Remarks on the Waterbag Model of Dispersionless Toda Hierarchy

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Abstract
We construct the free energy associated with the waterbag model of dToda. Also, relations for conserved densities are investigated.

1 Introduction
The dispersionless Toda hierarchy (dToda) is one of the most important 2+1-dimensional integrable models. The dToda hierarchy appears in several areas of mathematics and physics, such as the integrable structure of interface dynamics [16, 28], the solution of Dirichlet boundary value problems [15], multi-slit conformal mappings of the unit disc (the radial Löwner equation [24, 25]), two-dimensional string theory [22], the large-N limit of normal matrix models [1, 27, 25] and the WDVV equation coming from topological field theory [4]. On the other hand, the dToda equation (see below) is also known as the Boyer-Finley equation, and can be used to generate a scalar-flat Kahler metric with a Killing field [5]. It also appears in the classification of self-dual Einstein metrics [3, 5] and in the twistor construction of Einstein-Weyl spaces [11, 26].

We quickly review some facts about the dToda hierarchy. It is defined by [23]
\[
\begin{align*}
\frac{\partial \lambda}{\partial t_n} &= \{B_n(p), \lambda\}, & \frac{\partial \hat{\lambda}}{\partial \hat{t}_n} &= \{\hat{B}_n(p), \hat{\lambda}\}, \\
\frac{\partial \hat{\lambda}}{\partial t_n} &= \{B_n(p), \hat{\lambda}\}, & \frac{\partial \lambda}{\partial \hat{t}_n} &= \{\hat{B}_n(p), \lambda\},
\end{align*}
\]

where the Lax operators $\lambda$ and $\hat{\lambda}$ are
\[
\begin{align*}
\lambda &= e^p + \sum_{n=0}^{\infty} u_{n+1} e^{-np}, \\
\hat{\lambda}^{-1} &= \hat{u}_0 e^{-p} + \sum_{n=0}^{\infty} \hat{u}_{n+1} e^{np},
\end{align*}
\]
and
\[
B_n(p) = [\lambda^n]_{\geq 0}, \quad \hat{B}_n(p) = [\hat{\lambda}^{-n}]_{\leq -1}.
\]

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Here \([\cdots]_{\geq 0}\) and \([\cdots]_{\leq -1}\) denote the non-negative part and negative part of \(\lambda^n\) and \(\hat{\lambda}^{-n}\) respectively when expressed as Laurent series in \(e^p\). For example,

\[
B_1(p) = e^p + u_1, \quad \hat{B}_1(p) = \hat{u}_0 e^{-p}.
\]

Finally, the Poisson Bracket in (1.1) is

\[
\{f(t_0, p), g(t_0, p)\} = \frac{\partial f}{\partial t_0} \frac{\partial g}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial g}{\partial t_0}.
\]

One can view the complex-valued quantity \(\lambda\) as a local coordinate near \(\infty\) and \(\hat{\lambda}\) as a local coordinate near 0 \([13]\).

According to dToda theory \([23]\), there exist wave functions \(S, \hat{S}\) and the dispersionless \(\tau\) function \(F\) (or free energy), with

\[
S(\lambda) = \sum_{n=1}^{\infty} t_n \lambda^n + t_0 \ln \lambda - \sum_{n=1}^{\infty} \frac{\partial_{t_n} F}{n} \lambda^{-n},
\]

\[
\hat{S}(\hat{\lambda}) = \sum_{n=1}^{\infty} \hat{t}_n \hat{\lambda}^{-n} + t_0 \ln \hat{\lambda} - \sum_{n=1}^{\infty} \frac{\partial_{\hat{t}_n} F}{n} \hat{\lambda}^{n},
\]

such that

\[
B_n(\lambda) = \partial_{t_n} S(\lambda) = \lambda^n - \sum_{m=1}^{\infty} \frac{\partial_{t_m}^2 F}{m} \lambda^{-m},
\]

\[
B_n(\hat{\lambda}) = \partial_{\hat{t}_n} \hat{S}(\hat{\lambda}) = \hat{\lambda}^{-n} - \sum_{m=1}^{\infty} \frac{\partial_{\hat{t}_m}^2 F}{m} \hat{\lambda}^{m},
\]

\[
\hat{B}_n(\lambda) = \partial_{\hat{t}_n} S(\lambda) = - \sum_{m=1}^{\infty} \frac{\partial_{t_m}^2 F}{m} \lambda^{-m},
\]

\[
\hat{B}_n(\hat{\lambda}) = \partial_{\hat{t}_n} \hat{S}(\hat{\lambda}) = \hat{\lambda}^{-n} + \partial_{\hat{t}_m}^2 F - \sum_{m=1}^{\infty} \frac{\partial_{t_m}^2 F}{m} \hat{\lambda}^m.
\]

In particular, we have

\[
p(\lambda) = \partial_{t_0} S(\lambda) = \ln \lambda - \sum_{m=1}^{\infty} \frac{\partial_{t_m}^2 F}{m} \lambda^{-m},
\]

\[
p(\hat{\lambda}) = \partial_{\hat{t}_0} \hat{S}(\hat{\lambda}) = \ln \hat{\lambda} + \partial_{\hat{t}_m}^2 F - \sum_{m=1}^{\infty} \frac{\partial_{\hat{t}_m}^2 F}{m} \hat{\lambda}^m.
\]

(1.2)

Also,

\[
H^+_m = \partial_{t_m}^2 F = \frac{1}{m} \int_{-\infty}^{\infty} \lambda^m d\xi = \frac{1}{m} \int_{-\infty}^{\infty} \lambda^m \frac{d\xi}{\xi},
\]

\[
H^-_m = \partial_{\hat{t}_m}^2 F = \frac{1}{m} \int_{0}^{\infty} \hat{\lambda}^{-m} d\xi = \frac{1}{m} \int_{0}^{\infty} \hat{\lambda}^{-m} \frac{d\xi}{\xi}, \quad m \geq 1
\]

(1.3)
are the conserved densities of the dToda hierarchy, where \( p = \ln \xi \). Then the dToda hierarchy (1.1) can be expressed as
\[
\frac{\partial p(\lambda)}{\partial t_n} = \frac{\partial B_n(p(\lambda))}{\partial t_0}, \quad \frac{\partial p(\hat{\lambda})}{\partial t_n} = \frac{\partial B_n(p(\hat{\lambda}))}{\partial t_0},
\]
\[
\frac{\partial \hat{p}(\lambda)}{\partial t_n} = \frac{\partial \hat{B}_n(p(\lambda))}{\partial t_0}, \quad \frac{\partial \hat{p}(\hat{\lambda})}{\partial t_n} = \frac{\partial \hat{B}_n(p(\hat{\lambda}))}{\partial t_0},
\]
(1.4)
\[
\lambda, \hat{\lambda} \text{ being fixed. The systems (1.4) for all } n \text{ are the conservation laws for the dToda hierarchy.}
\]
From (1.2), one knows that
\[
B_1(p) = e^p + u_1 = e^p + \partial_{t_1}^2 F, \quad \hat{B}_1(p) = \hat{u}_0 e^{-p} = e^{\partial_{t_0}^2 F} e^{-p}.
\]
Then from (1.4), one has
\[
p_{t_1} = \partial_{t_0} [e^{\partial_{t_0}^2 F} e^{-p}], \quad p_{t_1} = \partial_{t_0} [e^p + \partial_{t_0}^2 F].
\]
Then \( p_{t_1 t_1} = p_{t_1 t_1} \) will imply
\[
\partial_{t_1 t_1}^2 F = -e^{\partial_{t_0}^2 F}.
\]
The latter is the dToda equation, also known as the Boyer-Finley equation.

This paper is organized as follows. In the next section, we construct the waterbag model of dToda type from the Hirota equation. Section 3 is devoted to finding the free energy associated with the waterbag model from the Landau-Ginzburg formulation in topological field theory. Also, equations for the conserved densities are obtained. In the final section, we discuss some further problems to be investigated.

2 Dispersionless Hirota Equation and Symmetry Constraints

The dToda hierarchy (1.1) (or (1.4)) is equivalent to the dispersionless Hirota equation [10]:
\[
D_{\mu} p(\lambda) = -\partial_{t_0} \ln[e^p(\lambda) - e^{\mu(\lambda)}], \quad D_{\mu} p(\hat{\lambda}) = -\partial_{t_0} \ln[1 - e^{\mu(\hat{\lambda}) - p(\hat{\lambda})}],
\]
\[
D_{\hat{\mu}} p(\lambda) = -\partial_{t_0} \ln[1 - e^{p(\mu) - p(\lambda)}], \quad D_{\hat{\mu}} p(\hat{\lambda}) = -\partial_{t_0} \ln[e^p(\hat{\lambda}) - e^{\hat{\mu}(\hat{\lambda})}],
\]
where
\[
D_{\mu} = \sum_{m=1}^{\infty} \mu^{-m} \partial_{t_m}, \quad D_{\hat{\mu}} = \sum_{m=1}^{\infty} \hat{\mu}^{-m} \partial_{t_m}.
\]
We can also express them in terms of the \( S \)-function
\[
D_{\mu} S(\lambda) = -\ln\left[\frac{e^{p(\lambda)} - e^{\mu(\lambda)}}{\mu}\right], \quad D_{\hat{\mu}} S(\lambda) = -\ln[1 - e^{p(\mu) - p(\lambda)}],
\]
\[
D_{\mu} \hat{S}(\hat{\lambda}) = -\ln[1 - e^{p(\mu) - p(\hat{\lambda})}], \quad D_{\hat{\mu}} \hat{S}(\hat{\lambda}) = -\ln\left[\frac{e^{p(\hat{\lambda})} - e^{\hat{\mu}(\hat{\lambda})}}{\mu}\right].
\]
(2.1)

Next, we consider the symmetry constraints. These symmetry constraints relate the “non-isospectral symmetry”, to the “isospectral symmetry” using the wave function \( S \) and the dispersionless \( \tau \) function \( F \), similar to the case with dispersion [14]. The point is that such a symmetry
constraint can reduce a 2+1-dimensional dispersionless equation to a set of 1+1-dimensional systems of integrable hydrodynamic type with a finite number of dependent variables. The details can be found in [2]. From this integrable hydrodynamic system, one can find the exact solutions using the generalized hodograph method [18, 21].

Given the infinitesimal symmetries $\delta F$, the symmetry constraints can be written as [2]

$$\delta F = \sum_{i=1}^{N} \epsilon_i S_i, \quad \text{where} \quad S_i = S(\lambda_i).$$

- **Case I:** $F_0 = \sum_{i=1}^{N} \epsilon_i S_i$.

Near $\infty$ we have, by (2.1),

$$p = \ln \lambda - D_\lambda F_0 = \ln \lambda - D_\lambda \sum_{i=1}^{N} \epsilon_i S_i$$

$$= \ln \lambda - \sum_{i=1}^{N} \epsilon_i D_\lambda S_i$$

$$= \ln \lambda - \sum_{i=1}^{N} \epsilon_i \ln \left[ \frac{e^{p(\lambda_i)} - e^{h_i}}{\lambda_i} \right], \quad h_i = p(\lambda_i)$$

$$= \ln \lambda - \sum_{i=1}^{N} \epsilon_i \ln [e^p - e^{h_i}] + (\sum_{i=1}^{N} \epsilon_i) \ln \lambda.$$

Let $(\sum_{i=1}^{N} \epsilon_i) = 0$. Then one gets

$$\lambda = e^{p} \prod_{i=1}^{N} (e^p - e^{h_i})^{-\epsilon_i}.$$

Moreover, near 0 we also have $\partial^2_{t_0} F = \sum_{i=1}^{N} \epsilon_i h_i$. Then

$$p(\lambda) = \ln \lambda + \partial^2_{t_0} F - D_\lambda F_0$$

$$= \ln \lambda + \sum_{i=1}^{N} \epsilon_i h_i + \sum_{i=1}^{N} \epsilon_i \ln [1 - e^{p(\lambda) - h_i}]$$

$$= \ln \lambda + \sum_{i=1}^{N} \epsilon_i h_i + \sum_{i=1}^{N} \epsilon_i \ln [e^{-p(\lambda)} - e^{-h_i}] + (\sum_{i=1}^{N} \epsilon_i) p(\lambda).$$

Then one obtains

$$\hat{\lambda} = e^{p - \sum_{i=1}^{N} \epsilon_i h_i} \prod_{i=1}^{N} (e^{-p} - e^{-h_i})^{-\epsilon_i}.$$

Actually, we can see that

$$\hat{\lambda} = e^{p - (\sum_{i=1}^{N} \epsilon_i) p - \sum_{i=1}^{N} \epsilon_i h_i} \prod_{i=1}^{N} (e^{-p} - e^{-h_i})^{-\epsilon_i}$$

$$= e^{p} \prod_{i=1}^{N} (e^{h_i} - e^{p})^{-\epsilon_i} = e^{p} \prod_{i=1}^{N} (e^p - e^{h_i})^{-\epsilon_i} = \lambda. \quad (2.2)$$
Also,

\[ H_1^+ = \sum_{i=1}^{N} \varepsilon_i e^{h_i}, \quad H_1^- = -e^{\sum_{i=1}^{N} \varepsilon_i h_i} \left( \sum_{i=1}^{N} \varepsilon_i e^{-h_i} \right). \]

The \( t_1 \) and \( \hat{t}_1 \) evolutions are

\[
\begin{align*}
\partial_{t_1} h_i &= \partial_{h_0} [e^{h_i} + \sum_{i=1}^{N} \varepsilon_i h_i], \\
\partial_{\hat{t}_1} h_i &= \partial_{h_0} [e^{-h_i} + \sum_{i=1}^{N} \varepsilon_i h_i].
\end{align*}
\] (2.3)

We can also express them in Hamiltonian form as

\[
\begin{bmatrix}
  h_1 \\
  h_2 \\
  \vdots \\
  h_N \\
\end{bmatrix}
_{t_1} = \eta^{ij} \partial_{h_0}
\begin{bmatrix}
  \frac{\partial H^+}{\partial h_1} \\
  \frac{\partial H^+}{\partial h_2} \\
  \vdots \\
  \frac{\partial H^+}{\partial h_N}
\end{bmatrix},
\]

\[
\begin{bmatrix}
  h_1 \\
  h_2 \\
  \vdots \\
  h_N \\
\end{bmatrix}
_{\hat{t}_1} = \eta^{ij} \partial_{h_0}
\begin{bmatrix}
  \frac{\partial H^-}{\partial h_1} \\
  \frac{\partial H^-}{\partial h_2} \\
  \vdots \\
  \frac{\partial H^-}{\partial h_N}
\end{bmatrix},
\]

where

\[
\eta^{ij} = \begin{bmatrix}
  1 + \frac{1}{\varepsilon_1} & 1 & \cdots & \cdots & 1 \\
  1 & 1 + \frac{1}{\varepsilon_2} & 1 & \cdots & 1 \\
  \vdots & \vdots & \ddots & \ddots & \vdots \\
  1 & 1 & \cdots & 1 & 1 + \frac{1}{\varepsilon_N}
\end{bmatrix}.
\]

As in the case of dispersionless discrete KP hierarchy [30], case(I) is called the waterbag model.

Once (I) has been done, we similarly have the following cases.

- **Case(II):** \( F_m = \sum_{i=1}^{N} \varepsilon_i S_i \).

In that case

\[
B_m(\lambda) = \lambda^m - D_\lambda F_m = \lambda^m - \sum_{i=1}^{N} \varepsilon_i D_\lambda S_i
\]

\[
= \lambda^m - \sum_{i=1}^{N} \varepsilon_i \ln(e^p - e^{h_i}).
\]

Hence the Lax operator is

\[
\lambda^m = e^{mp} + u_{m-1}e^{(m-1)p} + u_{m-2}e^{(m-2)p} + \cdots + u_1 e^p
\]

\[
+ u_0 + \sum_{i=1}^{N} \varepsilon_i \ln(e^p - e^{h_i}). \quad (2.4)
\]
• Case (III): \( F_m = \sum_{i=1}^{N} \varepsilon_i S_i \),

In that case

\[
\hat{B}_m(\hat{\lambda}) = \hat{\lambda} - m + \partial_{\hat{\lambda}}^2 F - \hat{D}_m F = \hat{\lambda} - m + \partial_{\hat{\lambda}}^2 F - \sum_{i=1}^{N} \varepsilon_i \hat{D}_i S_i
\]

Hence the Lax operator is

\[
\hat{\lambda} - m = \hat{u}_m e^{-mp} + \hat{u}_{m-1} e^{-(m-1)p} + \hat{u}_{m-2} e^{-(m-2)p} + \ldots + \hat{u}_1 e^{-p} + \hat{u}_0 + \sum_{i=1}^{N} \varepsilon_i \ln(e^{-p} - e^{-h_i}) \quad (2.5)
\]

### 3 Residue formula and free energy

In this section, we compute the free energy associated with the waterbag model of case (I) in the last section, given by equation (2.2). Also, relations for the conserved densities are investigated.

The free energy is a function \( F(t_1, t_2, \ldots, t_n) \) such that the associated functions

\[
c_{ijk} = \frac{\partial^3 F}{\partial t_i \partial t_j \partial t_k}
\]

satisfy the following conditions.

- The matrix \( \eta_{ij} = c_{1ij} \) is constant and non-degenerate. This together with the inverse matrix \( \eta^{ij} \) are used to raise and lower indices.
- The functions \( c'_{ijk} = \eta^{ir} c_{rjk} \) define an associative commutative algebra with a unity element (Frobenius algebra).

The equations of associativity give a system of non-linear PDEs for \( F(t) \)

\[
\frac{\partial^3 F(t)}{\partial t^\alpha \partial t^\beta \partial t^\gamma} \eta^\lambda \mu \frac{\partial^3 F(t)}{\partial t^\mu \partial t^\nu \partial t^\sigma} = \frac{\partial^3 F(t)}{\partial t^\alpha \partial t^\beta \partial t^\gamma} \eta^\lambda \mu \frac{\partial^3 F(t)}{\partial t^\mu \partial t^\nu \partial t^\sigma}.
\]

These equations constitute the Witten-Dijkgraaf-Verlinde-Verlinde (or WDVV) equations. In general, two dimensional topological field theories (TFTs) can be classified by the solutions of the WDVV equations of associativity \([7, 8, 29]\) in the sense that a particular solution of the WDVV equations provides the primary free energy of some topological model. In fact, various classes of solutions to the WDVV equations have been obtained \([9, 12, 13]\) and references therein), which turn out to be the tau-functions of dispersionless integrable hierarchies. Accordingly, investigating the solution space of the WDVV equations will deepen our understanding of 2d TFT.

The geometrical setting in which to understand the free energy \( F(t) \) is the Frobenius manifold \([8]\). Given any solution of the WDVV equations, one can construct a Frobenius manifold \( M \) associated with it. One way to construct such a manifold is derived via the Landau-Ginzburg formalism as the structure on the parameter space \( M \) of the appropriate form

\[
\lambda = \lambda(p; t^1, t^2, \ldots, t^n).
\]
The Frobenius structure is given by the flat metric
\[
\eta(\partial, \partial') = -\sum \text{res}_{d\lambda=0} \left\{ \frac{\partial(\lambda dp)\partial'(\lambda dp)}{d\lambda(p)} \right\}
\]
and the tensor
\[
c(\partial, \partial', \partial'') = -\sum \text{res}_{d\lambda=0} \left\{ \frac{\partial(\lambda dp)\partial'(\lambda dp)\partial''(\lambda dp)}{d\lambda(p)dp} \right\}
\]
defines a totally symmetric \((3, 0)\)-tensor \(c_{ijk}\).

Geometrically, a solution of WDVV equation defines a multiplication
\[
\circ : T M \times T M \longrightarrow T M
\]
of vector fields on the parameter space \(M\), i.e.
\[
\partial_i \circ \partial_j = c^\gamma_{ij}(t) \partial_\gamma.
\]
From \(c^\gamma_{ij}(t)\), one can construct integrable hierarchies of hydrodynamic type whose corresponding Hamiltonian densities are defined recursively by the formula [8, 9]
\[
\frac{\partial^2 \psi^{(l)}_{a\beta}}{\partial t^i \partial t^j} = c^\mu_{ij} \frac{\partial \psi^{(l-1)}_{a\alpha}}{\partial t^\mu},
\]
where \(l \geq 1, \alpha = 1, 2, \cdots, n, \) and \(\psi^{0}_{a\alpha} = \eta_{a\alpha}e^\xi\). The integrability conditions for this systems are automatically satisfied when the \(c^k_{ij}\) are defined as above.

In the proof of the following theorem, to make the computations more easily, one uses \(\ln \xi\) to replace \(\lambda\).

**Theorem 1.** Let the Lax operator be defined as in (2.2). Then

\(\eta(\partial_i, \partial_i) = \eta_{ii} = -\varepsilon_i \varepsilon_j, \quad i \neq j,\)

\(\eta(\partial_i, \partial_j) = \eta_{ij} = -\varepsilon_i^2 + \varepsilon_i,\)

\(c(\partial_i, \partial_j, \partial_k) = c_{ijk} = \varepsilon_i \varepsilon_j \varepsilon_k, \quad i \neq j \neq k,\)

\(c(\partial_i, \partial_i, \partial_j) = c_{ik} = \varepsilon_i \varepsilon_k \left[ \varepsilon_i + \frac{e^{h_i}}{e^{h_i} - e^{h_j}} \right], \quad i \neq k,\)

\(c(\partial_i, \partial_j, \partial_i) = c_{iij} = \varepsilon_i^3 + \varepsilon_i \left[ 1 - \varepsilon_i - \sum_{l=1, l \neq i}^N \frac{\varepsilon_l e^{h_l}}{e^{h_i} - e^{h_l}} \right].\)

**Proof.** We see that \(\frac{\partial \ln \xi}{\partial h_i} = \frac{\varepsilon_i}{\xi - e^{h_i}}\), where \(p = \ln \xi\). Also, we have
\[
\frac{d \ln \lambda}{dp} = 1 - \sum_{k=1}^N \varepsilon_k \frac{e^{h_k}}{\xi - e^{h_k}} = \prod_{k=1}^N \frac{(\xi - \omega_k)}{\prod_{k=1}^N (\xi - e^{h_k})},
\]
(3.4)
In the following proofs, we use the formula (3.4) and the fact that the residue at infinity is zero.

(I)

\[ \eta(\partial_{\eta_i}, \partial_{\eta_j}) = \sum_{d \ln \lambda = 0} \text{Res} \frac{\partial \ln \lambda \partial \ln \lambda}{\partial \eta \partial \eta} d\xi \]

\[ = \sum_{d \ln \lambda = 0} \text{Res} \frac{\epsilon_i \epsilon_j e^{\eta_i} e^{\eta_j}}{\xi (1 - \sum_{k=1}^{N} \epsilon_k e^{\eta_k})} d\xi \]

\[ = \sum_{d \ln \lambda = 0} \text{Res} \frac{\epsilon_i \epsilon_j e^{\eta_i} e^{\eta_j} \prod_{k=1}^{N} (\xi - e^{\eta_k})}{\xi (\xi - e^{\eta_i})(\xi - e^{\eta_j}) \prod_{k=1}^{N} (\xi - \omega_k)} \]

\[ = -\text{Res}_{\xi = 0} \frac{\epsilon_i \epsilon_j e^{\eta_i} e^{\eta_j} \prod_{k=1}^{N} (\xi - e^{\eta_k})}{\xi (\xi - e^{\eta_i})(\xi - e^{\eta_j}) \prod_{k=1}^{N} (\xi - \omega_k)} = -\epsilon_i \epsilon_j, \quad i \neq j. \]

(II)

\[ \eta(\partial_{\eta_i}, \partial_{\eta_i}) = \sum_{d \ln \lambda = 0} \text{Res} \frac{\epsilon_i^2 e^{2\eta_i} \prod_{k=1}^{N} (\xi - e^{\eta_k})}{\xi (\xi - e^{\eta_i})^2 \prod_{k=1}^{N} (\xi - \omega_k)} \]

\[ = -(\text{Res}_{\xi = 0} + \text{Res}_{\xi = e^{\eta_i}}) \sum_{d \ln \lambda = 0} \text{Res} \frac{\epsilon_i^2 e^{2\eta_i} \prod_{k=1}^{N} (\xi - e^{\eta_k})}{\xi (\xi - e^{\eta_i})^2 \prod_{k=1}^{N} (\xi - \omega_k)} \]

\[ = -\epsilon_i^2 - \frac{\epsilon_i^2 e^{2\eta_i} \prod_{k=1, k \neq i}^{N} (e^{\eta_i} - e^{\eta_k})}{e^{\eta_i} \prod_{k=1}^{N} (e^{\eta_i} - \omega_k)} \]

\[ = -\epsilon_i^2 - \epsilon_i e^{\eta_i} \frac{1}{-\epsilon_i e^{\eta_i}} = -\epsilon_i^2 + \epsilon_i. \]

(III)

\[ c(\partial_{\eta_i}, \partial_{\eta_i}, \partial_{\eta_j}) = \sum_{d \ln \lambda = 0} \text{Res} \frac{\epsilon_i e^{\eta_i} \epsilon_j e^{\eta_j} \epsilon_k e^{\eta_k} \prod_{k=1}^{N} (\xi - e^{\eta_k})}{\xi (\xi - e^{\eta_i})(\xi - e^{\eta_j})(\xi - e^{\eta_k}) \prod_{k=1}^{N} (\xi - \omega_k)} d\xi \]

\[ = -\text{Res}_{\xi = 0} \frac{\epsilon_i e^{\eta_i} \epsilon_j e^{\eta_j} \epsilon_k e^{\eta_k} \prod_{k=1}^{N} (\xi - e^{\eta_k})}{\xi (\xi - e^{\eta_i})(\xi - e^{\eta_j})(\xi - e^{\eta_k}) \prod_{k=1}^{N} (\xi - \omega_k)} d\xi \]

\[ = \epsilon_i \epsilon_j \epsilon_k, \quad i \neq j \neq k. \]

(IV)

\[ c(\partial_{\eta_i}, \partial_{\eta_i}, \partial_{\eta_i}) = \sum_{d \ln \lambda = 0} \text{Res} \frac{\epsilon_i^2 e^{2\eta_i} \epsilon_i e^{\eta_i} \prod_{k=1}^{N} (\xi - e^{\eta_k})}{\xi (\xi - e^{\eta_i})^2 (\xi - e^{\eta_i}) \prod_{k=1}^{N} (\xi - \omega_k)} d\xi \]

\[ = -[\text{Res}_{\xi = 0} + \text{Res}_{\xi = e^{\eta_i}}] \frac{\epsilon_i^2 e^{2\eta_i} \epsilon_i e^{\eta_i} \prod_{k=1}^{N} (\xi - e^{\eta_k})}{\xi (\xi - e^{\eta_i})^2 (\xi - e^{\eta_i}) \prod_{k=1}^{N} (\xi - \omega_k)} d\xi \]

\[ = -[\epsilon_i^2 + \frac{\epsilon_i^2 e^{2\eta_i} \epsilon_i e^{\eta_i} \prod_{k=1, k \neq i}^{N} (e^{\eta_i} - e^{\eta_k})}{e^{\eta_i} (e^{\eta_i} - e^{\eta_i}) \prod_{k=1}^{N} (e^{\eta_i} - \omega_k)}] \]

\[ = -\epsilon_i^2 \epsilon_k + \epsilon_i \epsilon_k e^{\eta_i} \frac{e^{\eta_i}}{e^{\eta_i} - e^{\eta_k}} = \epsilon_i \epsilon_k [1 + \frac{e^{\eta_i}}{e^{\eta_i} - e^{\eta_k}}]. \]
(V)

\begin{align*}
c(\partial_{\xi'}, \partial_{\eta'}, \partial_{\eta''}) &= \sum_{d \ln k = 0} \text{Res} \frac{e_i^3 e^{3\eta'} \prod_{l=1}^{N}(\xi - e^{\eta'})}{\xi (\xi - e^{\eta'})^3 \prod_{l=1}^{N}(\xi - \omega_l)} d\xi \\
&= -[\text{Res}_{\xi = 0} + \text{Res}_{\xi = e^{\eta'}}] \frac{e_i^3 e^{3\eta'} \prod_{l=1}^{N}(\xi - e^{\eta'})}{\xi (\xi - e^{\eta'})^3 \prod_{l=1}^{N}(\xi - \omega_l)} d\xi \\
&= -\{ -e_i^3 + e_i^3 e^{3\eta'} \frac{d}{d\xi} \frac{\prod_{l=1, l \neq i}^{N}(\xi - e^{\eta'})}{\xi \prod_{l=1}^{N}(\xi - \omega_l)} \}_{\xi = e^{\eta'}} \\
&= e_i^3 + e_i^3 e^{3\eta'} \frac{1 - e_i - \sum_{l=1, l \neq i}^{N} \frac{e^{\eta'}}{e^{\eta'} - e^{\eta'}}}{e_i^2 e^{3\eta'}} \\
&= e_i^3 + e_i(1 - e_i - \sum_{l=1, l \neq i}^{N} \frac{e^{\eta'}}{e^{\eta'} - e^{\eta'}}).
\end{align*}

Let’s define \( \Omega = \sum_{l=1}^{N} \frac{\partial}{\partial \eta^l} \). Then we can verify directly that

\[ \eta(\partial_{\partial_{\xi'}}, \partial_{\eta'}) = c(\partial_{\partial_{\eta'}}, \partial_{\eta'}, \Omega) = \sum_{k=1}^{N} c(\partial_{\partial_{\eta'}}, \partial_{\eta'}, \partial_{\eta'}) \]  
(3.5)

Also, from the Theorem, it’s not difficult to check directly the compatibility (or Egorov’s) condition

\[ \partial_{\eta'} c_{\eta \eta} = \partial_{\eta} c_{\eta \eta}, \quad i, l, m, n = 1 \cdots N. \]

Hence one can get the free energy associated with (2.2) as

\[ F(\tilde{\eta}) = \sum_{1 \leq i < j \leq N} \epsilon_i \epsilon_j \epsilon_k h^i h^j h^k + \frac{1}{6} \sum_{i=1}^{N} (\epsilon_i - e_i^2 + e_i^3)(h^i)^3 \\
+ \frac{1}{2} \sum_{i \neq k} e_i^2 \epsilon_k (h^i)^2 h^k \\
+ \frac{1}{2} \sum_{1 \leq i < k \leq N} \epsilon_i \epsilon_k [L_{3e}(e^{h^i - h^k}) + L_{3e}(e^{h^k - h^i})], \]  
(3.6)

where \( L_{3e}(e^x) = \sum_{k=1}^{N} \frac{e^x}{x} \) is the poly-logarithmic function. Moreover, from (3.5), one knows that \( t^1 = \sum_{k=1}^{N} h^k \).

This solution looks like (but is different to) the formula related to classical Lie algebras in [19]. In [6], the solution of the WDVV equation associated with the waterbag reduction for dispersionless KP(dKP) is found and, using this solution, one can construct the recursive operator of the conserved densities in (3.3). Hence the bi-Hamiltonian structure of the waterbag reduction of the dKP hierarchy can be found. For more solutions of the WDVV equation associated with integrable hydrodynamic systems, one should refer to [17]. We remark that the free energy (3.6) is invariant.
under any permutation of \((h^1, h^2, \ldots, h^N)\), which is different from the formula in \([8]\).

Furthermore, we have

\[
\begin{align*}
    c_{\alpha\beta}^\gamma &= 0, \quad \alpha \neq \beta \neq \gamma; \quad c_{\alpha\alpha}^\beta = \epsilon_{\alpha} - \frac{e^{h^\alpha}}{e^{h^\gamma} - e^{h^\beta}}, \quad \alpha \neq \beta; \\
    c_{\alpha\beta} = \epsilon_{\beta} - \frac{e^{h^\beta}}{e^{h^\alpha} - e^{h^\beta}}, \quad \alpha \neq \beta; \quad c_{\alpha\alpha} = 1 - \sum_{\gamma \neq \alpha} \epsilon_{\gamma} - \frac{e^{h^\gamma}}{e^{h^\alpha} - e^{h^\beta}}. \quad (3.7)
\end{align*}
\]

If we define \(\phi_i = \frac{\partial \ln \lambda}{\partial h^i} = \epsilon_{\gamma} - \frac{e^{\beta}}{e^{\gamma} - e^{\beta}}\), then we have

\[
\phi_i \phi_j = c_{ij} \phi_i + Q_{ij} \frac{\partial \ln \lambda}{\partial h^i}, \quad (3.8)
\]

where

\[
Q_{ij} = \begin{cases}
    -\phi_i, & i = j, \\
    0, & i \neq j.
\end{cases}
\]

From (3.5), one knows that \(\Omega\) is the unit element of the associative algebra (3.8).

Now, we have the following

**Theorem 2.** Let \(H_n^+\) and \(H_n^-\) be the conserved densities defined in (1.3). Then one has

\[
(\text{I}) \quad \frac{\partial^2 H_n^+}{\partial h^i \partial h^j} = c_{ij} \frac{\partial H_n^+}{\partial h^k}, \quad (\text{II}) \quad \frac{\partial^2 H_n^-}{\partial h^i \partial h^j} = c_{ij} \frac{\partial H_n^-}{\partial h^k}.
\]

**Proof.** (I)

\[
\begin{align*}
    \frac{\partial^2 H_n^+}{\partial h^i \partial h^j} &= \frac{\partial}{\partial h^i} \int_0^\infty \lambda(\xi)^{n-1} \frac{\partial \ln \lambda}{\partial h^i} \xi^{-1} d\xi = \frac{\partial}{\partial h^i} \int_0^\infty \lambda(\xi)^{n-1} \lambda \frac{\partial \ln \lambda}{\lambda h^j} \xi^{-1} d\xi \\
    &= \frac{\partial}{\partial h^i} \int_0^\infty \lambda(\xi)^n \epsilon_j \frac{e^{h^j}}{e^{h^i} - e^{h^j}} \xi^{-1} d\xi \\
    &= n \int_0^\infty \lambda(\xi)^n \epsilon_i \frac{e^{h^i}}{e^{h^i} - e^{h^j}} \xi^{-1} d\xi + \int_0^\infty \lambda(\xi)^n \epsilon_j \frac{\partial}{\partial h^j} \left( \frac{e^{h^j}}{e^{h^i} - e^{h^j}} \right) \xi^{-1} d\xi \\
    &= n c_{ij} \int_0^\infty \lambda(\xi)^n \phi_i \xi^{-1} d\xi + n \int_0^\infty \lambda(\xi)^n \phi_j \xi^{-1} d\xi + n \int_0^\infty \lambda(\xi)^n Q_{ij} d\lambda \\
    &= n c_{ij} \int_0^\infty \lambda(\xi)^n \phi_i \xi^{-1} d\xi + n \int_0^\infty \lambda(\xi)^n \left[ \frac{\partial}{\partial h^j} (\epsilon_j \frac{e^{h^j}}{e^{h^i} - e^{h^j}}) \right] \xi^{-1} d\xi \\
    &= c_{ij} \frac{\partial}{\partial h^k} \int_0^\infty \lambda(\xi)^n \xi^{-1} d\xi - \int_0^\infty \frac{\partial}{\partial h^k} \left[ \lambda(\xi)^n \epsilon_j \frac{e^{h^j}}{e^{h^i} - e^{h^j}} \right] d\lambda \\
    &= c_{ij} \frac{\partial}{\partial h^k}, \quad n \geq 1.
\end{align*}
\]

(II) The calculation is similar. \[\square\]
4 Concluding remarks

We find the free energy associated with the waterbag model (2.2) using the Landau-Ginzburg formulation. From the free energy, one can establish the equations for the conserved densities $H_+^n$ and $H_-^n$. Unlike the waterbag model of dKP [6], here one can’t construct the recursive operator for $H_+^n$ or $H_-^n$ from Theorem 3.2. Therefore, the bi-Hamiltonian structure of (2.3) is still unknown. On the other hand, we can construct an integrable hierarchy via (3.3) and (3.7); however, it won’t be the dToda hierarchy. Finally, finding the free energies associated with (2.4) and (2.5) would also be very interesting.

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