Journal of Nonlinear Mathematical Physics Volume 15, Supplement 3 (2008), 345–352 ARTICLE

On Differential Operators on Sequence Spaces

M Maldonado, J Prada [a] and M J Senosiain

[a] *Dpto. Matemáticas, Plaza de la Merced 1-4, 37008 Salamanca, Spain*
E-mail: prada@usal.es

Abstract

Two differential operators T_1 and T_2 on a space Λ are said to be equivalent if there is an isomorphism S from Λ onto Λ such that $S\,T_1 = T_2\,S$.

The notion was first introduced by Delsarte in 1938 [2] where T_1 and T_2 are differential operators of second order and Λ a space of functions of one variable defined for $x \geq 0$. From them on several authors studied generalizations, applications and related problems [6], [7], [8], [11], [12].

In 1957 Delsarte and Lions [3] proved that if T_1 and T_2 are differential operators of the same order without singularities on the complex plane, Λ being the space of entire functions, then they are equivalent. Using sequence spaces and the fact that the space of entire functions is isomorphic to a power series space of infinite type, we find the same result in a simpler way in our opinion. The same method gives the result for differential operators of the same order with analytic coefficients on the space of holomorphic function on a disc, considering that spaces of holomorphic functions on a disc are isomorphic to a finite power series space. The method can be applied as well to linear differential operators of the same order on other sequence spaces, finding conditions for them to be equivalent. Finally using the fact that the space $\mathscr{C}^\infty_{2\pi}(\mathbb{R})$ of all 2π-periodic $\mathscr{C}^\infty$-functions on $\mathbb{R}$ is isomorphic to s, the space of rapidly decreasing sequences, we prove that two linear differential operators of order one with constant coefficients are not equivalent; this result can be extended to linear differential operators of greater order but the proof is essentially the same.

1 Introduction

Let T_1 and T_2 be two linear differential operators defined on a space Λ and consider the following problem,

Is there an isomorphism S from Λ onto Λ such that $T_1S = ST_2$?.

The isomorphism S was called "operateur de transmutation" by Delsarte in 1938, to whom the notion is due, in a paper that deals with two differential operators of second order and a space of functions of one variable [2]. Applications and generalizations can be found in [7, 8, 9].

If the operators T_1 and T_2 are of order greater than two with infinitely differentiable coefficients and the considered space is $C^\infty(\mathbb{R})$, the space of infinitely differentiable functions on $\mathbb{R}$, a particular solution of the problem is given in [7]. In 1957, Delsarte and Lions in [3] studied the question in the complex plane, precisely taking two differential operators of the same order without singularities in the complex plane and the space of entire functions.

Later on the subject was taken up by some russian mathematicians calling the problem equivalence of differential operators. For instance, Viner studied transformations of differential operators in the space of holomorphic functions [12], Kushnirchuk, Nagnibida and Fishman considered the equivalence of differential operators with a regular singular point [6] while Nagnibida and Oliinyk studied the equivalence of differential operators of infinite order on analytic spaces [11].

The question of the equivalence of differential operators is a very general one and the problem can be treated in many different contexts and so the literature concerning the subject is very wide. Let us mention, for instance, that in 1989, Kamran and Olver in [4] determine when two second-order differential operators on the line are the same under a change of variables while Volchkov, in 2000, considered spaces of analytic functions over the Tate field and differential operators on them [13].

In this paper we assume that T_1 and T_2 are linear differential operators of the same order and its coefficients are elements of a sequence space Λ. When $\Lambda = \mathscr{H}(\mathbb{C})$, space of entire functions on the complex plane or $\Lambda = \mathscr{H}(\mathbb{D})$, space of holomorphic functions on the unit disc, we prove that T_1 and T_2 are equivalent; the first result is due to Delsarte and Lions [3] but our proof is, in our opinion, much simpler. If $\Lambda = s$, space of sequences rapidly decreasing to zero and T_1 and T_2 are linear differential operators of order one with constant coefficients, then they are equivalent. In fact, when T_1 and T_2 are linear differential operator of order one with constant coefficients we give a sufficient condition for them to be equivalent on any Köthe space. The condition is not necessary as the case $\Lambda = s$ shows. If $\Lambda = C^\infty_{2\pi}(\mathbb{R})$, space of all 2π-periodic C^∞-functions on $\mathbb{R}$ we prove that two operators of order one with constant coefficients are not equivalent using the fact that $C^\infty_{2\pi}(\mathbb{R})$ and s are isomorphic. Finally we generalize some results using the Gelfand-Leontev derivative.

For the terminology used in this paper see [1, 5, 10].

2 Differential operators on sequence spaces

A linear differential operator of order m, $D^m + p_{m-1}D^{m-1} + p_{m-2}D^{m-2} + \cdots + p_1 D + p_0 I$, can be represented by a matrix $(t_{j,n})_{j,n=0}^\infty$ with

$$\begin{aligned}
t_{j,n} &= \sum_{i=0}^{n} n(n-1)\ldots(i+1)p_{n-i,j-i} && m \geq n \\
t_{j,n} &= 0 && j < n-m,\, n > m \\
t_{n-m,n} &= n(n-1)\ldots(n-m+1) && n > m \\
t_{n-m+r,n} &= \sum_{i=1}^{r} n(n-1)\ldots(n-m+i+1)p_{m-i,r-i} && r \geq 1,\, n > m,
\end{aligned}$$

where p_i, $0 \leq i \leq m-1$ are elements of a sequence space (Köthe space) Λ considered as functions $p_i(z) = \sum_{k=0}^\infty p_{i,k} z^k$.

Given two linear differential operators

$$T_1 = D^m + p_{m-1}D^{m-1} + p_{m-2}D^{m-2} + \cdots + p_1 D + p_0 I$$

and

$$T_2 = D^m + q_{m-1}D^{m-1} + q_{m-2}D^{m-2} + \cdots + q_1 D + q_0 I$$

on a sequence space Λ to look for the isomorphism S that can be represented by a matrix $S = (S_{j,n})_{j,n=0}^{\infty}$ implies two things. First we have to solve the algebraic equations given by $T_1 S = ST_2$ and second to study the continuity of S and S^{-1}.

Assume that if $0 \le j \le m-1$, then $S_{j,n} = 1$ if $j = n$ and 0 if $j \neq n$.

Solving the equation $T_1 S = ST_2$ using the program Maple (or the program Mathematica) we get

$$\begin{aligned} S_{j+m,n} &= \frac{j!}{(j+m)!}\frac{n!}{(n-m)!} S_{j,n-m} \\ &\quad + \sum_{i=0}^{m-1}\sum_{l=0}^{j-n+i} \frac{j!\,n!}{(j+m)!(n-i)!} q_{i,l}\, S_{j,l+n-i} \\ &\quad - \sum_{i=0}^{m-1}\sum_{l=0}^{j-n+i} \frac{j!(j-l+i)!}{(j+m)!(j-l)!} p_{i,l}\, S_{j-l+i,n}, \qquad n \ge m \end{aligned}$$

$$\begin{aligned} S_{j+m,n} &= \sum_{i=0}^{n}\sum_{l=0}^{j-n+i} \frac{j!\,n!}{(j+m)!(n-i)!} q_{i,l}\, S_{j,l+n-i} \\ &\quad - \sum_{i=0}^{m-1}\sum_{l=0}^{j-n+i} \frac{j!(j-l+i)!}{(j+m)!(j-l)!} p_{i,l}\, S_{j-l+i,n}, \qquad n < m \end{aligned}$$

$$S_{j+m,n} = 0 \qquad j+m < n.$$

The matrix S is invertible and its inverse S^{-1} is given by the above formulas interchanging the $p's$ and $q's$ as $S^{-1}T_1 = T_2 S^{-1}$. Therefore it is enough to study the continuity of S.

When the order of the operators is $m = 1$ and the coefficients are constants, that is $T_1 = D + pI$ and $T_2 = D + qI$, then the formula for the elements of the matrix S is much simpler, in fact

$$\begin{aligned} S_{j,n} &= \frac{1}{(j-n)!}(q-p)^{j-n} \qquad && j \ge n \\ S_{j,n} &= 0 && j < n. \end{aligned}$$

2.1 Equivalence of differential operators

The space $\mathscr{H}(\mathbb{C})$ of entire functions with the open-compact topology (respectively the space of holomorphic functions on the unit disc $\mathscr{H}(\mathbb{D})$) is isomorphic to the sequence space

$$\Lambda = \{x = (x_n) : \sum_{n=0}^{\infty} |x_n| e^{kn} < \infty, \forall k \in \mathbb{N}\}$$

respectively to the sequence space

$$\Lambda = \{x = (x_n) : \sum_{n=0}^{\infty} |x_n| e^{-\frac{n}{k}} < \infty, \forall k \in \mathbb{N}\}$$

with its usual topology, that is the one given by the norms

$$\|x\|_k = \sum_{n=0}^{\infty} |x_n| e^{kn}$$

or equivalently by the norms

$$\|x\|_k = \sup\left\{|x_n|e^{kn}\right\}$$

respectively the norms

$$\|x\|_k = \sum_{n=0}^{\infty} |x_n|e^{-\frac{n}{k}}$$

or

$$\|x\|_k = \sup\left\{|x_n|e^{-\frac{n}{k}}\right\}.$$

We have the following result

Theorem 1. *Let T_1 and T_2 be two linear differential operators of order m with entire (respectively analytic on the unit disc) functions coefficients. Then they are equivalent on the space of entire functions $\mathscr{H}(\mathbb{C})$ (respectively in the space $\mathscr{H}(\mathbb{D})$), that is, there exits an isomorphism $S\colon \mathscr{H}(\mathbb{C}) \to \mathscr{H}(\mathbb{C})$ (respectively $S\colon \mathscr{H}(\mathbb{D}) \to \mathscr{H}(\mathbb{D})$) such that*

$$T_1 S = S T_2$$

and

$$b_i = a_i, \qquad i = 0, 1, 2, \ldots, m-1$$

where (a_i) (respectively (b_i)) are the Taylor coefficients of an entire or analytic function on the disc $f(z) = \sum_{i=0}^{\infty} a_i z^i$ (respectively of its image $Sf(z) = \sum_{i=0}^{\infty} b_i z^i$).

Proof. We present only the proof for the case $\mathscr{H}(\mathbb{C})$. The proof for the case $\mathscr{H}(\mathbb{D})$ is similar.

S is continuous if and only if the following condition is verified:

$\forall k \in \mathbb{N}$, $\exists N(k) \in \mathbb{N}$, $\exists W(k)$ such that

$$|S_{j+m,n}| \le W(k)\frac{e^{N(k)n}}{e^{k(j+m)}}, \forall j, n. \tag{2.1}$$

We prove the result by induction on j. Given k take $N(k) = k+1$ (any $N(k) > k$ would do) and $j = j(k)$ such that the condition (2.1) is fulfilled for all $j \le j(k)$. Assume $j = j(k)+1$ and check the condition.

Consider the terms $S_{j+m,n}$ where $n \ge m$. We have

$$\begin{aligned} |S_{j+m,n}| \le{}& \frac{j!}{(j+m)!}\frac{n!}{(n-m)!}|S_{j,n-m}| \\ &+ \sum_{i=0}^{m-1}\sum_{l=0}^{j-n+i} \frac{j!\,n!}{(j+m)!(n-i)!}\,|q_{i,l}|\,|S_{j,l+n-i}| \\ &+ \sum_{i=0}^{m-1}\sum_{l=0}^{j-n+i} \frac{j!(j-l+i)!}{(j+m)!(j-l)!}\,|p_{i,l}|\,|S_{j-l+i,n}| \end{aligned} \tag{2.2}$$

The first part of formula (2.2), that is $\frac{j!}{(j+m)!}\frac{n!}{(n-m)!}|S_{j,n-m}|$ verifies

$$\frac{j!}{(j+m)!}\frac{n!}{(n-m)!}|S_{j,n-m}| \le W(k)\frac{e^{(k+1)(n-m)}}{e^{kj}} = W(k)\frac{e^{(k+1)n}}{e^{k(j+m)}}e^{-m}.$$

With respect to the second and third part of formula (2.2) we have

$$\frac{j!\,n!}{(j+m)!(n-i)!} \le \frac{1}{j+m}$$
$$\frac{j!(j-l+i)!}{(j+m)!(j-l)!} \le \frac{1}{j+m}$$

and

$$\sum_{i=0}^{m-1}\sum_{l=0}^{j-n+i}\frac{j!\,n!}{(j+m)!(n-i)!}|q_{i,l}|\,|S_{j,l+n-i}| \le W(k)\frac{e^{(k+1)n}}{e^{k(j+m)}}\frac{1}{j+m}e^{km}\sum_{i=0}^{m-1}\|q_i\|_{k+1}$$

and

$$\sum_{i=0}^{m-1}\sum_{l=0}^{j-n+i}\frac{j!(j-l+i)!}{(j+m)!(j-l)!}|p_{i,l}|\,|S_{j-l+i,n}| \le W(k)\frac{e^{(k+1)n}}{e^{k(j+m)}}\frac{1}{j+m}e^{km}\sum_{i=0}^{m-1}\|p_i\|_k\,.$$

Taking $j(k)$ large enough we have

$$\frac{1}{j+m}e^{km}\sum_{i=0}^{m-1}\left(\|q_i\|_{k+1}+\|p_i\|_k\right) \le 1-e^{-m}$$

and so

$$|S_{j+m,n}| \le W(k)\frac{e^{(k+1)n}}{e^{k(j+m)}}.$$

The case $n<m$ is proved in an analogous way. ∎

Considering the space s we have

Theorem 2. *Let T_1 and T_2 be two linear differential operators of order one with constant coefficients. Then they are equivalent on the space of space of sequences rapidly decreasing to zero s.*

Proof. It is enough to prove that

$$\sup_{n\in\mathbb{N}}\sum_{j=n}^{\infty}\frac{|q-p|^{j-n}}{(j-n)!}\frac{j^k}{n^k} = M_k < \infty$$

In fact:

$$\begin{aligned}
\sum_{j=n}^{\infty}\frac{|q-p|^{j-n}}{(j-n)!}\frac{j^k}{n^k} &= \sum_{j=0}^{\infty}\frac{|q-p|^{j}}{j!}\frac{(j+n)^k}{n^k}\\
&\leq \sum_{j=0}^{\infty}\frac{|q-p|^{j}}{j!}e^{k(1+\frac{j}{n})} = e^k\sum_{j=0}^{\infty}\frac{|q-p|^{j}}{j!}(e^{\frac{k}{n}})^j\\
&= e^k e^{|q-p|e^{\frac{k}{n}}} \leq e^k e^{|q-p|e^k} = M_k
\end{aligned}$$

■

The space $C^{\infty}_{2\pi}(\mathbb{R})$ with the topology induced by the system of seminorms

$$\|f\|_k^2 = \sum_{0\leq p\leq k}\left\|f^{p)}\right\|^2_{L_2[-\pi,\pi]}$$

is isomorphic to s and using this fact it follows

Theorem 3. *Let T_1 and T_2 two linear differential operators of order one with constant coefficients. Then they are not equivalent on the space $C^{\infty}_{2\pi}(\mathbb{R})$ of all 2π-periodic C^{∞}-functions on $\mathbb{R}$.*

Proof. Let $f\in C^{\infty}_{2\pi}(\mathbb{R})$ be given and

$$f(x) = \sum_{n\in\mathbb{Z}}\widetilde{f}_n e^{inx}$$

be the Fourier series of f. The isomorphism between $C^{\infty}_{2\pi}(\mathbb{R})$ and s is given by the function

$$F:\ f\longmapsto(\widetilde{f}_0,\widetilde{f}_1,\widetilde{f}_{-1},\widetilde{f}_2,\widetilde{f}_{-2},\ldots).$$

Therefore given the operators $T_1 = D+pI$ and $T_2 = D+qI$ on $C^{\infty}_{2\pi}(\mathbb{R})$ they induce on s two operators A_1 (respectively A_2) given by the diagonal matrix

$$\begin{pmatrix}
p & & & & & & \\
& i+p & & & & & \\
& & -i+p & & & & \\
& & & 2i+p & & & \\
& & & & -2i+p & & \\
& & & & & 3i+p & \\
& & & & & & \ddots
\end{pmatrix}$$

(respectively by the diagonal matrix substituting p by q). If there is an isomorphism S such that $T_1S = ST_2$ then there is an isomorphism $M = FSF^{-1}$ such that $A_1M = MA_2$ which is not possible.

■

3 Generalizations

Considering the Gelfand-Leontev derivative, $D_\gamma x^n = \gamma_n x^{n-1}$, $n \geq 1$, $Dx^0 = 0$, where $\gamma_1 \leq \gamma_2 \leq \cdots \nearrow \infty$, (when $\gamma_n = n$, $D_\gamma = D$, the ordinary derivative), we can study the equivalence of linear differential operators with respect to this derivative. The operator D_γ is assumed to be continuous. We give below sufficient conditions for linear operators with constant coefficients to be equivalent.The proofs of the following theorems are similar to the case of analytic functions and therefore not given.

Theorem 4. *Assume that* $\forall\, k \in \mathbb{N}$

$$\sup_{n,j\geq 0} \left\{ \frac{(n+j)!}{n!} \frac{\gamma_n}{\gamma_{n+j}} \frac{a^k_{n+j}}{a^k_n} \right\} = M_k < \infty$$

where (a^k_n) *is a Köthe matrix . The the operators* $T_1 = D_\gamma + pI$ *and* $T_2 = D_\gamma + qI$ *are equivalent in the Köthe space* $\lambda^1(a^k_n)$.

Theorem 5. *Let* $\Lambda_\infty(\alpha)$ *be a nuclear infinite power space (respectively* $\Lambda_0(\alpha)$ *a nuclear finite power series space) such that:*

$$\sup_{n\in\mathbb{N}} \left\{ \frac{e^{\alpha_n}}{e^{\alpha_{n-1}}} \right\} < \infty$$

and $\forall k \in \mathbb{N}$, $\exists r = r(k)$, $r > k$ *such that*

$$\sup_{j\geq n, n\geq m} \left\{ \frac{e^{k\alpha_j}}{e^{k\alpha_{j-m}}} \frac{e^{r\alpha_{n-m}}}{e^{r\alpha_n}} \right\} < 1$$

respectively

$$\sup_{j>n, n\geq m} \left\{ \frac{e^{-\frac{\alpha_j}{k}}}{e^{-\frac{\alpha_{j-m}}{k}}} \frac{e^{-\frac{\alpha_{n-m}}{r}}}{e^{-\frac{\alpha_n}{r}}} \right\} < 1.$$

Then the operators $T_1 = D^m_\gamma + p_{m-1} D^{m-1}_\gamma + \cdots + p_0 I$ *and* $T_2 = D^m_\gamma + q_{m-1} D^{m-1}_\gamma + \cdots + q_0 I$ *are equivalent.*

Assuming that the Köthe space is an infinite (or finite) power series space algebra, that is $\exists W$ such that

$$e^{\alpha_{n+m}} \leq W e^{\alpha_n} e^{\alpha_m}, n, m \in \mathbb{N}$$

respectively

$$e^{-\alpha_{n+m}} \leq W e^{-\alpha_n} e^{-\alpha_m}, n, m \in \mathbb{N}$$

and considering differential operators with coefficients in the algebra we have

Theorem 6. *Let $\Lambda_\infty(\alpha)$ be a nuclear infinite power series algebra (respectively $\Lambda_0(\alpha)$ a nuclear finite power series algebra). Assume that $\forall k \in \mathbb{N}$, $\exists r = r(k) \in \mathbb{N}$, $r \geq k$ such that*

$$N_k = \sup \left\{ W \frac{e^{r\alpha_{n-m}}}{e^{r\alpha_n}} e^{k\alpha_m} \right\} < 1$$

(respectively $\forall k \in \mathbb{N}$, $\exists r = r(k) \in \mathbb{N}$, $r \geq k$ such that

$$N_k = \sup \left\{ W \frac{e^{-\frac{\alpha_{n-m}}{r}}}{e^{-\frac{\alpha_n}{r}}} e^{-\frac{\alpha_m}{k}} \right\} < 1.$$

Then the operators $T_1 = D_\gamma^m + p_{m-1} D_\gamma^{m-1} + \cdots + p_0 I$ and $T_2 = D_\gamma^m + q_{m-1} D_\gamma^{m-1} + \cdots + q_0 I$ are equivalent.

Acknowledgments. This work has been supported by MTM2006-07618.

References

[1] DI BUCCHIANICO A, Probabilistic and analytical aspects of the umbral calculus, *CW/TRACT* 1996.

[2] DELSARTE J, Sur certaines transformations. *C. R. Acad. Sci. Paris* **206** (1938) p. 1780.

[3] DELSARTE J and LIONS J L, Transmutations d'opérateurs différentiels dans le domaine complexe, *Comment. Mat. Helvetici* **32** (1957) 113–128.

[4] KAMRAN N and OLVER P J, Equivalence of differential operators. *Siam J. On Math. Analysis, SIMA* **20** (1989), 1172–1185.

[5] KÖTHE G, Topological Vector Spaces I, *Springer* 1969.

[6] KUSHNIRCHUK F, NAGNIBIDA N I and FISHMAN K M, Equivalence of differential operators with a regular singular point. *Funct. Analysis and its applications* **8** (1974) 169–171.

[7] LIONS J L, Quelques applications d'opérateurs de transmutation. (French) La théorie des équations aux dérivées partielles. Nancy, 9-15 Avril (1956) 125–137. *Colloques Internationaux du Centre National de la Recherche Scientifique, LXXI Centre National de la Recherche Scientifique, Paris.*

[8] LIONS J L, Opérateurs de Delsarte et problèmes mixtes. *Bull. Soc. Math. France* **84** (1956) 9–95.

[9] LIONS J L, Solutions élémentaires de certains opérateurs différentiels à coefficients variables. *J. Math. Pures Appl.* (1957).

[10] MEISE R and VOGT D, Introduction to Functional Analysis, *Oxford Sci. Pub.* 1997.

[11] NAGNIBIDA N I and OLIINYK N P, The equivalence of differential operators of infinite order in analytic spaces. *Translated from Matematicheskie Zametki* **21** (1977) 33–37.

[12] VINER I J, Transformation of differential operators in the space of holomorphic functions. *Uspehi Mat. Nauk.* **20** (1965) 185–188.

[13] VOLCHKOV V V, quivalence of differential operators in spaces of analytic functions over the Tate field. *Ukrainian Math. J.* **52** (2000), 1826–1833.