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Symmetry Reductions of Second Heavenly Equation and $2+1$-Dimensional Hamiltonian Integrable Systems

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Abstract

Second heavenly equation of Plebañski, presented in a two-component form, is known to be a $3+1$–dimensional multi-Hamiltonian integrable system. We show that one symmetry reduction of this equation yields a two component $2+1$–dimensional multi-Hamiltonian integrable system. For this system, we present Hamiltonian and recursion operators, point symmetries and integrals of motion. For another symmetry reduction, the reduced system is ”almost bi-Hamiltonian”, with two known Hamiltonian operators but the second Hamiltonian density missing.

1 Introduction

For a long time there were very few examples of $2+1$–dimensional and no examples of $3+1$–dimensional integrable multi-Hamiltonian systems. In [2] we have discovered that the second heavenly equation of Plebañski

$$u_{tt}u_{xx} - u_{tx}^2 + u_{xz} + u_{ty} = 0 \tag{1.1}$$

when being presented in a two-component form

$$u_t = q\,, \qquad q_t = \frac{1}{u_{xx}}\left(q_x^2 - q_y - u_{xz}\right) \tag{1.2}$$

is a $3+1$–dimensional multi-Hamiltonian integrable system. The physical significance of the single scalar equation (1.1) follows from the fact that it is equivalent to complex Einstein field equations for (anti-)self-dual gravitational fields [5], with u being the metric potential.

In [3], we studied all nonequivalent $2+1$–dimensional symmetry reductions of this system. In general, the reduced equations apparently have no Hamiltonian structure. Here we show that one particular symmetry reduction, with respect to a special combination of translations, yields a two-component $2+1$–dimensional multi-Hamiltonian integrable system. For this system, we present the Hamiltonian and recursion operators, point symmetries and integrals of motion.

In section 2, we give a complete list of Lie point symmetries of the system (1.2), specify a translational symmetry for its reduction and present a reduced system. In section 3, we present Hamiltonian operator and symplectic 2-form for the reduced system. In section 4, we derive a recursion operator for the reduced system. In section 5, we obtain second Hamiltonian structure by applying the recursion operator to the first Hamiltonian structure. In section 6, we generate first integrals by using point symmetries and the Hamiltonian structure. Finally, in section 7, we present another reduced Hamiltonian system that possesses two Hamiltonian operators and a recursion operator for symmetries but second Hamiltonian density is missing.

2 Translational symmetry reduction of the second heavenly equation

Basis generators of one-parameter subgroups of a total Lie group of point symmetries for the second heavenly system (1.2) have the form [2, 3]

$$\begin{aligned}
&X_1 = -2z\partial_t + tx\partial_u + x\partial_q, \quad X_2 = t\partial_t + z\partial_z + u\partial_u, \quad W_f = f(y,z)\partial_u \\
&X_3 = t\partial_t + x\partial_x + 3u\partial_u + 2q\partial_q, \quad Z_b = b(y)\partial_z - b'(y)x\partial_t - b''(y)\frac{x^3}{6}\partial_u \\
&Y_a = a\partial_y + a'(x\partial_x - t\partial_t - z\partial_z + q\partial_q) + a''\left(xz\partial_t - \frac{tx^2}{2}\partial_u - \frac{x^2}{2}\partial_q\right) \\
&+ a'''\frac{x^3 z}{6}\partial_u, \quad V_d = d_z(y,z)(t\partial_u + \partial_q) - d_y(y,z)x\partial_u \\
&U_c = c_y\partial_t + c_z\partial_x - c_{yz}x(t\partial_u + \partial_q) + c_{yy}\frac{x^2}{2}\partial_u + c_{zz}\left(\frac{t^2}{2}\partial_u + t\partial_q\right)
\end{aligned} \tag{2.1}$$

where $a(y), b(y), c(y,z), d(y,z)$, and $f(y,z)$ are arbitrary functions, primes denote ordinary derivatives of functions of one variable, subscripts signify partial derivatives with respect to corresponding variables and we used the shorthand notation $\partial_t = \partial/\partial t$ and so on.

In our recent paper [3] we have constructed an optimal system of one-dimensional Lie subalgebras of the total Lie algebra of symmetry generators with a certain choice of simplest representatives of equivalence classes that constitute an optimal system of subalgebras. For corresponding reduced $2+1$–dimensional equations we failed to determine their Hamiltonian structure, which was probably due to our bad choice. In particular, generators of translations were not included in the optimal system.

Generators of translations in all independent variables are obvious symmetries. They arise for particular choices of arbitrary functions $a(y)$, $b(y)$, and $c(y,z)$

$$X_y = \partial_y = Y_{a=1}, \quad X_z = \partial_z = Z_{b=1}, \quad X_t = \partial_t = U_{c=y}, \quad X_x = \partial_x = U_{c=z}. \tag{2.2}$$

Here we show that by choosing for the symmetry reduction one particular combination of the translational generators

$$X_{tr} = \partial_z - \frac{1}{\alpha}\partial_y \tag{2.3}$$

where α is an arbitrary real constant, we obtain a $2+1$–dimensional two-component Hamiltonian system. The invariants of X_{tr} are determined by a characteristic system as

$$X = x, \quad Y = z + \alpha y, \quad T = t, \quad U = u, \quad Q = q. \tag{2.4}$$

The symmetry reduction implies the ansatz: $u = U(X,Y,T)$, $q = Q(X,Y,T)$. Substituting this into the original system (1.2) and renaming $U \to u$, $Q \to q$, $Y \to y$, and $T \to t$, we obtain the two component reduced system

$$u_t = q, \quad q_t = \frac{1}{u_{xx}}\left(q_x^2 - \alpha q_y - u_{xy}\right) \equiv Q, \tag{2.5}$$

where Q is now a shorthand notation for the right-hand side of the second equation (2.5).

3 Hamiltonian structure of the reduced system

Symmetry reduction of Hamiltonian operator of the $3+1$–dimensional system (1.2), that was presented in [2], yields the first Hamiltonian operator for the reduced system (2.5)

$$J_0 = \begin{pmatrix} 0 & \dfrac{1}{u_{xx}} \\ -\dfrac{1}{u_{xx}} & \dfrac{1}{u_{xx}}\left\{q_x D_x + D_x q_x - \alpha D_y\right\}\dfrac{1}{u_{xx}} \end{pmatrix} \tag{3.1}$$

where D_x and D_y (and D_t, that will show up later) are operators of total derivatives with respect to corresponding variables. Operator (3.1) is obviously skew-symmetric. Moreover, a straightforward, though lengthy, calculation shows that it satisfies Jacobi identity. A shorter proof can be given by applying the criterion of P. Olver that involves functional multi-vectors [4]. A much easier way to verify the Jacobi identity is to check closeness of the two-form

$$\Omega = \frac{1}{2}\int du^i \wedge K_{ij} du^j \tag{3.2}$$

using the symplectic operator $K = J_0^{-1}$, inverse to the Hamiltonian operator J_0:

$$K = \begin{pmatrix} q_x D_x + D_x q_x - \alpha D_y & -u_{xx} \\ u_{xx} & 0 \end{pmatrix}. \tag{3.3}$$

The two-form (3.2) becomes

$$\Omega = \frac{1}{2}\int \left[2q_x du \wedge du_x - \alpha du \wedge du_y + 2u_{xx} dq \wedge du\right]. \tag{3.4}$$

It is easy to see that $d\Omega = 0$, so that Ω is a symplectic two-form, that is equivalent to satisfying the Jacobi identity.

The Hamiltonian form of the reduced system (2.5) is

$$\begin{pmatrix} u \\ q \end{pmatrix}_t = J_0 \begin{pmatrix} \delta_u \mathscr{H}_1 \\ \delta_q H_1 \end{pmatrix} \tag{3.5}$$

with the Hamiltonian density

$$H_1 = \frac{1}{2}(q^2 u_{xx} - u_x u_y). \tag{3.6}$$

From now on, $\mathscr{H} = \int_{-\infty}^{+\infty} H dx dy$ will denote an integral of the motion along the flow (2.5), with the conserved density H. Note that the Hamiltonian density (3.6) can be obtained by the symmetry reduction from the Hamiltonian H_1 in [2].

4 Recursion operator for symmetries

Lie equations for symmetries of the reduced system (2.5) have the form

$$\begin{pmatrix} u \\ q \end{pmatrix}_\tau = \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \equiv \Phi \tag{4.1}$$

where φ and $\psi = \varphi_t$ are the components of the symmetry characteristic and τ is a symmetry group parameter. From the Frechét derivative of the flow (2.5), we find the equation that determines its symmetries, $\mathscr{A}(\Phi) = 0$, where the operator $\mathscr{A}$ is

$$\mathscr{A} = \begin{pmatrix} D_t & -1 \\ \frac{Q}{u_{xx}} D_x^2 + \frac{1}{u_{xx}} D_x D_y & D_t - (\frac{2q_x}{u_{xx}}) D_x + \frac{\alpha}{u_{xx}} D_y \end{pmatrix}. \tag{4.2}$$

The recursion operator is defined as an operator that commutes with the operator $\mathscr{A}$ of the symmetry condition $\mathscr{A}(\Phi) = 0$ on solutions of the latter equation and equations (2.5). It is obtained by a symmetry reduction from the recursion operator for the four-dimensional second heavenly system (1.2), that was given in [2], and reads

$$\mathscr{R} = \begin{pmatrix} D_x^{-1}(q_x D_x - \alpha D_y) & -D_x^{-1} u_{xx} \\ Q D_x + D_y & -q_x \end{pmatrix}, \tag{4.3}$$

where D_x^{-1} is the inverse of D_x. The commutator of the recursion operator (4.3) and the operator (4.2) of the symmetry condition has the form

$$[\mathscr{R}, \mathscr{A}] = \begin{pmatrix} D_x^{-1}(q_t - Q)_{xx} - (q_t - Q)_x & D_x^{-1}(u_t - q)_{xx} \\ \frac{1}{u_{xx}} \{(\alpha D_y - 2q_x D_x)(q_t - Q) + Q(u_t - q)_{xx}\} D_x & (q_t - Q)_x \end{pmatrix} \tag{4.4}$$

which implies that $\mathscr{A}$ and $\mathscr{R}$ indeed commute on solutions of (2.5).

5 Second Hamiltonian structure

Second Hamiltonian operator is obtained by applying the recursion operator (4.3) to the first Hamiltonian operator, $J_1 = \mathscr{R} J_0$, with the result

$$J_1 = \begin{pmatrix} D_x^{-1} & -\frac{q_x}{u_{xx}} \\ \frac{q_x}{u_{xx}} & J_1^{22} \end{pmatrix} \tag{5.1}$$

where

$$J_1^{22} = \frac{1}{2} \left[(Q D_x + D_y) \frac{1}{u_{xx}} + \frac{1}{u_{xx}} (D_x Q + D_y) \right] - 2 \frac{q_x}{u_{xx}} D_x \frac{q_x}{u_{xx}}$$
$$+ \frac{\alpha}{2} \left(\frac{q_x}{u_{xx}} D_y \frac{1}{u_{xx}} + \frac{1}{u_{xx}} D_y \frac{q_x}{u_{xx}} \right)$$

This operator is manifestly skew. The proof of the Jacobi identity is again straightforward and lengthy. The calculations are simplified by using P. Olver's criterion, theorem 7.8 in his book [4], formulated in terms of functional multivectors. Moreover, J_0 and J_1 are compatible Hamiltonian operators, that is, every linear combination $\alpha J_0 + \beta J_1$ with constant coefficients α and β satisfies the Jacobi identity. We again note that operator (5.1) could be obtained by a symmetry reduction from the second Hamiltonian operator in [2]. Thus, we obtain the second Hamiltonian form of reduced system (2.5)

$$\begin{pmatrix} u \\ q \end{pmatrix}_t = J_1 \begin{pmatrix} \delta_u \mathscr{H}_0 \\ \delta_q H_0 \end{pmatrix} \tag{5.2}$$

with the Hamiltonian density

$$H_0 = (x+c)qu_{xx} \tag{5.3}$$

where c is a constant. Therefore, the reduced system (2.5) is a bi-Hamiltonian system, that is, it can be written in the two Hamiltonian forms

$$\begin{pmatrix} u \\ q \end{pmatrix}_t = J_0 \begin{pmatrix} \delta_u \mathscr{H}_1 \\ \delta_q H_1 \end{pmatrix} = J_1 \begin{pmatrix} \delta_u \mathscr{H}_0 \\ \delta_q H_0 \end{pmatrix}. \tag{5.4}$$

The second Hamiltonian operator is obtained by acting with the recursion operator $\mathscr{R}$ on the Hamiltonian operator J_0. We could try to generalize this relation as

$$J_n = \mathscr{R}^n J_0 \tag{5.5}$$

and hope that J_n is also a Hamiltonian operator [1]. In the case of (5.1) we have $n = 1$. In particular, if we act with the recursion operator (4.3) on the second Hamiltonian operator J_1, or use (5.5) for $n = 2$, we can generate a new Hamiltonian operator $J_2 = \mathscr{R} J_1 = J_1 J_0^{-1} J_1$. Here we have used the fact that, by construction, $\mathscr{R} = J_1 J_0^{-1}$. By the repeated application of the recursion operator (4.3) to Hamiltonian operators J_0, J_1 and so on, we could obtain multi-Hamiltonian representation of the reduced system (2.5).

6 Symmetries and integrals of motion

Hamiltonian operators provide a natural link between commuting symmetries in evolutionary form [4] and conserved quantities (integrals of motion) that are in involution with respect to Poisson brackets. Our two-component reduced system (2.5) is also a member of an infinite hierarchy of symmetries. Point symmetries generators of (2.5) read

$$
\begin{aligned}
&X_a = \alpha a\partial_t + a\partial_x + \frac{1}{2}(t-\alpha x)^2 a'\partial_u + (t-\alpha x)a'\partial_q\\
&Y_b = b(t-\alpha x)\partial_u + b\partial_q, \quad Z_c = c\partial_u\\
&X_1 = -y\partial_y + u\partial_u + q\partial_q\\
&X_2 = y(3\alpha x - 2t)\partial_t + xy\partial_x - 2y^2\partial_y\\
&+\left[\frac{1}{2}x(\alpha x - t)^2 + yu\right]\partial_u + [x(t-\alpha x) + 3yq]\partial_q\\
&X_3 = \alpha x\partial_t + x\partial_x + u\partial_u + q\partial_q\\
&X_4 = (\alpha x - t)\partial_t - 2y\partial_y + q\partial_q\\
&X_5 = 2y\partial_x + t(t-\alpha x)\partial_u + (2t-\alpha x)\partial_q\\
&X_6 = t\partial_u + \partial_q,\ X_7 = \partial_x,\ X_8 = \partial_y
\end{aligned}
$$

where $a(y), b(y)$ and $c(y)$ are arbitrary functions and the primes denote derivatives with respect to y. The generator of time translations is a combination of these basis generators $\partial_t = (X_{a=1} - X_7)/\alpha$.

Many point symmetries from this list are generated by some integrals of motion, that is, they are variational symmetries [4]. The relation between symmetries and integrals is given by the Hamiltonian form of Noether's theorem

$$
\begin{pmatrix} \hat{\eta}_u \\ \hat{\eta}_q \end{pmatrix} = J_0 \begin{pmatrix} \delta_u \mathscr{H} \\ \delta_q \mathscr{H} \end{pmatrix} \tag{6.1}
$$

where $\mathscr{H} = \int_{-\infty}^{+\infty} H dxdy$ is an integral of the motion along the flow (2.5), with the conserved density H, which generates the symmetry with the two-component characteristic [4] $\hat{\eta}_u, \hat{\eta}_q$. We choose here the Poisson structure determined by our first Hamiltonian operator J_0 since we know its inverse K given by (3.3) which is used in the inverse Noether's theorem

$$
\begin{pmatrix} \delta_u \mathscr{H} \\ \delta_q \mathscr{H} \end{pmatrix} = K \begin{pmatrix} \hat{\eta}_u \\ \hat{\eta}_q \end{pmatrix}, \tag{6.2}
$$

that determines the integral $\mathscr{H}$ corresponding to any known symmetry $\hat{\eta}_u, \hat{\eta}_q$.

We have used (6.2) for reconstructing conserved densities corresponding to all variational point symmetries. Symmetries Z_c, Y_b, and X_a are generated by the integrals

$$
H_c = cqu_{xx} - \alpha c'(y)u, \tag{6.3}
$$

$$
H_b = b\left[(t-\alpha x)qu_{xx} + \frac{1}{2}u_x^2\right] + b'(y)\alpha(\alpha x - t)u, \tag{6.4}
$$

$$
\begin{aligned}
H_a = {} & u_{xx}\left\{q\left[\frac{1}{2}a'(y)(t-\alpha x)^2 - au_x\right] - \frac{\alpha}{2}aq^2\right\}\\
& + \frac{1}{2}a'(y)(t-\alpha x)u_x^2 - \frac{\alpha}{2}a''(y)(t-\alpha x)^2 u.
\end{aligned} \tag{6.5}
$$

Symmetry X_2 is generated by the integral with the conserved density

$$
\begin{aligned}
H_2 = {} & u_{xx}\left\{q\left[\frac{1}{2}x(\alpha x - t)^2 + yu - xyu_x + 2y^2u_y\right] - \frac{1}{2}y(3\alpha x - 2t)q^2\right\}\\
& + (\alpha x - t)u_x\left(yu_y - \frac{1}{2}xu_x\right) + \alpha y^2 u_y^2 - \frac{1}{2}\alpha u^2,
\end{aligned} \tag{6.6}
$$

while symmetry X_5 is generated by the integral

$$H_5 = qu_{xx}(t^2 - \alpha tx - 2yu_x) + \frac{1}{2}(2t - \alpha x)u_x^2 - \alpha y u_x u_y. \tag{6.7}$$

Symmetry X_6 is generated by the integral

$$H_6 = tqu_{xx} + \frac{1}{2}u_x^2. \tag{6.8}$$

For the translational symmetries X_7 and X_8, the conserved densities read

$$H_7 = qu_x u_{xx} + \frac{\alpha}{2}u_x u_y, \quad H_8 = qu_y u_{xx} + \frac{\alpha}{2}u_y^2. \tag{6.9}$$

Note that the Hamiltonian H_1, in (3.6), generates the symmetry ∂_t, that is, our reduced system (2.5). The symmetries X_1, X_3 and X_4 are not variational symmetries because their generating integrals in (6.1) and (6.2) do not exist.

7 A different reduced Hamiltonian system

We obtain a different $2+1$–dimensional two-component Hamiltonian system if we choose for symmetry reduction another combination of the translational generators

$$X_{tr} = \partial_z - \frac{1}{\alpha}\partial_y + \alpha\partial_t, \tag{7.1}$$

where α is an arbitrary real constant. The invariants of X_{tr} are now determined by a characteristic system as

$$X = x, \quad Y = z + \alpha y, \quad T = t - \alpha z, \quad U = u, \quad Q = q. \tag{7.2}$$

Substituting $u = U(X,Y,T)$, $q = Q(X,Y,T)$ into the original system (1.2) and again renaming $U \to u$, $Y \to y$, $T \to t$ we obtain a different two-component reduced system

$$u_t = q, \quad q_t = \frac{1}{u_{xx}}\left[q_x^2 + \alpha(q_x - q_y) - u_{xy}\right] \equiv Q'. \tag{7.3}$$

The first Hamilton operator for the reduced system (7.3) has an obviously skew-symmetric form

$$J_0 = \begin{pmatrix} 0 & \frac{1}{u_{xx}} \\ -\frac{1}{u_{xx}} & \frac{1}{u_{xx}}\left\{q_x D_x + D_x q_x + \alpha(D_x - D_y)\right\}\frac{1}{u_{xx}} \end{pmatrix}. \tag{7.4}$$

The inverse operator $K = J_0^{-1}$ reads

$$K = \begin{pmatrix} q_x D_x + D_x q_x + \alpha(D_x - D_y) & -u_{xx} \\ u_{xx} & 0 \end{pmatrix} \tag{7.5}$$

and determines the symplectic two-form

$$\Omega = \frac{1}{2}\int[(2q_x + \alpha)du \wedge du_x - \alpha du \wedge du_y + 2u_{xx}dq \wedge du]. \tag{7.6}$$

Obviously, Ω is a closed form, $d\Omega = 0$, that is equivalent to J_0 satisfying the Jacobi identity.

Reduced system (7.3) has the Hamiltonian form

$$\begin{pmatrix} u \\ q \end{pmatrix}_t = J_0 \begin{pmatrix} \delta_u \mathscr{H}_1 \\ \delta_q H_1 \end{pmatrix} \tag{7.7}$$

with the same Hamiltonian density (3.6) as for the first reduced system (2.5).

A recursion operator for symmetries of reduced system (7.3) has the form

$$\mathscr{R} = \begin{pmatrix} D_x^{-1}[(q_x+\alpha)D_x - \alpha D_y] & -D_x^{-1}u_{xx} \\ & \\ Q'D_x + D_y & -q_x \end{pmatrix}. \tag{7.8}$$

Applying the recursion operator (4.3) to the first Hamiltonian operator, we obtain the second Hamiltonian operator $J_1 = \mathscr{R}J_0$:

$$J_1 = \begin{pmatrix} D_x^{-1} & -\dfrac{q_x}{u_{xx}} \\ & \\ \dfrac{q_x}{u_{xx}} & J_1^{22} \end{pmatrix} \tag{7.9}$$

where

$$J_1^{22} = \frac{1}{2}\left[(Q'D_x + D_y)\frac{1}{u_{xx}} + \frac{1}{u_{xx}}(D_xQ' + D_y)\right] - 2\,\frac{q_x}{u_{xx}}D_x\frac{q_x}{u_{xx}}$$
$$-\frac{\alpha}{2}\left[\frac{q_x}{u_{xx}}(D_x - D_y)\frac{1}{u_{xx}} + \frac{1}{u_{xx}}(D_x - D_y)\frac{q_x}{u_{xx}}\right]$$

It is easy to see that this operator coincides with the second Hamiltonian operator for the first reduced system (2.5).

However, we have failed to find a local Hamiltonian density which together with the second Hamiltonian operator J_1 would generate the new reduced Hamiltonian system (7.3). Another important difference is that, as opposed to the first reduced system, the new operators J_0, J_1, and $\mathscr{R}$ cannot be obtained from the corresponding operators of the four-dimensional system (1.2) by the symmetry reduction with respect to generator (7.1). The same is true for the Hamiltonian density H_1.

8 Conclusions

We have shown that a certain symmetry reduction of the $3+1$–dimensional second heavenly equation, taken in a two-component from, yields a two component $2+1$–dimensional multi-Hamiltonian integrable system. For this system, we have presented explicitly two Hamiltonian operators, a recursion operator for symmetries, a complete set of point symmetries and corresponding integrals of the motion. We have also shown that for a different symmetry reduction the reduced system is ”almost bi-Hamiltonian”, with two known Hamiltonian operators but the second Hamiltonian density missing.

The first impression of the major part of this paper could be that it is an easy and even trivial task to obtain a three-dimensional multi-Hamiltonian system by a symmetry reduction of the original four-dimensional second heavenly system. All the main objects, $J_0, K, \mathscr{A}, \mathscr{R}$ and H_1, H_0 could be obtained by the symmetry reduction. However, the results of the last section show that even a slight change in a symmetry chosen for the reduction, a different combination of translational symmetries in this case, ruins all these properties and creates a difficulty in discovering bi-Hamiltonian structure of the reduced system (second Hamiltonian density H_0). If we choose more general symmetries for the reduction, for example from the optimal system of one-dimensional subalgebras from [3], then we shall be unable to discover even a single Hamiltonian structure of reduced systems. The problem of conservation of multi-Hamiltonian structure under symmetry reductions seems to be an important and interesting subject for a future research.

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