Journal of Nonlinear Mathematical Physics Volume 15, Supplement 3 (2008), 310–322 ARTICLE

The Sato Grassmannian and the CH Hierarchy

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Abstract

We discuss how the Camassa-Holm hierarchy can be framed within the geometry of the Sato Grassmannian. We discuss the geometry of an extension of the negative flows of the CH hierarchy, recover the well-known CH equations, and frame the problem within the theory of pseudo-differential operators.

1 Introduction

In this paper we study some specific aspects of the Camassa Holm hierarchy. Since its appearance in the literature, it has been recognized that the CH equation possesses specific features, (e.g., peakon solutions, the appearance of third order Abelian differentials in finite gap solutions,...) that other more ”classical” soliton hierarchies (KdV, Boussinesq, NLS) do not exhibit. Among these, especially in view of the Dubrovin–Zhang classification scheme [8], the non-existence of a formulation via a τ function is, from our point of view, of particular interest. The Sato theory of the τ function basically views it as a section of the (dual) determinant bundle over the so–called Sato (or Universal) Grassmannian (UG). It is associated with any hierarchy of evolutionary PDEs that can be represented as a hierarchy of linear flows on UG. Thus it seems important to analyze whether (and which) flows of the CH hierarchy can be realized as linear flows in the Sato Grassmannian.

The main aim of this paper is to discuss this problem in the framework of a set up, introduced in [11, 2], relating the (bi)–Hamiltonian structures of soliton hierarchies of KdV type to the Sato Grassmannian. In particular, we will rely on some preliminary results presented in [15] and earlier in [4], concerning the relations between the CH hierarchy and such a representation of the Sato Grassmannian.

In [12] it was shown that the bi–Hamiltonian structures of the CH and KdV equations (as well of the Harry–Dym equation) are related, being geodesic motions on the Virasoro group with respect to different metrics. Actually, the relation with the evolution on the Sato Grassmannian has been studied for the KdV and the HD hierarchies, showing that they are related to linear flows in the big cell of UG. In this paper we try to complete this picture showing that the CH hierarchy too is related to the big cell of the Sato Grassmannian by means of its local (also called negative) flows. One of the basic differences among this representation of the three hierarchies is given by the relation between the local flows and the “time” of the hierarchy related to the conservation of the linear momentum.

This will show up, in the present paper, as the realization of the CH local hierarchy in a constrained subspace of the big cell. The path leading us to this result is the analysis of the evolution of the Noether currents associated with the bi–Hamiltonian recurrence relation of the local hierarchy. We will argue as, on more general grounds, they are associated with a two-field (albeit somehow trivial) bi–Hamiltonian extension of the CH local hierarchy.

The ordinary CH bi–Hamiltonian hierarchy is recovered – together with the non local part including the ”true” CH equation – by Dirac restricting this two-field hierarchy to a specific submanifold, namely those selected by these Noether currents satisfying a specific constraint. Thus the CH equation is realized, in the picture herewith presented, as an additional commuting flow of an infinite system of (suitably constrained) linear flows on the Sato Grassmannian.

The full interpretation of the whole nonlocal hierarchy to this Sato Grassmannian approach, as well as the problem of how far this picture could be useful to explain and understand the non–existence of the τ function for CH is still under consideration.

The scheme of the paper is as follows: in Sections 2 and 3 we will present, basically following [15] the application of the bi–Hamiltonian scheme that provides a representation of the (local or negative) flows of the Camassa Holm hierarchy as flows on a suitable subset of the Sato Grassmannian. For the readers’ convenience, we will present the “inductive” route starting from the CH Poisson pencil and arriving at the Grassmannian representation. However, the logical scheme of the present paper (namely of section 3) is somehow to revert (and extend) this point of view. That is, to start from a set of equations on UG, and arrive at CH. In passing, we seize the chance of discussing some of the content of Sections 4 and 5 of [15], (e.g., we discuss some conditions that insure its consistency, and other related topics), as well as providing proofs therein missing or just sketched.

In Section 4 we discuss the (bi)–Hamiltonian geometry of the two-field system associated with such a constrained subspace (that we call Extended CH hierarchy) of the Sato Grassmannian. In Section 5 we recover, from such a two-field bi–Hamiltonian system the bi-Hamiltonian geometry of the full Camassa Holm *hierarchy*, by means of a suitable Dirac reduction procedure. Finally, we show how the problem we are dealing with can be framed within the theory of a Lax system for a second order differential operator.

2 The geometry of the CH hierarchy ...

It is well known[1, 10] that the CH equation[1]

$$4v_t - v_{xxt} = 24v_x v - 4v_{xx}v_x - 2vv_{xxx}$$

is a bi–Hamiltonian evolutionary PDE on $C^\infty(S^1,\mathbb{R})$ w.r.t. the Poisson pencil

$$P_\lambda = (4\partial_x - \partial_x^3) + \lambda(2m\partial_x + 2\partial_x m) \qquad \lambda \in \mathbb{R}$$

[1]We have herewith chosen unusual normalizations because this somewhat simplifies some of the formulæ we are interested in.

where $m = 4v - v_{xx}$.

The densities of the conserved laws of the hierarchy can be obtained by recursively solving

$$h_x + h^2 = mz^2 + 1, \qquad z = \sqrt{\lambda} \tag{2.1}$$

where h is the generating function of the densities of the Casimir of P_λ [4, 5, 6, 14, 17].

This Riccati equation admits two different solutions

$$\begin{aligned} h &= h_{-1}z + h_0 + \frac{h_1}{z} + \frac{h_2}{z^2} + \dots \\ k &= k_0 + k_{-1}z + k_{-2}z^2 + k_{-3}z^3 + \dots \quad . \end{aligned}$$

The two families of coefficients $\{h_i\}_{i\geq -1}$ and $\{k_i\}_{i\leq 0}$ give, by means of the Lenard recursion, all the f CH hierarchy. In particular, the h_i's are the densities of the negative (or local) CH hierarchy, and can be algebraically found from (2.1), while the k_j's are the densities of the positive (or "non–local") CH hierarchy whose first two members are, respectively, x-translation and the CH equation itself.

The first flow of the local hierarchy is

$$\frac{\partial}{\partial t_3} m = (4\partial_x - \partial_x^3)\frac{1}{2\sqrt{m}}. \tag{2.2}$$

The key ingredient used in [11] to relate the Hamiltonian structure of Soliton hierarchies of KdV type to evolutions on the Sato Universal Grassmannian manifold is given by the Noether currents.

In particular, it has been shown in [4] that the Noether currents associated with the local CH hierarchy are characterized, in the space of formal Laurent series in the parameter z by the following two properties:

1. Their asymptotic behavior is given by

$$J^{(s)} = z^s + O(z), \quad s \geq 2 \tag{2.3}$$

2. They belong to the span

$$\langle(\partial_x + h)^n z^2\rangle_{n\geq 0} \tag{2.4}$$

   of the *Faà di Bruno* monomials associated with the generating function h, which solves (2.1) with asymptotic condition $h(z) = h_1 z + h_0 + \dfrac{h_1}{z} + \cdots$, with coefficients on $C^\infty(S^1, \mathbb{R})$.

The connection between the currents $J^{(s)}$ and the generating function h is given by the fact that, along the s-th time of the local CH hierarchy, they evolve as

$$\partial_s h = \partial_x J^{(s)} \qquad \text{where} \quad \partial_s = \frac{\partial}{\partial t_s}. \tag{2.5}$$

The asymptotic behavior of the local Noether currents and the presence of a "generator" h suggest, in analogy with what happens in the KdV case, that they can be associated with linear evolutions on the Sato Grassmannian.

3 .... and the Sato Grassmannian

In this section we shall look at the problem starting from a slightly different perspective.

Let us consider the space J_+ given by the span on $C^\infty(S^1,\mathbb{R})$ of the family

$$J^{(i)} = z^i + J^i_{-1}z + J^i_0 + J^i_1 z^{-1} + \dots \qquad i \geq 2$$

in the space J of Laurent series (with at most a pole singularity at $z=\infty$). J admits a direct splitting as

$$J = J_+ \oplus J_-, \quad \text{where} \quad J_- := \langle z^i \rangle_{i\leq 1}. \tag{3.1}$$

Therefore the collection $\{J^{(i)}\}_{i\geq 2}$ defines a point of the big cell $\mathscr{B}$ of the Sato Grassmannian translated by z^2 w.r.t. the standard Sato representation [18].
On this space we can define an infinite family of flows setting

$$(\partial_s + J^{(s)})J_+ \subset J_+ \qquad s \geq 2 \tag{3.2}$$

that, more explicitly, can be written as

$$(\partial_s + J^{(s)})J^{(r)} = J^{(s+r)} + \sum_{i=-1}^{r-2} J^s_i J^{r-i} + \sum_{i=-1}^{s-2} J^r_i J^{s-i} + J^r_{-1}J^s_{-1}J^{(2)} \; . \tag{3.3}$$

Proposition 1. *The flows (3.2) commute.*

Proof We have to show that $[\partial_s,\partial_r]J_+ = 0$, i.e.

$$[\partial_s,\partial_r]J^{(n)} = 0, \qquad \forall s,r,n \geq 2 \; . \tag{3.4}$$

Thanks to (3.2) the flows satisfy the the “symmetry condition” $\partial_s J^{(r)} = \partial_r J^{(s)}$ and then equation (3.4) can be written as

$$[\partial_s,\partial_r]J^{(n)} = [\partial_s + J^{(s)}, \partial_r + J^{(r)}]J^{(n)}. \tag{3.5}$$

From the explicit form of the currents it holds

$$[\partial_s,\partial_r]J^{(n)} \in J_-,$$

but from (3.2)

$$[\partial_s + J^{(s)}, \partial_r + J^{(r)}]J^{(n)} \in J_+.$$

□

Proposition 2. *The local currents of CH satisfy (3.2).*

Proof The currents (2.3) are elements of J_+. Moreover from the property (2.4) follows that every element of J_+ can be written as $J^{(i)}_{lCH} = \sum_k c^i_k(\partial_x + h)^k z^2$. Using this expansion (2.5) we see that

$$\begin{aligned}(\partial_s + J^{(s)})\sum_{k=0}^{r} c^r_k(\partial_x+h)^k z^2 &= \sum_{k=0}^{r}(\partial_s c^r_k)(\partial_x+h)^k z^2 + \sum_{k=0}^{r} c^r_k(\partial_s + J^{(s)})(\partial_x+h)^k z^2 \\ &= \sum_{k=0}^{r}(\partial_s c^r_k)(\partial_x+h)^k z^2 + \sum_{k=0}^{r}(\partial_x+h)^k z^2\, J^{(s)} \subset J_+ \oplus z^2 J_+ \; .\end{aligned}$$

In [4] it is shown that, for the local currents of CH, $z^2 J_+ \subset J_+$ and then they satisfy (3.2).

□

Therefore, taking into account the results of [11] we can conclude that the local (negative) flows of CH hierarchy are given, by means of the construction outlined above, linear flows on the big cell $\mathscr{B}$ of the Grassmannian.

Remark. The basic issue to recover a hierarchy of 1+1 dimensional PDEs from a dynamical system of the form (3.2) is to specify (or define) the "physical" space variable x.

For instance, in the ordinary KP-KdV case, x can be, as it is well known, identified with the first "time" of the hierarchy. As it was shown in [3], fractional KdV hierarchies can be obtained identifying x with a different time t_s of a system similar to (3.2). Actually, in our case, x is not contained in the dynamical system, and thus should be added by means of the introduction of another current $h = h_{-1}z + h_0 + \frac{h_1}{z} + \dots$. In turn, this additional current has to be related with the action of x-translation on the currents $J^{(s)}$ of the Grassmannian.

The most natural way to add this new current is to consider the enlargement of the system (3.2) to

$$(\partial_s + J^{(s)})J_+ \subset J_+, \qquad (\partial_s + J^{(s)})h \in J_+ \ (s \geq 2), \qquad (\partial_x + h)J_+ \subset J_+, \tag{3.6}$$

which explicitly is given, in addition to Eqn.s (3.3), by

$$\begin{aligned} (\partial_x + h)J^{(s)} &= \sum_{i=-1}^{s-2} h_i J^{(s-i)} + h_{-1}J^s_{-1}J^{(2)} \qquad s \geq 2 \\ (\partial_s + J^{(s)})h &= \sum_{i=-1}^{s-2} h_i J^{(s-i)} + h_{-1}J^s_{-1}J^{(2)} \qquad s \geq 2. \end{aligned} \tag{3.7}$$

However, these flows are not in general commuting, so that further conditions have to be imposed. It is outside the size of this paper to discuss this problem in full generality; we simply remark the restriction to the subspace of the translated big cell defined by

$$J^{(2)} = z^2 \qquad \textit{and} \qquad z^2 J_+ \subset J_+ \ . \tag{3.8}$$

is a consistent one[2].

The following Lemma helps clarifying the meaning of the constraint(3.8):

Lemma 3. *For any choice of $J^{(2)}$, the currents $J^{(i)}$ satisfying (3.7) are elements of $F = sp\langle(\partial_x + h)^n J^{(2)}\rangle_{n\geq 0}$.*

Proof Expanding the relation (3.7) it follows that

$$J^{(s+1)} = \frac{1}{h_{-1}}(\partial_x + h)J^{(s)} - \sum_{i=-1}^{s-2} \frac{h_i}{h_{-1}} J^{(s-i)} + J^s_{-1}J^{(2)}. \tag{3.9}$$

Since $(\partial_x + h)F \subset F$ and $J^{(2)} \in F$, then one can write recursively all the currents using elements of F.

□

[2] Another consistent solution to this problem is given by requiring that $(\partial_x + h)h \in J_+$. The resulting system of commuting PDEs leads to a $2+1$ dimensional extension of the HD hierarchy [13, 16].

In the light of this proposition, we can rephrase the first of equations (3.8) saying that we consider only the case $J^{(2)} = z^2$. The study of more general choices of the current $J^{(2)}$ is under consideration.

The basic reason for this choice of ours is that the space J_+ defined by (3.8) contains the currents of the CH hierarchy (see Proposition 2). Moreover, it turns out that J_+ is parameterized by three fields, namely h_{-1}, h_0, and h_1. This can be seen as follows. Since $z^2 J_+ \subset J_+$ and $J^{(2)} = z^2$, we get that $J^{(4)} = z^4$. The recursion relations (3.9) allow us to write all the currents, and namely $J^{(4)}$, as differential polynomials in the components h_k of the formal Laurent series h. Thus we arrive at

$$\frac{z^2}{{h_{-1}}^2}(h_x + h^2) - z^2\left(\frac{h_{-1_x}}{{h_{-1}}^3} + \frac{2{h_0}^2}{h_{-1}}\right)h - z^2\left(\frac{h_{0_x}}{{h_{-1}}^2} - \frac{{h_0}^2}{{h_{-1}}^2} + \frac{2h_1}{h_{-1}} - \frac{h_0\,(h_{-1_x})}{{h_{-1}}^3}\right) = z^4. \quad (3.10)$$

It is straightforward to check that this relation enables one to recover $h_2, h_3, \ldots$ as differential polynomials in h_{-1}, h_0, h_1. So the system (3.7), determines a hierarchy of 1+1 evolutionary PDEs in the three fields (dependent variables) h_{-1}, h_0, h_1. For instance, the first non trivial flow is [15]:

$$\begin{aligned}
\partial_3 h_{-1} &= -\frac{h_{-1_x} h_1}{{h_{-1}}^2} + \frac{h_{1_x}}{h_{-1}} \\
\partial_3 h_0 &= \frac{3}{2}\frac{h_{1_x} h_{-1_x}}{{h_{-1}}^3} - \frac{3}{2}\frac{h_1 (h_{-1_x})^2}{{h_{-1}}^4} - \frac{1}{2}\frac{h_{1_{xx}}}{{h_{-1}}^2} + \frac{1}{2}\frac{h_{-1_{xx}} h_1}{{h_{-1}}^3} \\
\partial_3 h_1 &= -\frac{3}{2}\frac{h_{-1_x} h_{1_{xx}}}{{h_{-1}}^4} + \frac{5}{2}\frac{h_{-1_x} h_{-1_{xx}} h_1}{{h_{-1}}^5} + \frac{15}{4}\frac{(h_{-1_x})^2 h_{1_x}}{{h_{-1}}^5} - \frac{15}{4}\frac{h_1\,(h_{-1_x})^3}{{h_{-1}}^6} \\
&\quad + \frac{{h_1}^2 h_{-1_x}}{{h_{-1}}^3} + \frac{1}{4}\frac{h_{1_{xxx}}}{{h_{-1}}^3} - \frac{1}{4}\frac{h_{-1_{xxx}} h_1}{{h_{-1}}^4} - \frac{h_{1_x} h_{-1_{xx}}}{{h_{-1}}^4} - \frac{h_{1_x} h_1}{{h_{-1}}^2} \qquad .
\end{aligned} \quad (3.11)$$

We notice that the field h_0 does not affect the dynamics. Actually, this is true for all the times of the hierarchy we are considering. This is a consequence of the fact that no currents depends on h_0, as one can see by recursion using (3.9), noticing that $J^{(2)} = z^2$ and $J^{(3)} = \frac{z^2}{h_{-1}}(h - h_0)$.

Therefore the constraint given by (3.10) do not depend on h_0 as well, and so we can limit ourselves to the study of the system in the two dependent variables h_{-1}, h_1. We will study further this two–field system, that we call *Extended CH system* in the next Section.

4 The geometry of the extended CH system

In this section we shall prove that the system (3.11), or better, the closed system defined by its first and third equation (see the remark above) is a bi–Hamiltonian system and it admits an iterable Casimir, that is, a Casimir of the pencil of Poisson bracket (to be found below) that generates, via the Lenard recursion relations, the commuting flows. Our proof will be done in a sequence of steps as follows.

First we notice that, if we perform the change of variables $h_{-1} = \alpha$ and $h_1 = \frac{\gamma}{\alpha}$ the first and third of equations (3.11) become:

$$\begin{aligned}
\partial_3 \alpha &= \left(\frac{\gamma}{\alpha^2}\right)_x \\
\partial_3 \gamma &= \frac{\alpha}{4}\left(\frac{1}{\alpha}\left(\frac{1}{\alpha}\left(\frac{\gamma}{\alpha^2}\right)_x\right)_x\right)_x .
\end{aligned} \quad (4.1)$$

From the general theory, and namely from the representation (2.5) of the PDEs, we see that this system has an infinite sequence of conserved quantities, whose densities are given by the coefficients of the formal Laurent series (3.10) with $h_0 = 0$, i.e.:

$$\frac{1}{\alpha^2}(h_x + h^2) - \frac{\alpha_x}{\alpha^3}h - \frac{2\gamma}{\alpha^2} = z^2. \tag{4.2}$$

It is worthwhile to remark again that this equation determines all the coefficients $h_i, i \geq 0$ as differential polynomials in α, γ. For instance we have, apart form the obvious relations $h_{-1} = -\alpha$, $h_1 = -\gamma/\alpha$, the expressions

$$\begin{aligned} &h_2 = \left(\frac{\gamma}{2\alpha^2}\right)_x, \quad h_3 = \frac{\gamma^2}{2\alpha^3} - \left(\frac{1}{\alpha}\left(\frac{\gamma}{\alpha^2}\right)_x\right)_x, \quad h_4 = \textit{ total derivative}, \\ &h_5 = \frac{\gamma^3}{2\alpha^5} - \frac{1}{12}\frac{\gamma^2\alpha_{xx}}{\alpha^6} + \frac{1}{8}\frac{\gamma\gamma_{xx}}{\alpha^5} - \frac{7}{24}\frac{\gamma\gamma_x\alpha_x}{\alpha^6} + \textit{ total derivative}, \dots \end{aligned} \tag{4.3}$$

and so on and so forth.

The motivation for the change of variables, as well as further hints for our program come from considering of the dispersionless limit of (4.1), that is,

$$\begin{aligned} &\partial_3\alpha = \left(\frac{\gamma}{\alpha^2}\right)_x \\ &\partial_3\gamma = 0. \end{aligned} \tag{4.4}$$

This equation is bi–Hamiltonian w.r.t. to the Poisson tensors

$$P_0^{disp} = \begin{pmatrix} 0 & \partial_x\alpha \\ \alpha\partial_x & \gamma\partial_x + \partial_x\gamma \end{pmatrix} \qquad P_1^{disp} = \begin{pmatrix} \partial_x & 0 \\ 0 & 0 \end{pmatrix} \tag{4.5}$$

with Hamiltonian densities $h_3 = \gamma^2/2\alpha^3, h_1 = -\gamma/\alpha$. This property suggests that the full dispersive hierarchy can be obtained by suitably deforming the pencil of Poisson tensors (4.5).

As a first step in this direction, one notices that the flow (4.1) can be obtained in a "Hamiltonian" way, via the action of the antisymmetric tensors

$$P_0 = \begin{pmatrix} 0 & \partial_x\alpha \\ \alpha\partial_x & \gamma\partial_x + \partial_x\gamma + \frac{\alpha}{4}\partial_x T_\alpha^2\alpha \end{pmatrix}, \qquad P_1 = \begin{pmatrix} \partial_x & \frac{1}{4}\partial_x T_\alpha^2\alpha \\ \frac{\alpha}{4}\partial_x T_\alpha^2 & \frac{\alpha}{16}\partial_x T_\alpha^4\alpha \end{pmatrix}, \tag{4.6}$$

where T_α is the operator[3] $\dfrac{1}{\alpha}\partial_x$, as

$$\begin{pmatrix} \partial_3\alpha \\ \partial_3\gamma \end{pmatrix} = P_0 d\int h_3\,dx = P_1\int h_1\,dx,$$

where h_1 and h_3 are the densities (4.3). Furthermore, a direct computation shows that h_1 is the density of a Casimir of P_0. Actually, our use of this terminology is justified by the following proposition, whose proof, that can be directly obtained via a straightforward albeit tedious computation, will be apparent from the sequel.

[3]Operator composition is here and in the following, understood.

Proposition 4. *The tensors (4.6) are a pair of compatible Poisson tensors.*

To push our analysis further, the following observation is important. We notice that the member P_1 of the pair (4.6) is greatly degenerate. Indeed one sees that vector fields $(\dot{\alpha}, \dot{\gamma})$ belong to its image if and only if the relation

$$\dot{\gamma} = \frac{\alpha}{4}\partial_x \frac{1}{\alpha}\partial_x \frac{1}{\alpha}\dot{\alpha} \left(= \frac{\alpha}{4}\left(T_\alpha^\dagger\right)^2 \dot{\alpha}\right). \tag{4.7}$$

This entail that the system (4.1), as well as *any* bi–Hamiltonian vector field associated with the pair (4.6) admits as an invariant submanifold the one defined by

$$\gamma - \frac{1}{4}\partial_x^2 \ln\alpha + \frac{1}{8}(\partial_x \ln\alpha)^2 \left(\equiv \gamma - \frac{1}{8}(\alpha(T_\alpha - T_\alpha^\dagger)T_\alpha(\alpha)\right) = const. \tag{4.8}$$

This fact (together with the particularly simple dependence on γ of the relation (4.8)) prompts us to consider the dependent variable $u = y - \frac{1}{4}\partial_x^2 \ln\alpha + \frac{1}{8}(\partial_x \ln\alpha)^2$. In the coordinates (α, u) the tensors of (4.6) become

$$P_0 = \begin{pmatrix} 0 & \partial_x \alpha \\ \alpha\partial_x & u\partial_x + \partial_x u - \frac{1}{4}\partial_x^3. \end{pmatrix}, \qquad P_1 = \begin{pmatrix} \partial_x & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.9}$$

The fact that the antisymmetric tensors we are considering indeed make up a Poisson pair is now apparent from the theory of affine Poisson structures on duals of Lie algebras. This new form of the pencil will also allow us to state that the hierarchy of commuting vector fields starting with (4.1) is indeed a bi–Hamiltonian hierarchy.

According to the Gel'fand–Zakharevich bi–Hamiltonian scheme, we look for a Casimir of the pencil (4.9). This amounts to finding an exact one-form $\Omega(\lambda) = (X(\lambda), Y(\lambda))$ that satisfies the equation

$$(P_1 - \lambda P_0)\Omega = 0, \quad \text{with asymptotics } \Omega(\lambda) = \Omega_0 + \frac{\Omega_1}{\lambda} + \cdots,$$

whose first element is the differential of the Casimir of P_0 (in particular, with obvious meaning of the notation, $Y_0 \simeq \dfrac{1}{\alpha}$). So we can trade the above equation for the system

$$X(\lambda) = \lambda\alpha Y(\lambda); \qquad \lambda\alpha^2 Y(\lambda)^2 + 2uY(\lambda)^2 - \frac{1}{2}Y_{xx}(\lambda)Y(\lambda) + \frac{1}{4}(Y_x(\lambda))^2 = \lambda. \tag{4.10}$$

In turn, the second of these equations is equivalent to the following system

$$h_x + h^2 = \lambda\alpha^2 + 2u, \qquad h = \frac{z}{Y(\lambda)} + \frac{1}{2}\frac{Y_x(\lambda)}{Y(\lambda)}, \tag{4.11}$$

where $z^2 = \lambda$, and $h = h_{-1}z + h_0 + \dfrac{h_1}{z} + \cdots$. It can be easily shown that the series h(z) solving the first of these equations is, in the sense of formal Laurent series, indeed the potentials of the one-form $\Omega(\lambda)$. Also, the coefficients h_i can be algebraically computed in a recursive way.

The comparison of this Riccati equation with the Riccati equation associated with the local CH hierarchy suggests a further minor coordinate change, namely to set $m = \alpha^2$. Indeed in the coordinates (m,u) the Poisson pencil $P_1 - \lambda P_0$ is (4.9)

$$\begin{pmatrix} 2(\partial_x m + m\partial_x) & 0 \\ 0 & 0 \end{pmatrix} - \lambda \begin{pmatrix} 0 & \partial_x m + m\partial_x \\ \partial_x m + m\partial_x & -\frac{1}{4}\partial_x^3 + \partial_x u + u\partial_x \end{pmatrix}, \tag{4.12}$$

and the corresponding Riccati equation is

$$h_x + h^2 = 2u + mz^2, \qquad z = \sqrt{\lambda}. \tag{4.13}$$

The vector field (4.1) becomes simply

$$\partial_3 m = (2u\partial_x + 2\partial_x u - \frac{1}{2}\partial_x^3)\frac{1}{\sqrt{m}} \qquad\qquad \partial_3 u = 0\,, \tag{4.14}$$

Summing up,the search for a Casimir of the pencil (4.12) is reduced to the problem of solving - in the space of formal Laurent series - the Riccati equation for $h(z) = h_{-1}z + \sum_{i=1}^{\infty} \frac{h_i}{z^i}$. This problem can be iteratively solved, and is equivalent, up to the total derivative h_0, to (4.2) written in the u,m variables.

Remarks. **1)** On $u = \frac{1}{2}$ the first of the equations (4.14) becomes the first nontrivial local CH flow (2.2).

2) In the coordinates (m,u) (as well as in the coordinates (α,u)), all vector fields of this hierarchy are somewhat trivial, since they read

$$\partial_{t_i} m = \partial_x(F_i(m,u)), \qquad \partial_{t_i} u = 0. \tag{4.15}$$

This fact can be, in a sense, understood also in the framework of the theory of reciprocal transformations. For instance, transforming the system (4.1) under the reciprocal transformation induced by its first element (seen as a conservation law) yield the triangular system

$$\begin{aligned} \partial_3 U &= \frac{1}{2}(UV)_z \\ \partial_3 V &= \frac{1}{4}(V_{zzz} + 6VV_z) \end{aligned}$$

where $dx = Udz + \frac{1}{2}UVdt_3$, $U = \frac{1}{\alpha}$, and $V = \frac{-2\gamma}{\alpha^2}$. To fully examine these equations in the light of the theory of reciprocal transformations, however, is outside the aim of the present paper [9].

3) As a final check of the bi–Hamiltonian analysis we performed, we notice the following We exchange the role of the Poisson tensors P_0 and P_1 and consider the Casimir function $K = \int (u+m)dx$ of P_1. Clearly enough, the vector field $P_0 dK$ is just x-translation. This Casimir does not give rise to a new Lenard sequence, since $P_0 dK_0$ does not lie in the image of P_1. However from the fact that x=translation is the image under P_0 of a Casimir of P_1 confirms that it commutes with all the vector field of the hierarchy, as it should be.

5 Back to the CH hierarchy: its bi–Hamiltonian structure and its Lax representation

As we have seen, the bi–Hamiltonian geometry of the manifold we are considering is particularly simple: indeed, it is stratified by the submanifolds given by $u=\kappa$ for some constant κ, and these submanifolds are left invariant by all vector fields that are Hamiltonian w.r.t P_1, and thus by all bi–Hamiltonian vector fields. Also, on the invariant submanifold $u=\dfrac{1}{2}$ we have that the first flow of our hierarchy coincides with the first local CH flow, and the Riccati equation (4.13) reduces to the Riccati equation associated with the CH hierarchy (2.1).

These facts suggest the opportunity to consider the Dirac reduction of the pencil (4.12).

Proposition 5. *The Dirac reduction of (4.12) on the constraint $u=\kappa$ gives a Poisson pencil for the Camassa Holm. The hierarchy restricts to this submanifold as a bi–Hamiltonian hierarchy.*

Proof. To prove the assertion, we find it more convenient to use the notation of Poisson brackets rather than that of Poisson tensors. According with Dirac's theory, the reduction on $u=const$ of the Poisson brackets associated with our pencil is given by

$$\begin{aligned}\{m(x),m(y)\}_0^D &:= \{m(x),m(y)\}|_{u=\kappa}\\ &\quad -\int dw\int dz\{m(x),u(w)\}(\{u(w),u(z)\})^{-1}\{u(z),m(y)\}|_{u=\kappa}\end{aligned}$$

where $\{u^i(x),u^j(y)\}_0 := \int dz \frac{\delta u^i(x)}{\delta u^k(x)}(P_\lambda)^{kl}\frac{\delta u^j(y)}{\delta u^l(x)}$.

A simple computation shows that

$$P_\lambda^D|_{u=\kappa} = 2(\partial_x m + m\partial_x) - \lambda(\partial_x m + m\partial_x)\left(2\ \kappa\ \partial_x - \frac{1}{4}\partial_x^3\right)^{-1}(\partial_x m + m\partial_x).$$

It is easy to recognize in the above formula (one of) the Poisson pencils of the CH hierarchy, namely the one given by the standard Lie Poisson tensor and the first nonlocal tensor with the suitable choice $\kappa=\frac{1}{2}$ [4]. The Dirac reduction of the Poisson structure (4.12) generates exactly the local part of the CH hierarchy. This follows from the fact that the Dirac deformation of the Poisson bracket associated with P_0 is achieved by means of Casimir functions of the other brackets. This entails that Lenard relations $P_0 dH = P_1 dK$ hold also for the corresponding Dirac reductions. On the manifold $u=\kappa$ (e.g., $u=\frac{1}{2}$) we can recover the standard nonlocal part of CH hierarchy using the solution of (4.13) whose asymptotic behavior is $1+O(z)$ as in [4], via the usual CH substitution $m=4v-v_{xx}$. In terms of the geometry of bi–Poisson manifolds, with reference to Remark #3 of the previous Section, the situation is the following. In the extended two-field system, the Casimir $K=\int(u+m)dx$ of P_1 does not give rise to any additional Lenard sequence. On the contrary, in the CH constrained submanifold $u=\kappa$, this Casimir is iterable, and gives rise (with all the proviso about non-locality in mind) to the positive CH hierarchy. In other words, in this picture, the flows of the positive CH hierarchy (and so, the CH equation as well) play the role of "additional" (commuting) symmetries of these flows, which are restrictions to $u=\frac{1}{2}$ of the linear flows defined by (3.2).

□

[4]Indeed, κ can be rescaled to $\frac{1}{2}$ without loss of generality. For $\kappa=0$, we get a Poisson pencil of HD.

A further outcome the previous construction is to provide a Lax representation of the (extended) local CH hierarchy as a suitable flow in the space of pseudodifferential operators. We will basically follow a construction presented in [2] for the KdV–KP case.

The Riccati constraint (3.8) can be read as the requirement that the function $\psi = \exp\left(\int h dx\right)$ be an eigenfunction of the operator $L = \frac{1}{m}\partial_x^2 - \frac{2u}{m}$ with eigenvalue z^2. Also, the equations of motion imply $\partial_s J^{(r)} = \partial_r J^{(s)}$ and $\partial_s h = \partial_x J^{(s)}$. Therefore, from the compatibility of equations $L\psi = z^2\psi$ and $\partial_s \psi = J^{(s)}\psi$, we get

$$\partial_{2s+1} L = \left[J^{(2s+1)}, L\right] \qquad s \geq 1, \tag{5.1}$$

while times and currents with even label $2s$ are trivial, as implied by the constraint (3.8). In order to obtain an operatorial version of the equations of motion we relate the currents $J^{(s)}$ with L.

First of all we need the following technical

Lemma 6. *Under the constraint (3.8) it holds* $J^{(s)} = \Pi_{J_+}(z^s)$.

Proof The space $J_+ = \Pi_{J_+}(J)$ is, by definition, the linear span of $J^{(i)}$. Therefore there is a unique way to write the element $\Pi_{J_+}(z^s)$ by means of the currents $J^{(s)}$. Since the leading term of $J^{(s)}$ is exactly z^s, the assertion is true.

□

Because of Lemma 3, J_+ is also the linear span of the $\{(\partial_x + h)^i z^2\}_{i \geq 0}$. Moreover, extending by recursion the definition of $(\partial_x + h)^i z^2$ to negative powers, the set $\{(\partial_x + h)^i z^2\}_{i \in \mathbb{Z}}$ is a basis of all the space J. The map

$$\begin{aligned} \phi : J &\to \Psi DO \\ (\partial_x + h)^i z^2 &\to \partial_x^i \cdot L \end{aligned}$$

by means of the basis $(\partial_x + h)^n z^2$ with $n \in \mathbb{Z}$ of the space J, gives the operatorial action of an element J on ψ

Proposition 7. *Under the constraint (3.8) it holds*

$$J^{(s)}\psi = \left(L^{s/2-1}\right)_+ L\psi$$

Proof The map ϕ intertwines between Π_{J_+} and the operator $(\,\cdot\, L^{-1})_+ L$ on the ΨDO space where $(\,\cdot\,)_+$ is the standard projection on the differential part of a ΨDO operator. This property can be easily proved remarking that it holds for any element $(\partial_x + h)^i z^2$ of the J basis. Therefore

$$J^{(s)}\psi = \phi(J^{(s)})\psi = \phi(\Pi_{J_+}(z^s))\psi = (L^{s/2-1})_+ L\psi.$$

□

The equations (5.1) become then

$$\partial_{2s+1} L = \left[(L^{s-1/2})_+ L, L\right]. \tag{5.2}$$

We finally notice that in the CH case that is, under the constraint $u = \dfrac{1}{2}$, the Lax operator is $L = \dfrac{1}{m}\partial_x^2 - \dfrac{1}{m}$; therefore $(L^{1/2})_+ = m^{-1/2}\partial_x - \frac{1}{2}(m^{-1/2})_x$ and the previous equation gives

$$\partial_3 \frac{1}{m} = -2m^{-2}\left(\partial_x - \frac{1}{4}\partial_x^3\right) m^{-1/2}$$

which is equivalent to the local CH (2.2).

We end this Section noticing that the integrability of the system constructed starting from the Lax operator for local CH can be proven also by means of a direct computation. Indeed it holds:

Proposition 8. *Let DO_2 be the space of second order differential operators of the form $\lambda = a\partial^2 + b\partial + c$, and let $()_+$ be the projection operator from ΨDO to DO. The equations*

$$\partial_s \lambda = \left[(\lambda^{\frac{s}{2}})_+ \lambda, \lambda\right]$$

define a family of commuting flows on DO_2, that is, $\partial_r\partial_s\lambda = \partial_s\partial_r\lambda$.

Proof. We start expanding

$$\begin{aligned}
\partial_r\partial_s\lambda &= \partial_r\left[(\lambda^{\frac{s}{2}})_+\lambda, \lambda\right] = \left[(\partial_r\lambda^{\frac{s}{2}})_+\lambda, \lambda\right] + \left[(\lambda^{\frac{s}{2}})_+\partial_r\lambda, \lambda\right] + \left[(\lambda^{\frac{s}{2}})_+\lambda, \partial_r\lambda\right] \\
&= \left[\left[(\lambda^{\frac{r}{2}})_+\lambda, \lambda^{\frac{s}{2}}\right]_+ \lambda, \lambda\right] + \left[(\lambda^{\frac{s}{2}})_+\left[(\lambda^{\frac{r}{2}})_+\lambda, \lambda\right], \lambda\right] + \left[(\lambda^{\frac{s}{2}})_+\lambda, \left[(\lambda^{\frac{r}{2}})_+\lambda, \lambda\right]\right],
\end{aligned}$$

as well as $\partial_s\partial_r\lambda$. Then the assertion follows using standard techniques in the ΨDO approach to KP-type equations (see e.g. [7]), with the crucial remarks that, since we are considering degree 2 operators,

$$\left(\left[(\lambda^{\frac{r}{2}})_-\lambda, (\lambda^{\frac{s}{2}})_-\right]\right)_+ = \left((\lambda^{\frac{r}{2}})_-\left[(\lambda^{\frac{s}{2}})_-, \lambda\right]\right)_+ = 0,$$

because the degrees of the operators appearing in these expression is less than zero.

□

Acknowledgments. The authors would like to thank Marco Pedroni, Boris Dubrovin, Andrew Hone, Paolo Lorenzoni, and Franco Magri for useful discussions and remarks. G.O also benefited from discussions with Paolo Casati and Boris Konopelchenko, and thanks the University of Milano Bicocca for the kind hospitality, as well as the organizer of the NEEDS 2007 Conference. This paper was partially supported by the European Community through the FP6 Marie Curie RTN *ENIGMA* (Contract number MRTN-CT-2004-5652), by the European Science Foundation project *MISGAM*, and by the Italian MIUR Cofin2006 project “Geometrical methods in the theory of nonlinear waves and applications”.

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