defined algorithm to obtain them for any Lie group.

The basic idea of **GAQ** consists in having *two* mutually *commuting* copies of the Lie algebra $\tilde{\mathscr{G}}$ of a group $\tilde{G}$ of *strict symmetry* (of a given physical system), that is,

$$\mathscr{X}^L(\tilde{G}) \approx \tilde{\mathscr{G}} \approx \mathscr{X}^R(\tilde{G})$$

in such a way that one copy, let us say $\mathscr{X}^R(\tilde{G})$, plays the role of *pre-Quantum Operators* acting (by usual derivation) on complex (wave) functions on $\tilde{G}$, whereas the other, $\mathscr{X}^L(\tilde{G})$, is used to *reduce* the representation in a manner *compatible* with the action of the operators, thus providing the *true quantization*.

In fact, from the group law $g'' = g' * g$ of any group $\tilde{G}$, we can read two different actions:

$$g'' = g' * g \equiv L_{g'} g$$

$$g'' = g' * g \equiv R_g g'$$

The two actions commute and so do the generators $\tilde{X}^R_a$ and $\tilde{X}^L_b$ of the *left* and *right* actions respectively, i.e.

$$[\tilde{X}^L_a, \tilde{X}^R_b] = 0 \ \ \forall a, b \, .$$

The generators $\tilde{X}^R_a$ are right-invariant vector fields closing a Lie algebra, $\mathscr{X}^R(\tilde{G})$, isomorphic to the tangent space to $\tilde{G}$ at the identity, $\tilde{\mathscr{G}}$. The same, changing $L \leftrightarrow R$, applies to $\tilde{X}^L \in \mathscr{X}^L(\tilde{G})$.

Another manifestation of the commutation between left an right translations corresponds to the invariance of the left-invariant canonical 1-forms, $\{\theta^{L^a}\}$, dual to $\{\tilde{X}^L_b\}$, with respect to the right-invariant vector fields, that is,

$$L_{\tilde{X}^R_a} \theta^{L^b} = 0 \, , \quad \{\theta^{L^a}\} \text{ dual base to } \{\tilde{X}^L_a\}$$

and the other way around ($L \leftrightarrow R$). In particular, we may resort to a natural *invariant volume* $\omega$ on the group manifold in order to build up an actual scalar product of wave functions. In fact:

$$L_{\tilde{X}^R_a} (\theta^{L^1} \wedge \theta^{L^2} \wedge \cdots \wedge \theta^{L^d}) \equiv L_{\tilde{X}^R_a} \omega = 0 \, , \forall \tilde{X}^R_a \in \mathscr{X}^R(\tilde{G}), \ d = \dim(G)$$

We should then be able to recover all *physical ingredients* of quantum systems out of *algebraic structures*. In particular, the Poincaré-Cartan form $\Theta_{PC}$ and the phase space itself $M \equiv (x^i, p_j)$ should be regained from a group of *strict symmetry* $\tilde{G}$. In fact, in the special case of a Lie group which bears a central extension structure with structure group $U(1)$ parameterized by $\zeta \in C$ such that $|\zeta|^2 = 1$, as we are in fact considering, the group manifold $\tilde{G}$ itself can be endowed with the structure of a principal bundle with an invariant connection, thus generalizing the notion of *quantum manifold*.

More precisely, the $U(1)$-component of the left-invariant canonical form (dual to the *vertical* generator $\tilde{X}^L_\zeta$, i.e. $\theta^{L(\zeta)}(\tilde{X}^L_\zeta) = 1$) will be named *Quantization Form*

$$\Theta \equiv \theta^{L^{(\zeta)}}$$

and generalizes the *Poincaré-Cartan form* $\Theta_{PC}$ of Classical Mechanics. The quantization form remains *strictly invariant* under the group $\tilde{G}$ in the sense that

$$L_{\tilde{X}^R_a} \Theta = 0 \ \ \forall \tilde{X}^R_a \in \mathscr{X}^R(\tilde{G})$$

whereas $\Theta_{PC}$ is, in general, only *semi-invariant*, that is to say, it is invariant except for a total differential.

It should be stressed that a true *Quantum Manifold* in the sense of *Geometric Quantization* [10, 11] could be achieved by taking in the pair $\{\tilde{G}, \Theta\}$ the quotient by the action of the left subgroup generated by the characteristic module of $\Theta$, and a further quotient by the structure subgroup $U(1)$ provides the *Classical Solution Manifold M* or classical *Phase Space*. Even more, those left-invariant vector fields

$$X^L \ / \ i_{X^L} d\Theta = 0 = i_{X^L} \Theta$$

constitute the *Classical Equations of Motion*. On the other hand, the right-invariant vector fields are used to provide classical functions on the phase space. In fact, the functions

$$F_a \equiv i_{\tilde{X}^R_a}\Theta$$

are stable under the action of the left-invariant vector fields in the characteristic module of $\Theta$, the equations of motion, and then constitute the *Noether Invariants*.

As far as the quantum theory is concerned, the above-mentioned quotient by the classical equations of motion are really not needed . We consider the space of complex functions $\Psi$ on the whole group $\tilde{G}$ and restrict them to only $U(1)$-functions, that is, those which are homogeneous of degree one on the argument $\zeta \equiv e^{i\phi} \in U(1)$. Wave functions thus satisfy

$$\tilde{X}^L_\zeta \Psi = \Psi\,.$$

On these functions the right-invariant vector fields act as *pre-Quantum Operators* by ordinary derivation. They are, in fact, Hermitian operators with respect to the invariant volume $\omega$ defined above. However, this action is not a proper quantization of the Poisson algebra of the Noether Invariants (associated with the symplectic structure given by $d\Theta$) since there is a set of non-trivial operators commuting with this representation. In fact, all the left-invariant vector fields do commute with the right-invariant ones, i.e. the (pre-quantum) operators. According to Schur's Lemma those operators must be trivialized. To this end we define a *Polarization* subalgebra $\mathscr{P}$ as: *a maximal left subalgebra containing the Characteristic subalgebra and excluding the central generator*. The role of a Polarization is then that of *reducing* the representation which now constitutes a true *Quantization*. We then impose on wave functions the Polarization condition:

$$\tilde{X}^L_b \Psi = 0\,, \;\; \forall \tilde{X}^L_b \in \mathscr{P}$$

The integration volume $\omega$ can be restricted to the Hilbert space of polarized wave functions with a canonical procedure a bit technical for the scope of the present contribution. We refer the reader to Ref. [8].

## 3 Linear Systems

Typical linear systems are those for which the classical Hamiltonian describing the motion closes a *finite-dimensional* (if the system has a finite number of degree of freedom) algebra with the positions $x^i$ and the canonical momenta $p_j$, that is to say, $\{H, x^i, p_j, 1\}$ is a finite Poisson subalgebra of the general Poisson algebra of Noether invariants. Canonical momenta are defined in terms of the Lagrangian (for a regular Hamiltonian) $\mathscr{L} \equiv p_i\dot{x}^i - H$ through the standard expression $p_i = \frac{\partial \mathscr{L}}{\partial \dot{x}^i}$. Since in this case the basic Noether invariants enter a finite-dimensional algebra, we only have to proceed according to the general scheme reported above by direct application. We expose briefly the

**Example of the free non-relativistic particle**

This system is described by the classical *Lagrangian* $\mathscr{L} = \frac{1}{2}m\dot{\vec{x}}^2$, from which we derive the *canonical momenta* $p_i = \frac{\partial \mathscr{L}}{\partial \dot{x}^i} = m\dot{x}_i$. Fortunately, for this linear system the canonical Poisson bracket of $x^i$, $p_j$, and $H = \frac{\vec{p}^2}{2m}$ close the Heisenberg-Weyl algebra with evolution

$$
\begin{aligned}
\{x^i, p_j\} &= \delta^i_j 1 \\
\{H, x^i\} &= -\frac{p^i}{m} \\
\{H, p_i\} &= 0\,,
\end{aligned}
$$

which completed with the additional Noether invariants generating the rotations ($J_i \equiv \eta_{ij\cdot}^{\ \ k} x^j p_k$)

$$
\begin{aligned}
\{J_i, J_j\} &= \eta_{ij\cdot}^{\ \ k} J_k \\
\{J_i, H\} &= 0 \\
\{J_i, p_j\} &= \eta_{ij\cdot}^{\ \ k} p_k \\
\{J_i, x^j\} &= \eta_{i\cdot k}^{\ j} x^k
\end{aligned}
$$

constitutes the Lie algebra of the *centrally extended Galilei group*.

Since we dispose of a well-defined, finite-dimensional group of symmetry we only have to apply step by step the algorithm GAQ just described. To this end we exponentiate the algebra above and find the following group law in terms of the parameters $t, \vec{x}, \vec{p}, \vec{\varepsilon}$:

$$
\begin{aligned}
t'' &= t' + t \\
\vec{x}'' &= \vec{x}' + R'(\vec{\varepsilon})\vec{x} + \frac{\vec{p}'}{m} t \\
\vec{p}'' &= \vec{p}' + R'(\vec{\varepsilon})\,\vec{p} \\
\vec{\varepsilon}'' &= \sqrt{1 + \frac{\vec{\varepsilon}'^2}{4}}\,\vec{\varepsilon} + \sqrt{1 + \frac{\vec{\varepsilon}^2}{4}}\,\vec{\varepsilon}' + \frac{1}{2}\vec{\varepsilon}' \times \vec{\varepsilon} \\
\zeta'' &= \zeta'\zeta e^{i[\vec{x}' R' \vec{p} + \frac{t}{m}(\vec{p}' R' \vec{p} + \vec{p}'^2/2)]/\hbar}\,,
\end{aligned}
$$

where $R$ is the rotation matrix around the direction of $\vec{\varepsilon}$ through the angle $2\sin^{-1}\frac{|\vec{\varepsilon}|}{2}$. We have introduced the constant $\hbar$ to turn the exponent dimensionless.

The left-invariant vector fields are:

$$
\begin{aligned}
\tilde{X}^L_t &= \frac{\partial}{\partial t} + \frac{\vec{p}}{m} \cdot \frac{\partial}{\partial \vec{x}} + \frac{1}{\hbar}\frac{\vec{p}^2}{2m}\,\Xi \\
\tilde{X}^L_{\vec{x}} &= R\frac{\partial}{\partial \vec{x}} \\
\tilde{X}^L_{\vec{p}} &= R\left(\frac{\partial}{\partial \vec{p}} + \frac{1}{\hbar}\vec{x}\,\Xi\right) \\
\tilde{X}^L_{\vec{\varepsilon}} &= X^{L\ SU(2)}_{(\vec{\varepsilon})} \\
\tilde{X}^L_{\phi} &= i\zeta\frac{\partial}{\partial \zeta} + h.c. \equiv \Xi\,,
\end{aligned}
$$

and those invariant on the right:

$$\begin{aligned}
\tilde{X}^R_t &= \frac{\partial}{\partial t} \\
\tilde{X}^R_{\vec{x}} &= \frac{\partial}{\partial \vec{x}} + \frac{1}{\hbar}\vec{p}\,\Xi \\
\tilde{X}^R_{\vec{p}} &= \frac{\partial}{\partial \vec{p}} + \frac{t}{m}\frac{\partial}{\partial \vec{x}} + \frac{1}{\hbar}t\vec{p}\,\Xi \\
\tilde{X}^R_{\vec{\varepsilon}} &= X^{R\ SU(2)}_{(\vec{\varepsilon})} + \vec{x}\times\frac{\partial}{\partial \vec{x}} + \vec{p}\times\frac{\partial}{\partial \vec{p}} \\
\tilde{X}^R_{\phi} &= i\zeta\frac{\partial}{\partial \zeta} + h.c. \equiv \Xi\,,
\end{aligned}$$

where $X^{L\ SU(2)}_{(\vec{\varepsilon})}$ and $X^{R\ SU(2)}_{(\vec{\varepsilon})}$ are the left- and right-invariant generators of the subgroup $SU(2)$ itself and their actual expression does really not matter at this moment; see the last section.

The quantization 1-form, dual to $\Xi$, is

$$\Theta = -\vec{x}\cdot d\vec{p} - \frac{p^2}{2m}dt + \hbar\frac{d\zeta}{i\zeta}\,,$$

from which we can compute the characteristic module:

$$\mathscr{G}_\Theta = \langle \tilde{X}^L_t, \tilde{X}^L_{\vec{\varepsilon}}\rangle\,.$$

As a module it is generated by the vector fields associated with time translation and rotations.

There exists a Polarization, which is:

$$\mathscr{P} = \langle \tilde{X}^L_t, \tilde{X}^L_{\vec{\varepsilon}}, \tilde{X}^L_{\vec{x}}\rangle\,.$$

By imposing the polarization conditions, $\tilde{X}^L\Psi = 0, \forall \tilde{X}^L \in \mathscr{P}$, along with the $U(1)$-function condition, we arrive at a wave function of the form:

$$\Psi(\zeta,\vec{x},\vec{p},t,\vec{\varepsilon}) = \zeta e^{-\frac{i}{\hbar}\frac{p^2}{2m}t}\varphi(\vec{p})\,,$$

where $\varphi(\vec{p})$ is an arbitrary function, save for normalization.

The quantum operators will be now proportional to the right-invariant vector fields, that is:

$$\begin{aligned}
\hat{E} &= i\hbar\tilde{X}^R_t \\
\hat{\vec{P}} &= -i\hbar\tilde{X}^R_{\vec{x}} \\
\hat{\vec{K}} &= i\hbar\tilde{X}^R_{\vec{p}} \\
\hat{\vec{J}} &= -i\hbar\tilde{X}^R_{\vec{\varepsilon}}\,.
\end{aligned}$$

Their action on the wave functions, once they are restricted to only $\varphi(\vec{p})$, is:

$$\begin{aligned}
\hat{E}\varphi(\vec{p}) &= \frac{p^2}{2m}\varphi(\vec{p}) \\
\hat{\vec{P}}\varphi(\vec{p}) &= \vec{p}\varphi(\vec{p}) \\
\hat{\vec{K}}\varphi(\vec{p}) &= i\hbar\frac{\partial}{\partial \vec{p}}\varphi(\vec{p}) \\
\hat{\vec{J}}\varphi(\vec{p}) &= -i\hbar\vec{p}\times\frac{\partial}{\partial \vec{p}}\varphi(\vec{p})\,.
\end{aligned} \tag{3.1}$$

It is clearly observed that these operators take the usual expressions in Quantum Mechanics. It should be also remarked that the operators $\hat{E}$ y $\hat{\vec{J}}$ are written as a function of the basic ones $\hat{\vec{P}}$ y $\hat{\vec{K}}$:

$$\begin{aligned}\hat{E} &= \frac{1}{2m}\hat{\vec{P}}^2 \\ \hat{\vec{J}} &= -\hat{\vec{P}}\times\hat{\vec{K}}.\end{aligned}$$

# 4 Simple "Non-Linear" Systems

As a preliminary step towards non-linearity, we shall consider the simpler example of the *free relativistic particle* with classical Lagrangian $\mathscr{L} = -mc^2\sqrt{1-\frac{\dot{\vec{x}}^2}{c^2}}$. As was previously commented, even though the corresponding equations of motion are linear, the "non-linearity" (non-quadratic, indeed) of the Lagrangian causes the impossibility of closing a finite-dimensional algebra containing $H, x^i, p_j$. Let us present the problem very quickly. The canonical momenta are derived in the standard way from $\mathscr{L}$:

$$p_i = \frac{\partial\mathscr{L}}{\partial\dot{x}^i} = \frac{m\dot{x}_i}{\sqrt{1-\frac{\dot{\vec{x}}^2}{c^2}}}$$

as well as the Hamiltonian:

$$H = \dot{x}^i p_i - \mathscr{L} = \frac{mc^2}{\sqrt{1-\frac{\dot{\vec{x}}^2}{c^2}}} = \sqrt{\vec{p}^{\,2}c^2+m^2c^4} \equiv p_0 c$$

where we make use of the traditional definition of four-momentum $\{p_\mu\} = \{p_0, p_i\}$.

Let us try to close the basic Poisson subalgebra including $H$. We find

$$\begin{aligned}\{x^i, p_j\} &= \delta^i_j 1 \ \text{(canonical Poisson bracket)} \\ \{H, p_i\} &= 0 \\ \textbf{but}\ \{H, x^i\} &= -c\frac{p^i}{p_0}\ \left(\approx -\frac{p_i}{m} - \frac{1}{\mathbf{c^2}}\frac{\vec{p}^{\,2}p^i}{2m^3} + \ldots\right)\end{aligned}$$

So, the classical functions $\langle H, x^i, p_j, 1\rangle$ *do not close* a finite Lie algebra. They "close" an algebra with structure constants *depending* on the *energy* $H \equiv p_0 c$.

Here, two different options arise. One option consists in trying to close an infinite-dimensional Poisson subalgebra by defining new functions which are quadratic and beyond in the basic functions $x^i$ and $p_j$. This can be done order by order in the parameter $\frac{1}{c^2}$ which now plays the role of "coupling constant". But there is another, far simpler when possible, which consists in looking for new "basic functions", closing a finite-dimensional subalgebra, in terms of which we can rewrite the old basic functions. Here we proceed along this last line.

To this end we define the classical functions $k^i \equiv \frac{p_0}{mc}x^i$. Then, the new algebra $\langle H, p_i, k^j, J^k, 1\rangle$ *does close* on the Poincaré algebra:

$$\begin{aligned}\{p_0, k^i\} &= \{p_0, \frac{p_0}{mc}x^i\} = -\frac{p^i}{m} \\ \{p_0, p_i\} &= 0 \\ \{k^i, k^j\} &= -\eta_{ij\cdot}^{\ \ k}\eta^k_{\cdot mn}x^m p^n \equiv -\eta_{ij\cdot}^{\ \ k}J^k \\ \{k^i, p_j\} &= \{\frac{p_0}{mc}x^i, p_j\} = \frac{p_0}{mc}\delta^i_j\end{aligned}$$

If we quantize the Poincaré group, we regain the original (*infinite-dimensional*) algebra of quantum operators inside its enveloping algebra:

$$\langle \hat{H}, \hat{k}^i, \hat{x}^j \sim \frac{mc}{2}(\hat{p}_0^{-1}\hat{k}^j + \hat{k}^j\hat{p}_0^{-1}), \hat{J}^k, \hat{1}\rangle$$

To do that we parameterize a central extension of the Poincaré group by (abstract) variables $\{a^0, \vec{a}, \vec{v}, \vec{\varepsilon}, \zeta\}$ so that the, let us say, right-invariant generators reproduce the respective functions $\{H, \vec{p}, \vec{k}, \vec{J}, 1\}$ as Noether invariants as well as the Poisson brackets above.

We are not going to present here the precise details of the quantization of the Poincaré group nor insist in those typical problems related to the "position operator" in quantum relativity that can be read from, for instance, Ref. [12]. Let us just comment that the *wave functions* are solutions of a higher-order Polarization $\mathscr{P}\psi = 0$, with

$$\mathscr{P} = \langle \tilde{X}^{LHO}_{a^0} \equiv (\tilde{X}^L_{a^0})^2 - c^2(\tilde{X}^L_{\vec{a}})^2 - \frac{2imc^2}{\hbar}\tilde{X}^L_{a^0},\, \tilde{X}^L_{\vec{v}},\, \tilde{X}^L_{\vec{\varepsilon}}\,\rangle$$

from which we arrive at a wave function which depends only on $a^0$ and $\vec{a}$ and satisfies the ordinary Klein-Gordon equation in variables $x^0$ and $\vec{x}$:

$$(\Box + m^2)\psi = 0\ ,$$

after the change $a^\mu = \frac{p_0}{mc}x^\mu$.

## 5 (True) Non-Linear Systems: Dynamics on $SU(2)$

We end the presentation by sketching the way in which the idea demonstrated in the previous section can really apply to a non-trivial problem. Let us start by parameterizing rotations with a vector $\vec{\chi}$ in the rotation-axis direction and with modulus

$$|\vec{\chi}| = 2\sin\frac{\varphi}{2}$$

$$R(\vec{\chi})^i_j = (1 - \frac{\vec{\chi}^{\,2}}{2})\delta^i_j - \sqrt{1 - \frac{\vec{\chi}^{\,2}}{4}}\,\eta^i_{\cdot jk}\chi^k + \frac{1}{2}\chi^i\chi_j$$

In these coordinates the canonical left-invariant 1-forms read:

$$\theta^{L(i)}_j = \left[\sqrt{1 - \frac{\vec{\chi}^{\,2}}{4}}\,\delta^i_j + \frac{\chi^i\chi_j}{4\sqrt{1 - \frac{\vec{\chi}^2}{4}}} + \frac{1}{2}\eta^i_{\cdot jm}\chi^m\right]$$

and in terms of these the *particle-σ-Model Lagrangian* acquires the following expression:

$$\Lambda = \frac{1}{2}\delta_{ij}\theta^{L(i)}_m\theta^{L(j)}_n\dot{\chi}^m\dot{\chi}^n = \frac{1}{2}\left[\delta_{ij} + \frac{\chi_i\chi_j}{4(1 - \frac{\vec{\chi}^2}{4})}\right]\dot{\chi}^i\dot{\chi}^j \equiv \frac{1}{2}g_{ij}\dot{\chi}^i\dot{\chi}^j$$

Proceeding much in the same way followed in the previous section, we compute the canonical momenta:

$$\pi_i = \frac{\partial\Lambda}{\partial\dot{\chi}^i} = g_{ij}\dot{\chi}^j$$

and the Hamiltonian:

$$\mathscr{H} = \pi_i \dot{\chi}^i - \Lambda = \frac{1}{2} g^{-1ij} \pi_i \pi_j$$

We asume the canonical bracket between the basic functions $\chi^i$ and $\pi_j$:

$$\begin{aligned} \{\chi^i, \pi_j\} &= \delta^i_j \quad \text{added with} \\ \{\mathscr{H}, \chi^i\} &= -g^{-1ij}\pi_j \\ \{\mathscr{H}, \pi_i\} &= \frac{1}{2}(\vec{\chi}\cdot\vec{\pi})\pi_i \,, \end{aligned}$$

so that $\langle \mathscr{H}, \chi^i, \pi_j, 1\rangle$ *do not close* a finite-dimensional Lie algebra.

However, we may define the following set of new *"coordinates"*, *"momenta"* and even, *"energy"* and *"angular momenta"*:

$$\langle\, p^i \equiv 2g^{-1ij}\pi_j,\, k^j \equiv \sqrt{2\mathscr{H}}\chi^j,\, E \equiv 2\sqrt{2\mathscr{H}},\, J^k \equiv \eta^{k}_{\cdot mn}\chi^m\pi^n \,\rangle \,.$$

They close the Lie algebra of $SO(3,2)$ i.e. an Anti-de Sitter algebra. That is, the basic brackets:

$$\begin{aligned} \{E, p^i\} &= k^i \\ \{E, k^j\} &= -p^j \\ \{k^i, p_j\} &= \delta^i_j E \,, \end{aligned}$$

along with the induced ones:

$$\begin{aligned} \{k^i, k^j\} &= -\eta^{ij\cdot}_{\ \ k} J^k \quad \{p^i, p^j\} = -\eta^{ij\cdot}_{\ \ k} J^k \\ \{J^i, J^j\} &= \eta^{ij\cdot}_{\ \ k} J^k \qquad \{J^i, k^j\} = \eta^{ij\cdot}_{\ \ k} k^k \\ \{J^i, p^j\} &= \eta^{ij\cdot}_{\ \ k} p^k \qquad \{E, \vec{J}\} = 0, \end{aligned}$$

close a finite-dimensional Lie algebra to which we may apply the GAQ. (*Note* the minus sign in the first line, which states that the involved group is $SO(3,2)$ and not $SO(4,1)$).

We then quantize the Anti-de Sitter group so that the original operators $\hat{\mathscr{H}}$, $\hat{\pi}_i$, $\hat{\chi}^j$ can be found in its *enveloping algebra* through the expression:

$$\langle \hat{\mathscr{H}} \equiv \frac{1}{8}\hat{E}^2, \hat{\pi}^i \equiv \frac{1}{4}\left(\hat{g}^{ij}\hat{p}_j + \hat{p}_j\hat{g}^{ij}\right), \hat{\chi}^i \equiv \frac{1}{2\sqrt{2}}\left(\hat{\mathscr{H}}^{-1/2}\hat{k}^i + \hat{k}^i\hat{\mathscr{H}}^{-1/2}\right), \hat{1}\rangle$$

In so doing we parameterize a central extension of the Anti-de Sitter group by (abstract) variables $\{a^0, \vec{a}, \vec{v}, \vec{\varepsilon}, \zeta\}$, which mimic those for the Poincaré group. In the same way we hope that the right-invariant generators associated with those parameters reproduce corresponding functions $\{E, p_i, k^j, J^k, 1\}$ as Noether invariants satisfying the Poisson brackets above.

At this point it should be stressed that the parameters $\vec{\varepsilon}$ and the corresponding quantum operators (essentially the right generators $\tilde{X}^R_{\vec{\varepsilon}}$) are associated with ordinary rotations on Anti-de Sitter space-time, whereas the $\vec{\chi}$ parameters correspond to "translations" on the $SU(2)$ manifold.

We shall not give here the explicit group law for the group variables (which can be found in Refs. [13, 14]), limiting ourselves to write the explicit expression for the left-invariant vector fields on the extended $SO(3,2)$ group:

$$\begin{aligned}
\tilde{X}^L_{a^0} &= \frac{1}{q_a}\{(\frac{\omega^2}{2c^2}a^0(\vec{a}\cdot\vec{v})q_v + (1-\frac{\omega^2}{4c^2}(a^0)^2)k_v)\frac{\partial}{\partial a^0} + (2q_v\vec{v} + \frac{\omega^2}{4c^2}(2q_v(\vec{a}\cdot\vec{v}) - a^0k_v)\vec{a})\cdot\frac{\partial}{\partial\vec{a}} \\
&- \frac{1}{q_v}\frac{\omega^2}{4c^2}[((2(1+\vec{v}^2)(\vec{a}\cdot\vec{v} - 2a^0q_v) + \vec{a}\cdot\vec{v})\vec{v} + \vec{a})\cdot\frac{\partial}{\partial\vec{v}} \\
&+ (2q_\varepsilon(\vec{a}\times\vec{v}) - \vec{\varepsilon}\times(\vec{a}\times\vec{v}))\cdot\frac{\partial}{\partial\vec{\varepsilon}}]\} + \{\frac{1}{q_a}(\frac{\omega^2}{2c^2}a^0(\vec{a}\cdot\vec{v})q_v + (1-\frac{\omega^2}{4c^2}(a^0)^2)k_v) - 1\}\Xi \\
\tilde{X}^L_{\vec{a}} &= \frac{1}{q_a}R(\vec{\varepsilon})\{(\frac{\omega^2}{4c^2}a^0(\vec{a} + 2(\vec{a}\cdot\vec{v})\vec{v}) + 2q_v(1-\frac{\omega^2}{4c^2}(a^0)^2)\vec{v})\frac{\partial}{\partial a^0} \\
&+ \frac{\partial}{\partial\vec{a}} + (2+\frac{\omega^2}{4c^2}(1+2\vec{a}\cdot\vec{v}))\vec{v}\vec{v} - \frac{\omega^2}{2c^2}a^0q_v\vec{v}\vec{a})\cdot\frac{\partial}{\partial\vec{a}} \\
&- \frac{1}{q_v}\frac{\omega^2}{4c^2}[(2q_v(\vec{a}\cdot\vec{v}) - a^0k_v)(\frac{\partial}{\partial\vec{v}} + \vec{v}(\vec{v}\cdot\frac{\partial}{\partial\vec{v}})) \\
&+ (-2q_\varepsilon q_v\vec{a} + 2q_\varepsilon a^0\vec{v} + q_v(\vec{\varepsilon}\times\vec{a}) - a^0(\vec{\varepsilon}\times\vec{v}))\times\frac{\partial}{\partial\vec{\varepsilon}}] \\
&+ (\frac{\omega^2}{4c^2}a^0(\vec{a} + 2(\vec{a}\cdot\vec{v})\vec{v}) + 2q_v(1-\frac{\omega^2}{4c^2}(a^0)^2)\vec{v})\Xi\} \\
\tilde{X}^L_{\vec{v}} &= \frac{1}{q_v}R(\vec{\varepsilon})\{\frac{\partial}{\partial\vec{v}} + \vec{v}\vec{v}\cdot\frac{\partial}{\partial\vec{v}} - (2q_\varepsilon\vec{v} - \vec{\varepsilon}\times\vec{v})\times\frac{\partial}{\partial\vec{\varepsilon}}\} \\
\tilde{X}^L_{\vec{\varepsilon}} &= X^{R\ SU(2)}_{(\vec{\varepsilon})} \\
\tilde{X}^L_{\phi} &= i\zeta\frac{\partial}{\partial\zeta} + h.c. \equiv \Xi\,,
\end{aligned}$$

where:

$$q_a = \sqrt{1+\frac{\omega^2}{4c^2}(\vec{a}^2 - (a^0)^2)}\,,\quad q_v = \sqrt{1+\vec{v}^2}\,,\quad k_v = 1+2\vec{v}^2 \;\text{ and }\; q_\varepsilon = \sqrt{1-\frac{\vec{\varepsilon}^{\,2}}{4}}\,.$$

Following the same steps as those given in the case of the Poincaré group we look for a higher-order polarization leading to the configuration-space “representation”. In fact, the simplest possibility turns out to be generated by a combination of generators formally analogous to that of the Poincaré case, although now the generators are obviously different.

From the polarization condition $\mathscr{P}\psi = 0$ we again arrive at a wave function depending only on $a^0, \vec{a}$ and satisfying a Klein-Gordon-like equation with $SO(3,2)$ D'Alembertian operator given by

$$\begin{aligned}
\Box &= \frac{1}{16q_a^2}\{[16-(a^0)^2\frac{\omega^2}{c^2}(8+\frac{\omega^2}{c^2}(r_a^2-(a^0)^2))]\frac{\partial^2}{\partial a^{02}} - a^0\frac{\omega^2}{c^2}[40+7\frac{\omega^2}{c^2}(r_a^2-(a^0)^2)]\frac{\partial}{\partial a^0} \\
&- [16+r_a^2\frac{\omega^2}{c^2}(8+\frac{\omega^2}{c^2}(r_a^2-(a^0)^2))]\frac{\partial^2}{\partial r_a^2} - \frac{1}{r_a}[32+\frac{\omega^2}{c^2}(40+7\frac{\omega^2}{c^2}(r_a^2-(a^0)^2))]\frac{\partial}{\partial r_a} \\
&- 2a^0r_a\frac{\omega^2}{c^2}(8+\frac{\omega^2}{c^2}(r_a^2-(a^0)^2))\frac{\partial^2}{\partial a^0\partial r_a}\} + \frac{\vec{L}^2}{q_a^2r_a^2}\,,
\end{aligned}$$

where

$$\vec{L}^2 = -\frac{1}{\sin\theta_a}\frac{\partial}{\partial\theta_a}(\sin\theta_a\frac{\partial}{\partial\theta_a}) - \frac{1}{\sin^2\theta_a}\frac{\partial^2}{\partial\varphi_a^2}$$

is the square of the standard orbital angular momentum operator (save for a factor $\hbar$) and $r_a \equiv \sqrt{\vec{a}\cdot\vec{a}}$.

The *wave functions* are

$$\phi(\vec{a},a^0) = e^{-2ic\lambda_{nl}\arcsin\left(\frac{\omega q_a a^0}{\sqrt{4c^2+\omega^2 q_a^2 r_a^2}}\right)}\, Y_m^l(\theta_a,\varphi_a)\left(1+\frac{\omega^2}{c^2}q_a^2 r_a^2\right)^{-\frac{\lambda_{nl}}{2}}(q_a r_a)^l \phi_l^{\lambda_{nl}}(r_a)$$

where

$$\begin{aligned}
\lambda_{nl} &\equiv \frac{E}{\hbar\omega}\\
E &\equiv \left(\frac{3}{2}+2n+l+\frac{1}{2}\sqrt{9+4\frac{m^2c^2}{\hbar^2\omega^2}-48\xi}\right)\hbar\omega\\
\phi_l^{\lambda_{nl}} &= {}_2F_1\left(-n,n+l+\frac{3}{2}-\lambda_{nl},l+\frac{3}{2};-\frac{\omega^2}{c^2}q_a^2 r_a^2\right)
\end{aligned}$$

and $\xi$ is a free parameter related to the *"zero-point energy"*. On this representation the operators corresponding to the original functions $\chi^i$, $\pi_j$ and the energy $\mathscr{H}$ can be realized.

Although one can verify by an explicit computation that $\phi(\vec{a},a^0)$ is actually a solution of the wave equation, a large amount of time can be saved by performing in Ref. [15], eqn. (64)-(66), the change of variables

$$t = \frac{2c}{\omega}\arcsin\left(\frac{\omega q_a a^0}{\sqrt{4c^2+\omega^2 q_a^2 \vec{a}^2}}\right), \qquad \vec{x} = q_a\vec{a},$$

which leads to expressions analogous to ours, although intended to describe the quantum evolution of a free particle moving on an Anti-de Sitter space-time.

As a general comment, we would like to bring the attention of the reader to a potential normal-ordering problem appearing in going from the enveloping algebra of the auxiliary group to the quantum version of the canonical variables in the original Lagrangian formalism. We are referring in particular to the "change" of quantum variables of the form $\hat{x}^j \sim \frac{mc}{2}(\hat{p}_0^{-1}\hat{k}^j+\hat{k}^j\hat{p}_0^{-1})$ (in the case of the free particle). A more general prescription for normal-ordering can be addressed following some sort of "perturbative" group approach to quantization. In fact, it is possible to close order by order in some constants (like $1/c^2$ in the relativistic particle, the structure constants themselves for the sigma model, or coupling constants in general) a Lie algebra which joins together the original variables and those in the auxiliary group. Then, applying the group-quantization technique up to a certain order we arrive at the correct prescription of any operator at the given order.

To conclude we must recognize that at present we are unable to state the class of non-linear systems to which this mechanism can be applied, although much effort is being done in this direction.

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