Turing-Computability of Solution of Hirota Equation

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Abstract. In this paper, we mainly discuss the Turing-computable of the Solution operation of Hirota Equation. Firstly, we prove the existence and uniqueness of the solution operator of equation by principle of contraction mapping, and acclaim that its local solution is Turing-Computable in use of TTE theory. By computable functions constructing, we extend the solution from the internal to the entire space. Then the solution of this equation is Turing-computable. The result enlarges the application in computing differential equations on digital computers.

1. Introduction

Differential equation is an important mathematical model of real problems, but not all have good properties of solutions of differential equations. In reality, we are concerned about how the solution of this equation is calculated. As a result, the computability of solutions for operator becomes a key issue in the study. K. Weihrauch and N. Zhong [1-2] have studied the computability of the solution operator of the wave equation and KDV equation. Dianchen Lu and others have studied the computability of the nonlinear Kawahara equation [3]. In this paper, we will study the initial problem of Hirota equation:

\[
\begin{align*}
\partial_t u + i\alpha \partial_x^3 u + \beta \partial_x^3 u + \mu \partial_x(u^2 u) + i\gamma |u|^2 u &= 0, & x, t \in \mathbb{R}, \\
u(x,0) &= u_0(x), & x \in \mathbb{R}
\end{align*}
\]

where \(\alpha, \beta, (\mu, \gamma)\) are real (complex) constants and \(\alpha \beta \neq 0\), \(u\) is complex valued function. Hirota Equation (1) is a typical model in mathematical physics, which encompasses the well-know nonlinear Schrödinger equation. Hasegawa and Kodama [4-5] proposed (1) as a model for propagation of pulse in optical fiber.

The Cauchy problem of (1) changes as follows if \(\mu = 0\),

\[
\begin{align*}
\partial_t u + i\alpha \partial_x^3 u + \beta \partial_x^3 u + i\gamma |u|^2 u &= 0, & x, t \in \mathbb{R}, \\
u(x,0) &= u_0(x), & x \in \mathbb{R}
\end{align*}
\]

The structure of the article is that: In Part 2, we mainly introduce some basic definitions and lemmas a, which are relevant to the proof of Part 3; In Part 3, we prove the main theorem of the paper mainly.

2. Preliminaries

Lemma 1[1] (type conversion) Let \(\delta_i \subseteq \sum^w \rightarrow X_i\) be a representation of the set \(X_i\), \(f_i \subseteq X_1 \times \cdots \times X_k \rightarrow X_0\), define \(L(x_1, \cdots, x_k) = f(x_1, \cdots, x_k)\), then if \(f\) is \((\delta_1, \cdots, \delta_k, \delta_0)\)-computable if and only if \(L\) is \((\delta_1, \cdots, \delta_k, [\delta_k \rightarrow \delta_0])\)-computable.

Lemma 2[1] Let \(\gamma \subseteq Y \rightarrow M\) and \(\gamma' \subseteq Y \rightarrow M'\) are two representations, \(\nu_N\) is admissible representation of \(N\). Then we have the following propositions:
1) If $f \subseteq M \rightarrow M'$ is $(\gamma, \gamma')$-computable, then $f' \subseteq N \times M' \times M \rightarrow M'$ is $(v_N, \gamma', \gamma', \gamma) \cdot$-computable.

We define a function $g' \subseteq N \times M \rightarrow M'$ as following:

$$g'(0, x) = f(x), \ g'(n+1, x) = f'(n, g(n, x), x),$$

Where $x \in M$, $n \in N$, then $g'$ is $(v_N, \gamma', \gamma') \cdot$-computable.

2) Assume that $h \subseteq M \rightarrow M$ is $(\gamma, \gamma)$-computable and define a function

$$H \subseteq N \times M \rightarrow M : \quad H(0, x) = x, H(n+1, x) = h \circ H(n, x) = h^{\circ n}(x).$$

So, the function $H$ is $(v_N, \gamma, \gamma) \cdot$-computable.

**Definition 1** For $s, \tau \in R$, the Bourgain space $X_{s, b}$ is the complete space about Schwartz space $S(R^2)$ with the following norm:

$$\|u\|_{X_{s, b}} = \|\langle \xi \rangle^s \langle \tau - \beta \xi^3 - \alpha \xi^2 \rangle^b Fu\|_{L^2(\xi \tau)}$$

where $\langle \cdot \rangle = (1 + |\cdot|)$. $\|u\|_{X_{s, b}} = \|f\|_{X_{s, b}}$.

**Lemma 3** If $s \geq -\frac{1}{4}, \frac{1}{2} < b < \frac{7}{12}, b' > \frac{1}{2}$. Then

$$\|u \|_{X_{s, b}} \leq C\|u \|_{X_{s, b}}, \|u \|_{X_{s, b}}, \|u \|_{X_{s, b}}. \quad (3)$$

Let $\psi \in C^\infty_0(R)$ with $\psi = 1$ on $[-\frac{1}{2}, \frac{1}{2}]$ and $\sup p\psi \subset [-1, 1]$. We denote $\psi_\delta(\bullet) = \psi(\delta^{-1}(\bullet))$ for some non-zero $\delta \in R$.

**Lemma 4** If $s \geq -\frac{1}{4}, \frac{1}{2} < b < b' < 1, 0 < \delta \leq 1$. Then

$$\|\psi_\delta(t)S(t)\|_{X_{s, b}} \leq C\delta^{b-b'}\|\phi\|_{H^s}, \quad (4)$$

$$\|\psi_\delta(t)F\|_{X_{s, b}} \leq C\delta^{b-b'}\|F\|_{X_{s, b}}, \quad (5)$$

$$\|\psi_\delta(t)\int_0^t S(t-\tau)F(\tau)dt\|_{X_{s, b}} \leq C\delta^{\frac{1}{2}+b}\|F\|_{X_{s, b}}, \quad (6)$$

$$\|\psi_\delta(t)\int_0^t S(t-\tau)F(\tau)\psi_\delta(t-\tau)dt\|_{L^2(\xi \tau)} \leq C\delta^{\frac{1}{2}+b}\|F\|_{X_{s, b}}, \quad (7)$$

**Lemma 5** If $s \geq -\frac{1}{4}, \frac{1}{2} < b < b' < 1, 0 < \delta \leq 1$. Then

$$\|u - \varphi\|_{X_{s, b}} \leq \delta^{b-b'}(\|u\|_{X_{s, b}} + \|\varphi\|_{X_{s, b}})\|u - \varphi\|_{X_{s, b}}. \quad (8)$$

### 3. Main Result

From the problem (2), we establish a nonlinear map $K_R : H^s \rightarrow C(\mathbb{R}; H^s(\mathbb{R}))$, which translate the initial data $u_0$ to the solution $u$. The map $K_R$ is the solution operator of the problem (2).

**Theorem 1** $\forall t \in R$, when $s > -\frac{1}{4}$, the solution operator of the problem (2) $K_R : H^s \rightarrow C(\mathbb{R}; H^s(\mathbb{R}))$ is $(\delta_{H^s}, [\rho \rightarrow \delta_{H^s}])$-computable.

**Proof** We can get problem (2) equivalent integral equation by the Duhamel principle.
\[ u(x,t) = S(t)u_0 - i\int_0^t S(t-t')\gamma |u(t')|^2 u(t')dt', \]

where \( S(t) = F_X^{-1}e^{i(\alpha x^2 + \beta t^2)}F_X \), \( F_X \) represents fourier transform.

For \( u_0 \in H^s \left( s \geq -\frac{1}{4} \right) \), define operator

\[ G(u, u_0, t) = \psi_i(t) \left( S(t)u_0 - i\int_0^t S(t-t')\psi_{i\epsilon}(t')\gamma |u(t')|^2 u(t')dt' \right), \]

\[ \overline{G}(u, u_0)(t) = G(u, u_0, t). \]

According the lemma 3.2 in [1], it is easy way to prove the operator \( G \) is \( ([\rho \to \delta_\gamma], \delta_\gamma, \rho, \delta_\gamma) \)-computable. By lemma 1, the operator \( \overline{G} \) is \( ([\rho \to \delta_\gamma], \delta_\gamma, [\rho \to \delta_\gamma]) \)-computable.

Define function \( v: S(R) \times N \to C(R; S(R)), \)

\[ v(u_0, 0) = \overline{G}(0, u_0), v(u_0, j + 1) = \overline{G}(v(u_0, j), u_0). \]

By Lemma 2 \( v \) is \( (\delta_\gamma, v, [\rho \to \delta_\gamma]) \)-computable.

Let \( \omega(x, t) = u(x, t_0 + t), t \in [0, T], t_0 \geq 0. \)

If \( u(x, t_0) = \phi(x) \), then

\[
\begin{aligned}
\{ \partial_t + i\alpha \partial_x^2 \omega + \beta \partial_x^3 \omega + iy |\omega|^2 \omega = 0, & \quad x, t \in R, \\
\omega(x, 0) = \phi(x), & \quad x \in R.
\end{aligned}
\]

We assume that the initial value \( \phi \in H^s(R) \) is given by a \( \tilde{\delta}_H \) -name, i.e., \( p = \langle p_0, p_1, \cdots \rangle \) which is obtained by \( \delta_{\omega_1}(p_i) = \omega_i \) and \( \|\phi_n - \phi\| \leq 2^{-n-2}. \) For \( \forall k \in N \), there exists computable \( n_k \) satisfying \( \|\phi_{n_k} - \phi\|_{H^s} \leq 2^{-n_k-2} \leq 2^{-k-2}. \)

We define functions

\[ \omega_n^0 := \overline{G}(0, \phi_n), \quad \omega_n^{j+1} := \overline{G}(\omega_n^j, \phi_n). \]

It is easy to prove \( \omega_n^j \to \omega_n \quad (j \to \infty), \) there \( \omega_n \) satisfies the following integral equation:

\[
\begin{aligned}
\{ \frac{\partial \omega_n}{\partial t} + i\alpha \frac{\partial^2 \omega_n}{\partial x^2} + \beta \frac{\partial^3 \omega_n}{\partial x^3} + iy |\omega_n|^2 \omega_n = 0, & \quad x, t \in R, \\
\omega_n(x, 0) = \mu_n(x), & \quad x \in R.
\end{aligned}
\]

Since \( \lim_{j \to \infty} \omega_n^j = \omega_n \quad (j \to \infty), \) we can select suitable integer \( n_k, j_k \) to contract a sequence \( \{ \omega_{n_k}^j \}_{j \in N} \) satisfying \( \|\omega_{n_k}^j - \omega_{n_k}\| \leq 2^{-j-k}. \) Then \( \{ \omega_{n_k}^j \}_{j \in N} \) is computable sequence.

In following, we prove \( \{ \omega_{n_k}^j \}_{j \in N} \) fastly converges to \( \omega. \)

From (3)-(8), we obtain
Choosing sufficient small $\delta$ such that $\frac{C_1}{1-4CrC_4\delta^{\nu-b}} < 2$, then
\[
\left\| \omega_n - \omega \right\|_{X,\frac{1}{2}} \leq 2^{-k-1}.
\]

Thus
\[
\left\| \omega_n - \omega \right\|_{X,\frac{1}{2}} \leq \left\| \omega_n - \omega \right\|_{X,\frac{1}{2}} + \left\| \omega_n - \omega \right\|_{X,\frac{1}{2}} \leq 2^{-k-1} + 2^{-k-1} \leq 2^{-k}.
\]

Then we have proved \( \{ \omega_n \}_{n=N} \) fastly converges to $\omega$ and $\omega$ is computable.

We have known \( \{ \omega_n \}_{n=N} \) is computable sequence, if $\delta_n(q_s) = \omega_n(t)$, then $\widetilde{\delta}_{\mu'}(q_0, q_1, \cdots) = \omega(t)$, i.e., $\langle q_0, q_1, \cdots \rangle$ is the $\widetilde{\delta}_{\mu'}$-name of $\omega(t)$. Hence the solution $\omega$ of the initial problem (9) is computable on $[0, t]_T$.

We define a \((\rho, \delta_{\mu'}, \rho, \delta_{\mu'})\)-computable map $I : (t_0, u_0, t) \rightarrow u(t), t \in [t_0, t_0 + T]$ where $\omega(t_0) = u_0$, $\omega(t)$ is the solution of the initial problem (2) on $\mathbb{R}$.

Then we prove the solution $u(n \cdot T)$ is computable. The function $H : H(\phi, n) = u(nT)$ defined by
\[
H(u_0, 0) = u_0, \quad H(u_0, n+1) = I(nT, H(u_0, n), (n+1)T).
\]

It is easy to prove the function $H$ is computable since $H$ is derived by primitive recursion from computable function $I$.

In the end, we prove $u(t)$ is computable. Let $n \cdot T \leq t \leq (n+1) \cdot T$, we firstly compute $u(n \cdot T)$, then compute $I(nT, u(nT), t)$, so $u(t) = I(nT, u(nT), t)$ is computable.

In this way, we have get the computable solution on $t \in \mathbb{R}$. For $\forall t \in \mathbb{R}, s \geq -\frac{1}{4}$, the solution operator $K_s : H^s \rightarrow C(R; H^s(R))$ of Hirota equation (2) is $(\delta_s, [\rho \rightarrow \delta_s])$-computable.

Similarly, we can prove the solution operator of Hirota equation is computable.

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References