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Hopf Bifurcations in a Watt Governor with a Spring

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Abstract

This paper pursues the study carried out in [10], focusing on the codimension one Hopf bifurcations in the hexagonal Watt governor system. Here are studied Hopf bifurcations of codimensions two, three and four and the pertinent Lyapunov stability coefficients and bifurcation diagrams. This allows to determine the number, types and positions of bifurcating small amplitude periodic orbits. As a consequence it is found an open region in the parameter space where two attracting periodic orbits coexist with an attracting equilibrium point.

1 Introduction

The centrifugal governor is a device that automatically controls the speed of an engine. The most important one, the Watt governor, a landmark of the Scientific Revolution, is taken as the starting point for the Theory of Automatic Control. The centrifugal governor design received several important modifications as well as other types of governors were also developed. From MacFarlane [5], p. 251, we quote:
"*Several important advances in automatic control technology were made in the latter half of the 19th century. A key modification to the flyball governor was the introduction of a simple means of setting the desired running speed of the engine being controlled by balancing the centrifugal force of the flyballs against a spring, and using the preset spring tension to set the running speed of the engine"*.

In this paper the system coupling the Watt governor with a spring, defined in section 2, will be abbreviated as WGSS.

The stability analysis of the stationary states and small amplitude oscillations of this system will be pursued here. The case of no spring, the standard Watt Governor System (WGS) has been considered by the authors in [9], where several historical facts on the subject have been mentioned.

Landmarks of the mathematical analysis of the stability conditions at the equilibrium of the WGS are Maxwell [6], Vyshnegradskii [13] and Pontryagin [7].

From the mathematical point of view, the oscillatory, small amplitude, behavior in the WGS can be associated to a periodic orbit that appears from a Hopf bifurcation at the above mentioned equilibrium. This was established by Hassard et al. in [2], Al-Humadi and Kazarinoff in [1] and by the authors in [8, 9].

In [8] we characterized the surface of Hopf bifurcations in a WGS, which is more general than that presented by Pontryagin [7] and Al-Humadi and Kazarinoff [1]. In [9] restricting ourselves to Pontryagin's system of differential equations for the WGS, we carried out a deeper investigation of the stability of the equilibrium along the critical Hopf bifurcations up to codimension 3, happening at a unique point at which the bifurcation diagram was established. A conclusion derived from the diagram implied the existence of parameters where the WGS has an attracting periodic orbit coexisting with an attracting equilibrium. In [10] we characterized the hypersurface of Hopf bifurcations in a WGSS. See Theorem 2 and Fig. 2 for a review of the critical surface where the first Lyapunov coefficient vanishes.

In the present paper we go deeper investigating the stability of the equilibrium along the above mentioned critical surface. To this end the second Lyapunov coefficient is calculated and it is established that it vanishes along two curves. The third Lyapunov coefficient is calculated on these curves and it is established that it vanishes at a unique point. The fourth Lyapunov coefficient is calculated at this point and found to be negative. See Theorem 3. The pertinent bifurcation diagrams are established. This leads to the existence of an open set in the space of parameters where two attracting periodic orbits coexist with an attracting equilibrium.

This paper is organized as follows. In Section 2 we introduce the differential equations that model the WGSS. The Hopf bifurcations in the WGSS differential equations are studied in Sections 3 and 4. Expressions for the second, third and fourth Lyapunov coefficients, which fully clarify their sign, are obtained, pushing forward the method found in the works of Kuznetsov [3, 4].

The extensive calculations involved in Theorem 3 have been corroborated with the software MATHEMATICA 5 [15], the main computational steps have been posted in the site [14] and the expressions, too long to be displayed in print here, appear in [11].

2 The Watt governor system with a spring

The WGSS studied in this paper is shown in Fig. 1. There, $\varphi \in \left(0, \frac{\pi}{2}\right)$ is the angle of deviation of the arms of the governor from its vertical axis S_1, $\Omega \in [0, \infty)$ is the angular velocity of the rotation of the engine flywheel D, θ is the angular velocity of the rotation of S_1, l is the length of the arms, m is the mass of each ball, H is a sleeve which supports the arms and slides along S_1, T is a set of transmission gears and V is the valve that determines the supply of steam to the engine. The spring along S_1, is compressed as the arms raise, i.e. as φ increases.

The WGSS differential equations can be found as follows. For simplicity, we neglect the mass of the sleeve and the arms. There are four forces acting on the balls at all times. They are the tangential component of the gravity $-mg\sin\varphi$, where g is the standard acceleration of gravity; the tangential component of the centrifugal force $m\, l\, \sin\varphi\, \theta^2 \cos\varphi$; the tangential component of the restoring force due to the spring $-2kl(1-\cos\varphi)\sin\varphi$, $2l$ is the natural length of the spring and $k \geq 0$ is the spring elasticity constant; and the force of friction $-bl\dot{\varphi}$, $b > 0$ is the friction

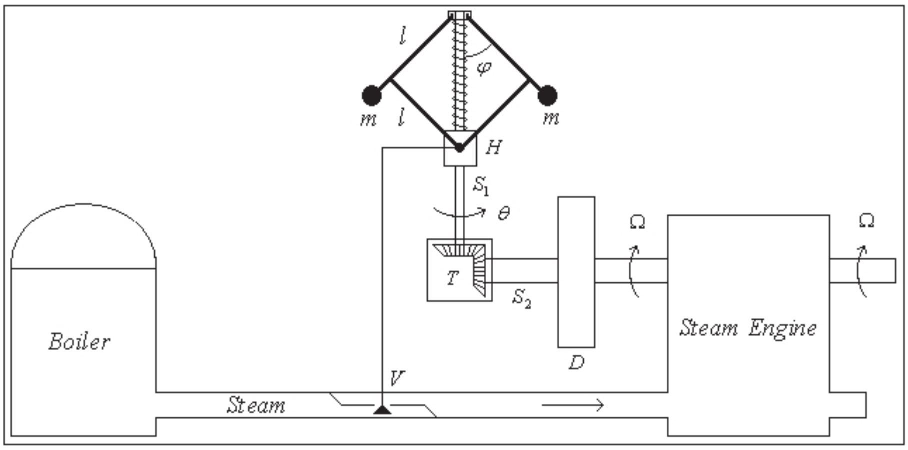

Figure 1: Watt centrifugal governor with a spring – steam engine system.

coefficient.

From Newton's Second Law of Motion, and using the transmission function $\theta = c\,\Omega$, where $c > 0$, one has

$$\ddot{\varphi} = \left(\frac{2k}{m} + c^2\Omega^2\right)\sin\varphi\cos\varphi - \frac{2kl+mg}{ml}\sin\varphi - \frac{b}{m}\,\dot{\varphi}. \tag{2.1}$$

The torque acting upon the flywheel D is

$$I\,\dot{\Omega} = \mu\,\cos\varphi - F, \tag{2.2}$$

where I is the moment of inertia of the flywheel, F is an equivalent torque of the load and $\mu > 0$ is a proportionality constant. See [7], p. 217, for more details.

From Eq. (2.1) and (2.2) the differential equations of our model are given by

$$\begin{aligned} \frac{d\,\varphi}{d\tau} &= \psi \\ \frac{d\,\psi}{d\tau} &= \left(\frac{2k}{m} + c^2\Omega^2\right)\sin\varphi\cos\varphi - \frac{(2kl+mg)}{ml}\sin\varphi - \frac{b}{m}\,\psi \\ \frac{d\,\Omega}{d\tau} &= \frac{1}{I}\,(\mu\cos\varphi - F) \end{aligned} \tag{2.3}$$

where τ is the time.

The standard Watt governor differential equations in Pontryagin [7], p. 217, are obtained from (2.3) by taking $k = 0$.

Defining the following changes in the coordinates, parameters and time

$$x = \varphi,\ y = \left(\frac{ml}{2kl+mg}\right)^{1/2}\psi,\ z = c\left(\frac{ml}{2kl+mg}\right)^{1/2}\Omega,\ t = \left(\frac{2kl+mg}{ml}\right)^{1/2}\tau,$$

$$\kappa = \frac{2kl}{2kl+mg},\ \varepsilon = \frac{b}{m}\left(\frac{ml}{2kl+mg}\right)^{1/2},\ \alpha = \frac{c\mu}{I}\left(\frac{ml}{2kl+mg}\right),\ \beta = \frac{F}{\mu},$$

where $0 \leq \kappa < 1$, $\varepsilon > 0$, $\alpha > 0$ and $0 < \beta < 1$, (2.3) can be written as

$$\begin{aligned} x' = \frac{dx}{dt} &= y \\ y' = \frac{dy}{dt} &= (z^2+\kappa)\,\sin x\,\cos x - \sin x - \varepsilon\,y \\ z' = \frac{dz}{dt} &= \alpha\,(\cos x - \beta) \end{aligned} \tag{2.4}$$

or equivalently by

$$\mathbf{x}' = f(\mathbf{x}, \zeta), \tag{2.5}$$

where

$$f(\mathbf{x}, \zeta) = \left(y, (z^2+\kappa)\,\sin x\,\cos x - \sin x - \varepsilon\,y, \alpha\,\left(\cos x - \beta\right)\right),$$

$$\mathbf{x} = (x,y,z) \in \left(0, \frac{\pi}{2}\right) \times \mathbb{R} \times [0,\infty),\ \zeta = (\beta, \alpha, \varepsilon, \kappa) \in (0,1) \times (0,\infty) \times (0,\infty) \times [0,1)\,.$$

The WGSS differential equations (2.4) have only one admissible equilibrium point

$$P_0 = (x_0, y_0, z_0) = \left(\arccos\beta, 0, \left(\frac{1}{\beta} - \kappa\right)^{1/2}\right). \tag{2.6}$$

The following theorem follows from the linear analysis of (2.4) and was proved in [10].

Theorem 1. *If*

$$\varepsilon > \varepsilon_c = 2\,\alpha\,\beta^{3/2}(1-\kappa\beta)^{1/2}, \tag{2.7}$$

then the WGSS differential equations (2.4) have an asymptotically stable equilibrium point at P_0. If $0 < \varepsilon < \varepsilon_c$ then P_0 is unstable.

In section 4 we study the stability of P_0 under the condition $\varepsilon = \varepsilon_c$, that is, on the Hopf hypersurface complementary to the range of validity of Theorem 1.

3 Lyapunov coefficients

The beginning of this section is a review of the method presented in [3] and in [4] for the calculation of the first and second Lyapunov coefficients. The calculation of the third Lyapunov coefficient can be found in [9].

The authors have not found the calculation of the fourth Lyapunov coefficient in the current literature. The extensive calculations and the long expressions for these coefficients have been obtained with the software MATHEMATICA 5 [15].

Consider the differential equations

$$\mathbf{x}' = f(\mathbf{x}, \mu), \tag{3.1}$$

where $\mathbf{x} \in \mathbb{R}^n$ and $\mu \in \mathbb{R}^m$ are respectively vectors representing phase variables and control parameters. Assume that f is of class C^∞ in $\mathbb{R}^n \times \mathbb{R}^m$. Suppose (3.1) has an equilibrium point $\mathbf{x} = \mathbf{x_0}$ at $\mu = \mu_\mathbf{0}$ and, denoting the variable $\mathbf{x} - \mathbf{x_0}$ also by $\mathbf{x}$, write

$$F(\mathbf{x}) = f(\mathbf{x}, \mu_\mathbf{0}) \tag{3.2}$$

as

$$\begin{aligned} F(\mathbf{x}) = A\mathbf{x} + \frac{1}{2}B(\mathbf{x},\mathbf{x}) + \frac{1}{6}C(\mathbf{x},\mathbf{x},\mathbf{x}) + \frac{1}{24}D(\mathbf{x},\mathbf{x},\mathbf{x},\mathbf{x}) + \frac{1}{120}E(\mathbf{x},\mathbf{x},\mathbf{x},\mathbf{x},\mathbf{x}) + \\ \frac{1}{720}K(\mathbf{x},\mathbf{x},\mathbf{x},\mathbf{x},\mathbf{x},\mathbf{x}) + \frac{1}{5040}L(\mathbf{x},\mathbf{x},\mathbf{x},\mathbf{x},\mathbf{x},\mathbf{x},\mathbf{x}) + \frac{1}{40320}M(\mathbf{x},\mathbf{x},\mathbf{x},\mathbf{x},\mathbf{x},\mathbf{x},\mathbf{x},\mathbf{x}) \\ + \frac{1}{362880}N(\mathbf{x},\mathbf{x},\mathbf{x},\mathbf{x},\mathbf{x},\mathbf{x},\mathbf{x},\mathbf{x},\mathbf{x}) + O(||\mathbf{x}||^{10}), \end{aligned} \tag{3.3}$$

where $A = f_\mathbf{x}(0, \mu_\mathbf{0})$ and, for $i = 1, \ldots, n$,

$$B_i(\mathbf{x},\mathbf{y}) = \sum_{j,k=1}^{n} \frac{\partial^2 F_i(\xi)}{\partial \xi_j \, \partial \xi_k}\bigg|_{\xi=0} x_j \, y_k, \; C_i(\mathbf{x},\mathbf{y},\mathbf{z}) = \sum_{j,k,l=1}^{n} \frac{\partial^3 F_i(\xi)}{\partial \xi_j \, \partial \xi_k \, \partial \xi_l}\bigg|_{\xi=0} x_j \, y_k \, z_l,$$

and so on for D_i, E_i, K_i, L_i, M_i and N_i.

Suppose $(\mathbf{x_0}, \mu_\mathbf{0})$ is an equilibrium point of (3.1) where the Jacobian matrix A has a pair of purely imaginary eigenvalues $\lambda_{2,3} = \pm i\omega_0$, $\omega_0 > 0$, and admits no other eigenvalue with zero real part. Let T^c be the generalized eigenspace of A corresponding to $\lambda_{2,3}$.

Let $p, q \in \mathbb{C}^n$ be vectors such that

$$Aq = i\omega_0 \, q, \; A^\top p = -i\omega_0 \, p, \; \langle p, q\rangle = \sum_{i=1}^{n} \bar{p}_i \, q_i \; = 1, \tag{3.4}$$

where $A^\top$ is the transposed of the matrix A. Any vector $y \in T^c$ can be represented as $y = wq + \overline{wq}$, where $w = \langle p, y\rangle \in \mathbb{C}$. The two dimensional center manifold can be parameterized by $w, \bar{w}$, by means of an immersion of the form $\mathbf{x} = H(w, \bar{w})$, where $H : \mathbb{C}^2 \to \mathbb{R}^n$ has a Taylor expansion of the form

$$H(w, \bar{w}) = wq + \overline{wq} + \sum_{2 \le j+k \le 9} \frac{1}{j!k!} h_{jk} w^j \bar{w}^k + O(|w|^{10}), \tag{3.5}$$

with $h_{jk} \in \mathbb{C}^n$ and $h_{jk} = \bar{h}_{kj}$. Substituting this expression into (3.1) we obtain the following differential equation

$$H_w w' + H_{\bar{w}} \bar{w}' = F(H(w, \bar{w})), \tag{3.6}$$

where F is given by (3.2).

The complex vectors h_{ij} are obtained solving the system of linear equations defined by the coefficients of (3.6), taking into account the coefficients of F, so that system (3.6), on the chart w for a central manifold, writes as follows

$$w' = i\omega_0 w + \frac{1}{2} G_{21} w|w|^2 + \frac{1}{12} G_{32} w|w|^4 + \frac{1}{144} G_{43} w|w|^6 + \frac{1}{2880} G_{54} w|w|^8 + O(|w|^{10}),$$

with $G_{jk} \in \mathbb{C}$.

The *first Lyapunov coefficient* l_1 is defined by

$$l_1 = \frac{1}{2}\operatorname{Re} G_{21}, \tag{3.7}$$

where $G_{21} = \langle p, \mathscr{H}_{21}\rangle$, and $\mathscr{H}_{21} = C(q,q,\overline{q}) + B(\overline{q}, h_{20}) + 2B(q, h_{11})$.

The complex vector h_{21} can be found solving the nonsingular $(n+1)$-dimensional system

$$\begin{pmatrix} i\omega_0 I_n - A & q \\ \overline{p} & 0 \end{pmatrix} \begin{pmatrix} h_{21} \\ s \end{pmatrix} = \begin{pmatrix} \mathscr{H}_{21} - G_{21} q \\ 0 \end{pmatrix},$$

with the condition $\langle p, h_{21}\rangle = 0$. See Remark 3.1 of [9]. The procedure above can be adapted in connection with the determination of h_{32} and h_{43}.

The expression for $\mathscr{H}_{32}$ can be found in equation (36) of [9], p. 28. From the coefficients of the terms $w^3\overline{w}^2$ in (3.6), one has a singular system for h_{32} given by $(i\omega_0 I_n - A)h_{32} = \mathscr{H}_{32} - G_{32}q$, which has solution if and only if

$$\langle p, \mathscr{H}_{32} - G_{32} q\rangle = 0. \tag{3.8}$$

The *second Lyapunov coefficient* is defined by

$$l_2 = \frac{1}{12}\operatorname{Re} G_{32}, \tag{3.9}$$

where, from (3.8), $G_{32} = \langle p, \mathscr{H}_{32}\rangle$.

The complex vector h_{32} can be found solving the nonsingular $(n+1)$-dimensional system

$$\begin{pmatrix} i\omega_0 I_n - A & q \\ \overline{p} & 0 \end{pmatrix} \begin{pmatrix} h_{32} \\ s \end{pmatrix} = \begin{pmatrix} \mathscr{H}_{32} - G_{32} q \\ 0 \end{pmatrix},$$

with the condition $\langle p, h_{32}\rangle = 0$.

The expression for $\mathscr{H}_{43}$ can be found in equation (44) of [9], p. 30. From the coefficients of the terms $w^4\overline{w}^3$ in (3.6), one has a singular system for h_{43} given by $(i\omega_0 I_n - A)h_{43} = \mathscr{H}_{43} - G_{43}q$, which has solution if and only if

$$\langle p, \mathscr{H}_{43} - G_{43} q\rangle = 0. \tag{3.10}$$

The *third Lyapunov coefficient* is defined by

$$l_3 = \frac{1}{144}\operatorname{Re} G_{43}, \tag{3.11}$$

where, from (3.10), $G_{43} = \langle p, \mathscr{H}_{43}\rangle$.

The complex vector h_{43} can be found solving the nonsingular $(n+1)$-dimensional system

$$\begin{pmatrix} i\omega_0 I_n - A & q \\ \overline{p} & 0 \end{pmatrix} \begin{pmatrix} h_{43} \\ s \end{pmatrix} = \begin{pmatrix} \mathscr{H}_{43} - G_{43} q \\ 0 \end{pmatrix},$$

with the condition $\langle p, h_{43}\rangle = 0$. Defining $\mathscr{H}_{54}$ from an expression displayed in [11], since is too long to be put in print, and from the coefficients of the terms $w^5\overline{w}^4$ in (3.6), one has a singular system for h_{54} given by $(i\omega_0 I_n - A)h_{54} = \mathscr{H}_{54} - G_{54}q$, which has solution if and only if

$$\langle p, \mathscr{H}_{54} - G_{54} q\rangle = 0. \tag{3.12}$$

The *fourth Lyapunov coefficient* is defined by

$$l_4 = \frac{1}{2880}\,\mathrm{Re}\,G_{54}, \tag{3.13}$$

where, from (3.12), $G_{54} = \langle p, \mathscr{H}_{54}\rangle$.

For the definitions of Hopf points of codimension 1, 2 and 3 see [9], pp 31-32. A *Hopf point of codimension 4* is a Hopf point of codimension 3 where l_3 vanishes. A Hopf point of codimension 4 is called *transversal* if $\eta = 0$ ($\eta(\mu)$ is the real part of the critical eigenvalues), $l_1 = 0$, $l_2 = 0$ and $l_3 = 0$ have transversal intersections. In a neighborhood of a transversal Hopf point of codimension 4 —H4 point, for concision— with $l_4 \neq 0$ the dynamic behavior of the system (3.1), reduced to the family of parameter-dependent continuations of the center manifold, is orbitally topologically equivalent to

$$w' = (\eta + i\omega_0)w + \tau w|w|^2 + \nu w|w|^4 + \sigma w|w|^6 + l_4 w|w|^8,$$

where η, τ, ν and σ are unfolding parameters. See [12].

4 Hopf bifurcations in the WGSS

The following theorem was proved by the authors in [10].

Theorem 2. *Consider the four-parameter family of differential equations (2.4). The first Lyapunov coefficient at the point (2.6) for parameter values satisfying $\varepsilon = \varepsilon_c$ is given by*

$$l_1(\beta,\alpha,\kappa) = -\frac{G_1(\beta,\alpha,\kappa)}{4\beta\varepsilon_c\omega_0^4\omega_1^2(\varepsilon_c^4 + 5\varepsilon_c^2\omega_0^2 + 4\omega_0^4)}, \tag{4.1}$$

where $G_1(\beta,\alpha,\kappa)$ in given by

$$-3 + 5\kappa\beta - (\alpha^2 - 5)\beta^2 + \kappa(\alpha^2 - 7)\beta^3 - 2\alpha^2\kappa^2\beta^4 - (\alpha^4 - 2\alpha^2\kappa^2)\beta^6 + \alpha^4\kappa\beta^7. \tag{4.2}$$

If G_1 is different from zero then the system (2.4) has a transversal Hopf point at P_0 for $\varepsilon = \varepsilon_c$. More specifically, if $(\beta,\alpha,\kappa) \in S \cup U$ and $\varepsilon = \varepsilon_c$ then the system (2.4) has an H1 point at P_0; if $(\beta,\alpha,\kappa) \in S$ and $\varepsilon = \varepsilon_c$ then the H1 point at P_0 is asymptotically stable and for each $\varepsilon < \varepsilon_c$, but close to ε_c, there exists a stable periodic orbit near the unstable equilibrium point P_0; if $(\beta,\alpha,\kappa) \in U$ and $\varepsilon = \varepsilon_c$ then the H1 point at P_0 is unstable and for each $\varepsilon > \varepsilon_c$, but close to ε_c, there exists an unstable periodic orbit near the asymptotically stable equilibrium point P_0. The meanings and positions of the regions S and U are illustrated in Fig. 2.

The following is the main result of this paper. The meanings of the regions U_1, S_1, S_2 and curves C_1, C_2 are illustrated in Fig. 4 and 5.

Theorem 3. *For the four-parameter family of differential equations (2.4) there is unique point $Q = (\beta,\alpha,\kappa,\varepsilon_c)$, with coordinates*

$$\beta = 0.93593\ldots,\ \alpha = 1.02753\ldots,\ \kappa = 0.90164\ldots,\ \varepsilon_c = 0.73522\ldots,$$

where the surfaces $l_1 = 0$, $l_2 = 0$ and $l_3 = 0$ on the critical hypersurface intersect and there do it transversally. Moreover, the codimension 4 Hopf point at P_0 is asymptotically stable since

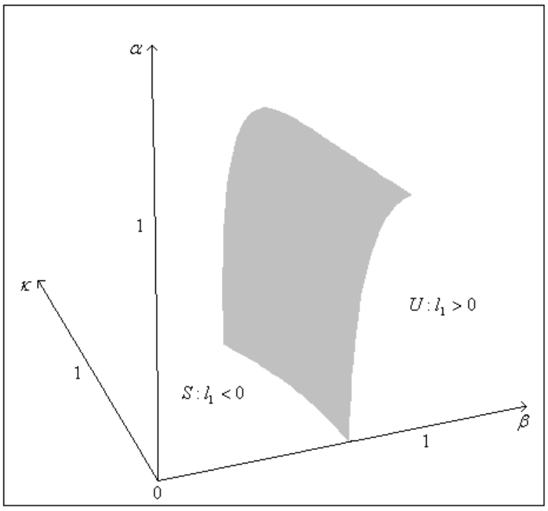

Figure 2: Signs of l_1 for system (2.4).

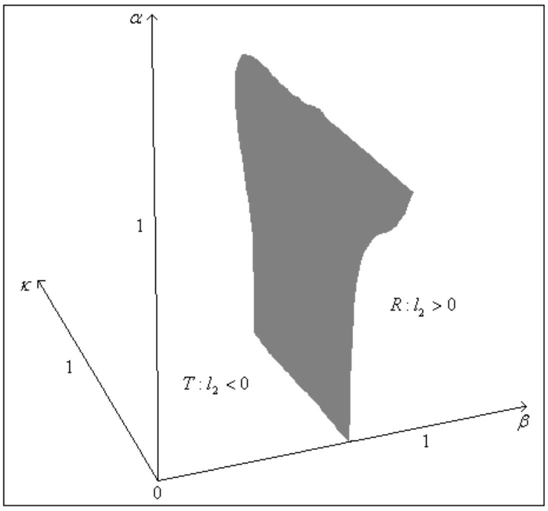

Figure 3: Signs of l_2 for system (2.4).

$l_4(Q) < 0$. *More specifically, if* $(\beta,\alpha,\kappa) \in S_1 \cup S_2 \cup U_1$ *and* $\varepsilon = \varepsilon_c$ *then the system (2.4) has an H2 point at* P_0*; if* $(\beta,\alpha,\kappa) \in S_1 \cup S_2$ *and* $\varepsilon = \varepsilon_c$ *then the H2 point at* P_0 *is asymptotically stable; if* $(\beta,\alpha,\kappa) \in U_1$ *and* $\varepsilon = \varepsilon_c$ *then the H2 point at* P_0 *is unstable. Along the curves* C_1 *and* $C_2 = C_{21} \cup C_{22} \cup \{Q\}$ *of Fig 4,* l_2 *vanishes.*

If $(\beta,\alpha,\kappa) \in C_1 \cup C_{21} \cup C_{22}$ *(see Fig. 5) and* $\varepsilon = \varepsilon_c$ *then the four-parameter family of differential equations (2.4) has a transversal Hopf point of codimension 3 at* P_0*; if* $(\beta,\alpha,\kappa) \in C_1 \cup C_{22}$ *and* $\varepsilon = \varepsilon_c$ *then the H3 point at* P_0 *is asymptotically stable and the bifurcation diagram for a typical point H is draw in Fig. 6; if* $(\beta,\alpha,\kappa) \in C_{21}$ *and* $\varepsilon = \varepsilon_c$ *then the H3 point at* P_0 *is unstable and the bifurcation diagram for a typical point G can be found in [9], p. 41.*

Proof. Outline of the computer assisted proof. The algebraic expression for the second Lyapunov coefficient can be obtained in [14]. This is too long to be put in print. The surface where the second Lyapunov coefficient vanishes is illustrated in Fig. 3.

The intersections of the surfaces $l_1 = 0$ and $l_2 = 0$ determine the curves C_1 and C_2 (see Fig. 4). The signs of the second Lyapunov coefficient on the surface $l_1 = 0$ complementary to the curves C_1 and C_2, that is on $S_1 \cup S_2 \cup U_1$ (see Fig. 5), are the following: l_2 is negative on $S_1 \cup S_2$ and is positive on U_1 and they can be viewed as extensions of the signs of the second Lyapunov coefficient at points on the curve determined by the intersection of the surface $l_1 = 0$ and the plane $\kappa = 0$ studied by the authors in [9]. The third Lyapunov coefficient is negative on $C_1 \cup C_{22}$ and is positive on C_{21}. The bifurcation diagram for a typical point G where $l_3(G) > 0$ can be viewed in [9]. In Fig. 6 and 7 are illustrated the bifurcation diagrams for a typical point H where $l_3(H) < 0$.

The point Q is the intersection of the surfaces $l_1 = 0$, $l_2 = 0$ and $l_3 = 0$. The existence and uniqueness of Q with the above coordinates has been established numerically with the software MATHEMATICA 5.

For the point Q take five decimal round-off coordinates $\beta = 0.93593$, $\alpha = 1.02753$, $\kappa = 0.90164$ and $\varepsilon_c = 0.73522$. For these values of the parameters one has

$$p = (-i/2, 0.27041 - 0.54618i, 0.40395 + 0.20000i)\,,\; q = (-i, 0.36401, 0.99407)\,,$$

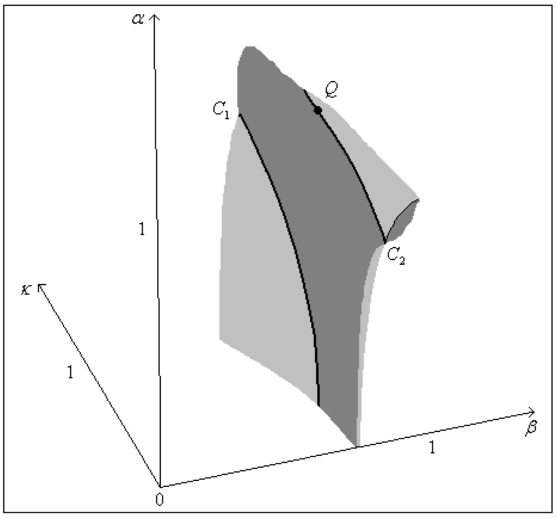

Figure 4: Intersections of $l_1 = 0$ and $l_2 = 0$.

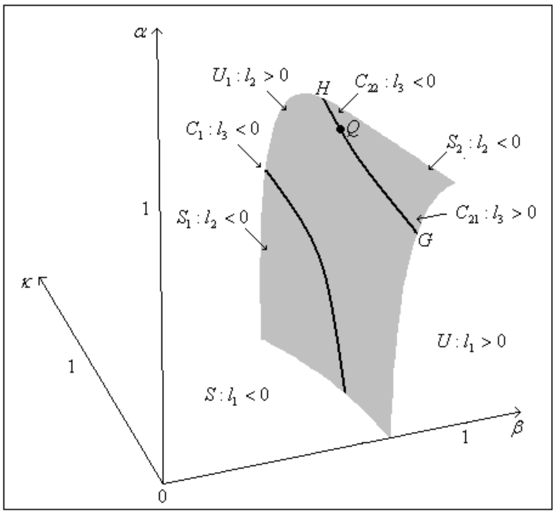

Figure 5: Signs of l_1, l_2 and l_3.

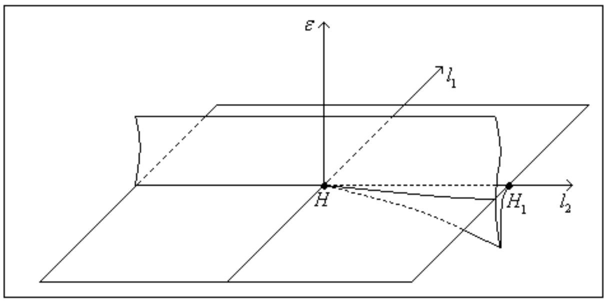

Figure 6: Bifurcation diagram for a typical point H where $l_3(H) < 0$.

$$\omega_0 = \sqrt{(1-\beta^2)/\beta},\ h_{11} = (-2.65769, 0, 0.19650)\,,$$
$$h_{20} = (-4.11029 - 0.18429i, 0.13416 - 2.99241i, 0.09159 - 3.36395i)\,,$$
$$h_{30} = (-3.63589 + 23.03616i, -25.15645 - 3.97054i, -18.16113 - 1.69167i)\,,$$
$$G_{21} = -3.91814i, \tag{4.3}$$
$$h_{21} = (3.24775 + 1.67247i, -4.52694 + 1.18222i, 4.85950 + 3.71541i)\,,$$
$$h_{40} = \begin{pmatrix} 160.39204 + 51.10539i \\ -74.41230 + 233.53975i \\ -25.03366 + 127.34049i \end{pmatrix},\ h_{50} = \begin{pmatrix} 702.48693 - 1263.93346i \\ 2300.44688 + 1278.57511i \\ 1054.20770 + 363.36145i \end{pmatrix},$$
$$h_{31} = (-69.44664 - 38.56274i, 25.90851 - 2.24484i, 36.10391 - 65.85524i)\,,$$
$$h_{22} = (-64.50829, 0, 10.76131)\,,$$
$$G_{32} = -153.21726i, \tag{4.4}$$
$$h_{32} = \begin{pmatrix} 178.24934 + 273.66781i \\ -233.17715 + 26.70966i \\ 395.89053 + 272.77265i \end{pmatrix},\ h_{41} = \begin{pmatrix} -521.71430 + 1074.26121i \\ -631.58388 - 484.25803i \\ -865.10385 - 413.20000i \end{pmatrix},$$

$$h_{60} = \begin{pmatrix} -10130.73267 - 9995.21750i \\ 21830.38995 - 22126.36639i \\ 5429.65950 - 9557.27148i \end{pmatrix}, h_{51} = \begin{pmatrix} 14227.43860 + 8237.49829i \\ -9991.87299 + 14431.55078i \\ -5753.08267 + 11280.54380i \end{pmatrix},$$

$$h_{42} = \begin{pmatrix} -4351.45992 - 4936.33553i \\ 2272.08822 + 1527.90723i \\ 4841.97866 - 5445.36779i \end{pmatrix}, h_{33} = \begin{pmatrix} -5969.63958 \\ 0 \\ 1764.47230 \end{pmatrix},$$

$$h_{70} = \begin{pmatrix} -146941.54096 + 63522.80004i \\ -161862.28504 - 374421.36634i \\ -86069.40319 - 83969.45215i \end{pmatrix}, h_{61} = \begin{pmatrix} 140223.18890 - 184094.16057i \\ 260780.07852 + 213929.28545i \\ 151116.49070 + 92225.27059i \end{pmatrix},$$

$$h_{52} = \begin{pmatrix} -105557.32750 + 127994.80577i \\ -41289.02476 - 79039.91108i \\ -106857.14273 - 88122.45467i \end{pmatrix}, h_{43} = \begin{pmatrix} 26579.27090 + 62051.16515i \\ -36944.56779 + 2499.10743i \\ 78144.32459 + 54070.14624i \end{pmatrix},$$

$$G_{43} = -22328.21224i. \tag{4.5}$$

$$h_{80} = \begin{pmatrix} -247681.58290 + 2173895.03048i \\ -6330624.44741 - 721276.35507i \\ -1324248.15135 + 594661.38331i \end{pmatrix}, h_{71} = \begin{pmatrix} -2230744.30930 - 2511854.85381i \\ 4663683.99275 - 4038564.75411i \\ 1618564.33911 - 2037646.14488i \end{pmatrix},$$

$$h_{62} = \begin{pmatrix} 2540059.79128 + 2277848.86298i \\ -2385453.21697 + 1869088.06376i \\ -1708253.47087 + 2025268.53034i \end{pmatrix}, h_{53} = \begin{pmatrix} -633499.15640 - 1125590.51413i \\ 390598.08062 + 466226.40735i \\ 1219484.73373 - 1101283.41903i \end{pmatrix},$$

$$h_{44} = (-1118100.12194, 0.00138, 546721.10946),$$

$$G_{54} = -22071.41115 - 5991090.52119i. \tag{4.6}$$

From (3.7), (3.9), (3.11), (3.13), (4.3), (4.4), (4.5) and (4.6) one has

$$l_1(Q) = 0,\ l_2(Q) = 0,\ l_3(Q) = 0,\ l_4(Q) = \frac{1}{2880}\,\mathrm{Re}\,G_{54} = -7.66368.$$

The calculations above have also been corroborated with 100 decimals round-off precision performed using the software MATHEMATICA 5 [15]. See [14].

Some values of $(\alpha, \beta, \kappa) \in C_1 \cup C_2$ as well as the corresponding values of $l_3(\alpha, \beta, \kappa)$ are listed in the table below. There the columns 1, 2, 3 and 4 correspond to C_1 while columns 5, 6, 7 and 8 correspond to C_2. The calculations leading to these values can be found in [14].

κ	α	β	$l_3(\alpha,\beta,\kappa)$ on C_1	κ	α	β	$l_3(\alpha,\beta,\kappa)$ on C_2
0.45	0.33319	0.72216	-0.91310	0	0.85050	0.86828	0.39050
0.5	0.42968	0.71770	-0.92567	0.2	0.90524	0.87760	0.46294
0.55	0.50934	0.71257	-0.88152	0.3	0.93123	0.88397	0.50684
0.6	0.57913	0.70665	-0.82064	0.4	0.95511	0.89159	0.55538
0.65	0.64241	0.69983	-0.75810	0.5	0.97602	0.90042	0.60637
0.7	0.70113	0.69201	-0.70006	0.6	0.99330	0.91029	0.65253
0.75	0.75659	0.68309	-0.64900	0.7	1.00674	0.92071	0.66963
0.8	0.80972	0.67302	-0.60580	0.8	1.01697	0.93045	0.56860
0.85	0.86120	0.66177	-0.57054	0.9	1.02731	0.93592	0.01665
0.9	0.91154	0.64940	-0.54288	0.92	1.03020	0.93585	-0.20674
0.95	0.96114	0.63600	-0.52217	0.98	1.04319	0.93201	-1.09289

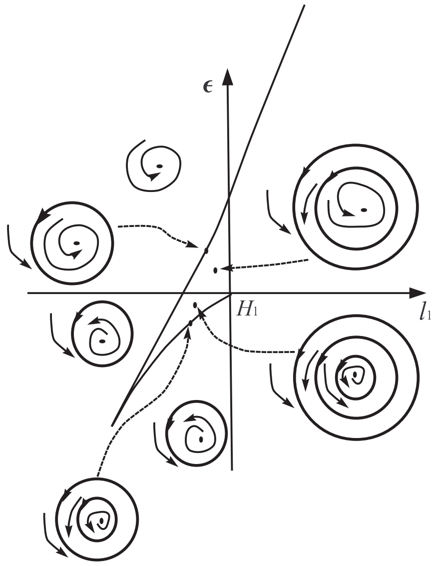

Figure 7: Bifurcation diagram for a typical point H_1. See Fig. 6.

The gradients of the functions l_1, l_2 and l_3, given in (3.7), (3.9), (3.11) at the point Q are $(-0.46264, 0.13437, -0.97565)$, $(-12.44701, 2.66791, -19.19345)$, $(-266.77145, 41.80505, -372.84969)$, respectively. The transversality condition at Q is equivalent to the non-vanishing of the determinant of the matrix whose columns are the above gradient vectors, which is evaluated gives -33.31133.

The main steps of the calculations that provide the numerical evidence for this theorem have been posted in [14]. ■

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