Turing computability of the Solution Operator of a higher order modified Camassa-Holm equation

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Abstract. In this paper, we mainly discuss the Turing computability of the solution operator of a higher order modified Camassa-Holm equation. Firstly, we transform the equation to its integral equation by Duhamel principle. Then applying the TTE theory, we prove that the solution operator of the integral equation is computable in a short interval. Finally, by constructing computable function, we extend the solution from partial internal to the entire space. The result enlarge the application in computing differential equations on digital computers.

Introduction

Camassa-Holm equation is an important equation. In the shallow water wave theory, different nonlinear completely integrable partial differential equation can be obtained by different levels of approximation. These equations with soliton solution, that is, they tend to be the determine constant in the infinity of space. Not disappear when they collide with each other and waveform, and speed also have no or only a small change. In addition, they show similar particle scattering behavior.

In this paper, for $m \geq 3$ odd, we will prove that the solution operator of the initial problem(1) is computable by the TTE theory [1-3].

$$\begin{cases} \partial_t u + \partial_x^m u + \dfrac{1}{2}\partial_x\left(u^2\right) + \left(1-\partial_x^2\right)^{-1}\partial_x\left[u^2 + \dfrac{1}{2}\left(\partial_x u\right)^2\right] = 0 \\ u\left(x,0\right) = \varphi\left(x\right), t \in R, x \in R. \end{cases} \tag{1}$$

Erika A.Olson show in [4] that the periodic initial value problem for Eq.(1) is well posed.

At present, the computability of solutions of the nonlinear evolution equations have become an important topic to the workers of physics and mathematics [5-6]. Researching boundedness and computability of the solutions of the nonlinear equations will offer effective tools for the application of equations, enrich theoretical foundation of computer science and promote the development of computer software.

The structure of the article is that: In part 2, we introduce some basic definitions and lemmas, which are relevant to the proof of part3; In part 3, we prove the main theorem of the paper mainly.

Preliminaries

Definition 1[4] For any $\forall s, b \in R$, $X^{s,b}$ denotes the completion of the Schwartz space $S\left(R^2\right)$ with respect to the norm

$$\|u\|_{s,b}^{'} = \left(\int_R\int_R \left(1+|\xi|\right)^{2s}\left(1+\left|\tau-\xi^m\right|\right)^{2b}\left|\hat{u}\left(\xi,\tau\right)\right|^2 d\xi d\tau\right)^{\frac{1}{2}}.$$

Definition 2[4] In the sequel, let $\psi = \psi(t) \in C_0^\infty(-1,1)$ be a cut-off function with $0 \le \psi \le 1$ and $\psi(t) \equiv 1$ for $|t| < \frac{1}{2}$.

Definition 3[4] For $0 < \delta < 1$, define $\psi_\delta(t) = \psi\left(\frac{t}{\delta}\right), T_\delta u = \psi_\delta T(\psi_\delta u)$.

Lemma 1[1] **(type conversion)**

Let $\delta_i :\subseteq \Sigma^\omega \to X_i$ be a representation of the set $X_i\ (0 \le i \le k)$, $f_i :\subseteq X_1 \times \cdots \times X_k \to X_0$, and define $L(x_1, \cdots, x_{k-1})(x_k) := f(x_1, \cdots, x_k)$, then if f is $(\delta_1, \cdots, \delta_k, \delta_0)$-computable if and only if L is $(\delta_1, \cdots, \delta_{k-1}, [\delta_k \to \delta_0])$-computable.

Lemma 2[1] Let $\gamma :\subseteq Y \to M$ and $\gamma' :\subseteq Y \to M'$ are two representations, v_N is admissible representation of N. Then we have the following propositions:

1) If $f :\subseteq M \to M'$ is (γ, γ')- computable, then $f' :\subseteq N \times M' \times M \to M'$ is $(v_N, \gamma', \gamma, \gamma')$- computable.

We define a function $g' :\subseteq N \times M \to M'$ as follow:

$$g'(0,x) = f(x), g'(n+1,x) = f'(n, g'(n,x), x),$$

where $x \in M$, $n \in N$, then g' is (v_N, γ, γ')- computable.

2) Assuming that $h :\subseteq M \to M$ is (γ, γ)- computable, Define a function

$$H :\subseteq N \times M \to M$$

$$H(0,x) = x, H(n+1,x) = h \circ H(n,x) = h^{n+1}(x).$$

So, the function H is (v_N, γ, γ)- computable.

Lemma 3[4] Given $s \in \left(\frac{1}{2} \cdot \frac{m^2 - 3m + 1}{m^2 - 3m + 2}, \frac{1}{2}\right)$, there exists $\beta > 0$ such that if $b \in \left(\frac{1}{2}, \frac{1}{2} + \beta\right]$, $b' \in \left[\frac{1}{2} - \beta, \frac{1}{2}\right)$, and $b + b' \le 1$, then there exists $c > 0$ such that for any $f, g \in X^{s,b'}$, $\|w_{fg}\|'_{s,b-1} \le c\|f\|'_{s,b'}\|g\|'_{s,b'}$, where w_{fg} satisfies

$$\hat{w}_{fg}(\xi,\tau) \sim \left(\xi + \frac{2\xi}{1+\xi^2}\right)\hat{f} * \hat{g}(\xi,\tau) + \frac{\xi}{1+\xi^2}\widehat{\partial_x f} * \widehat{\partial_x g}(\xi,\tau).$$

Main Result

From the problem (1), we establish a nonlinear map $K_R : H^s \to C(R; H^s(R))$, which translate the initial data φ to the solution u. The map K_R is the solution operator of the initial problem.

Theorem 3.1 For any $t \in R$, when $\frac{1}{2} \cdot \frac{m^2 - 3m + 1}{m^2 - 3m + 2} < s < \frac{1}{2}$, the solution operator $K_R : H^s \to C(R; H^s(R))$ is $(\delta_{H^s}, [\rho \to \delta_{H^s}])$-computable, where $m \ge 3$ odd.

Proof Defining $w = \frac{1}{2}\partial_x(u^2) + (1 - \partial_x^2)^{-1}\partial_x\left[u^2 + \frac{1}{2}(\partial_x u)^2\right]$, We can get Eq.(1) equivalent integral equation by Duhamel principle:

$$u(x,t) = W(t)\varphi(x) - \int_0^t W(t-\tau)w(x,\tau)d\tau. \tag{2}$$

where $W(t)\varphi(x) := \int_R \hat{\varphi}(\xi) e^{i(\xi x+\xi^m t)} d\xi$.

For $\varphi \in H^s\left(\frac{1}{2}.\frac{m^2-3m+1}{m^2-3m+2} < s < \frac{1}{2}\right)$, define operator

$$S(u,\varphi,t) = \psi(t)W(t)\varphi(x) - \psi_\delta(t)\int_0^t W(t-\tau)w(x,\tau)d\tau ,$$

$$\bar{S}(u,\varphi)(t) := S(u,\varphi,t).$$

According the Lemma 3.2 in [1], it is easy to prove S is $([\rho \to \delta_s], \delta_s, \rho, \delta_s)$-computable.

By lemma 1, $\bar{S}$ is $([\rho \to \delta_s], \delta_s, [\rho \to \delta_s])$-computable.

Define function $v : S(R) \times N \to C(R; S(R))$

$$v(\varphi,0) = \bar{S}(0,\varphi),\ v(\varphi, j+1) = \bar{S}(v(\varphi, j),\varphi).$$

The function v is defined by primitive recursion from computable functions. By Lemma 2 v is $(\delta_s, v_N, [\rho \to \delta_s])$-computable.

Let $\omega(x,t) = u(x,t_0+t)$, where $t \in [0,T]$, $t_0 \geq 0$.

If $u(x,t_0) = \mu(x)$, then

$$\begin{cases} \partial_t \omega + \partial_x^m \omega + \frac{1}{2}\partial_x(\omega^2) + (1-\partial_x^2)^{-1}\partial_x\left[\omega^2 + \frac{1}{2}(\partial_x \omega)^2\right] = 0 \\ \omega(x,0) = \mu(x), t \in R, x \in R. \end{cases} \tag{3}$$

We assume that the initial value $\mu \in H^s(R)$ is given by a $\tilde{\delta}_{H^s}$ -name , i.e., $p = \langle p_0, p_1, \cdots \rangle$ which is obtained by $\delta_{sc}(p_i) = \mu_i$ and $\|\mu_n - \mu\|_s \leq 2^{-n-2}$. For $\forall k \in N$, there exist appropriate computable n_k satisfying $\|\mu_{n_k} - \mu\|_{H^s} \leq 2^{-n_k-2} \leq 2^{-k-2}$.

Define

$$\omega_n^0 := \bar{S}(0,\mu_n),\quad \omega_n^{j+1} := \bar{S}(\omega_n^j, \mu_n).$$

It is easy to prove $\omega_n^j \to \omega_n \quad (j \to \infty)$, there ω_n satisfies the following integral equation:

$$\omega_n(t) = \bar{S}(\omega_n, \mu_n) = \psi(t)W(t)\mu_n(x) - \psi_\delta(t)\int_0^t W(t-\tau)w(x,\tau)d\tau,$$

where $w = \frac{1}{2}\partial_x(\omega_n{}^2) + (1-\partial_x^2)^{-1}\partial_x\left[\omega_n{}^2 + \frac{1}{2}(\partial_x \omega_n)^2\right]$.

Since $\omega_n^j \to \omega_n \quad (j \to \infty)$, we can select suitable integer n_k, j_k to constrct a sequence $\{\omega_{n_k}^{j_k}\}_{k\in N}$, satisfying $\|\omega_{n_k}^{j_k} - \omega_{n_k}\|_s \leq 2^{-k-1}$.

In the following , we prove $\{\omega_{n_k}^{j_k}\}_{k\in N}$ fastly converges to ω.

By the Lemma 3, we obtain

$$\|\omega_{n_k} - \omega\|'_{s,b} = \left\|\psi(t)W(t)(\mu_{n_k}(x) - \mu(x)) - \psi_\delta(t)\int_0^t W(t-\tau)\left[w_{\omega_{n_k}}(x,\tau) - w_\omega(x,\tau)\right]d\tau\right\|'_{s,b}$$

$$\leq \|\psi(t)W(t)(\mu_{n_k}(x) - \mu(x))\|'_{s,b} + \left\|\psi_\delta(t)\int_0^t W(t-\tau)\left[w_{\omega_{n_k}}(x,\tau) - w_\omega(x,\tau)\right]d\tau\right\|'_{s,b}$$

$$\leq \|\psi(t)W(t)(\mu_{n_k}(x) - \mu(x))\|'_{s,b} + \left\|\psi_\delta(t)\int_0^t W(t-\tau)w_{fg}(x,\tau)d\tau\right\|'_{s,b}$$

$$\leq C_1\|\mu_{n_k}(x) - \mu(x)\|_{H^s} + C\delta^{\frac{1}{2}-b}\|\omega_{n_k} + \omega\|'_{s,b}\|\omega_{n_k} - \omega\|'_{s,b}$$

$\le C_1 2^{-k-2} + 2rC\delta^{\frac{1}{2}-b} \left\| \omega_{n_k} - \omega \right\|'_{s,b}$, where $f = \omega_{n_k} + \omega, g = \omega_{n_k} - \omega$.

Choosing sufficient small δ such that $\dfrac{C_1}{1 - 2rC\delta^{\frac{1}{2}-b}} < 2$, then $\left\| \omega_{n_k} - \omega \right\|'_{s,b} \le 2^{-k-1}$.

Thus $\left\| \omega_{n_k}^{j_k} - \omega \right\|'_{s,b} \le \left\| \omega_{n_k}^{j_k} - \omega_{n_k} \right\|'_{s,b} + \left\| \omega_{n_k} - \omega \right\|'_{s,b} \le 2^{-k-1} + 2^{-k-1} \le 2^{-k}$.

Then we have proved $\left\{ \omega_{n_k}^{j_k} \right\}_{k \in N}$ fastly converges to ω and ω is computable.

We known $\left\{ \omega_{n_k}^{j_k} \right\}_{k \in N}$ is computable sequence , if $\delta_S(q_k) = \omega_{n_k}^{j_k}(t)$, then $\tilde{\delta}_{H^s} \langle q_0, q_1, \ldots \rangle = \omega(t)$, i.e., $\langle q_0, q_1, \ldots \rangle$ is the $\tilde{\delta}_{H^s}$-name of $\omega(t)$. Hence the solution ω of the initial problem (3) is computable on $t \in [0, T]$, that is solution operator map S is computable.

Define a $\left(\rho, \tilde{\delta}_{H^s}, \rho, \tilde{\delta}_{H^s}\right)$-computable map $P: (t_0, \mu, t) \to u(t), t \in [t_0, t_0 + T]$,

where $\omega(t_0) = \mu$, $\omega(t)$ is the solution of the initial problem (1) on $t \in [t_0, t_0 + T]$.

Then we prove the solution $u(n \cdot T)$ is computable.

Define function H: $H(\varphi, n) = u(nT)$

$$H(\varphi, 0) = \varphi, H(\varphi, n+1) = P(nT, H(\varphi, n), (n+1)T).$$

Then H is computable since H is derived by primitive recursion from computable function P.

In the end, we prove $u(t)$ is computable. let $n \cdot T \le t \le (n+1) \cdot T$, we first compute $u(n \cdot T)$, then compute $P(nT, u(nT), t)$, so $u(t) = P(nT, u(nT), t)$ is computable.

In this way, we have get the computable solution on $t \in R$. For $\forall t \in R$, $\frac{1}{2} \cdot \frac{m^2 - 3m + 1}{m^2 - 3m + 2} < s < \frac{1}{2}$, the solution operator $K_R : H^s \to C(R; H^s(R))$ of the higher order modified Camassa-Holm equation (1) is $(\delta_s, [\rho \to \delta_s])$ -computable.

Summary and Outlook

The paper study computability of the solution operator of the higher order modified Camassa-Holm equation. On the basis of computablity theory, whether problem can be implemented on computer is an important problem. Computational complexity theory just can be used to solve the problem. The topic we will study in the future.

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