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Symmetries and Differential Forms

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Abstract

The method for writing a differential equation or system of differential equations in terms of differential forms and finding their symmetries was devised by Harrison and Estabrook (1971). A modification to the method is demonstrated on a wave equation with variable speed, and the modified method is extended to determine approximate and potential symmetries.

1 Introduction

The use of differential forms in differential geometry is now well-known and the literature is abundant (see, e.g., [1]). Also, the analysis of invariance properties of differential equations is presented in a number of texts, for example, [2]. A specific method for writing the differential equation or system of differential equations in terms of differential forms and finding their symmetries was devised by Harrison and Estabrook (see [3]).

In the sequel, a modification to the method is demonstrated with particular reference to a wave equation with variable speed. This modified method is extended to determine approximate and potential symmetries (for a detailed discussion on the latter, see [4] or, for 'nonlocal symmetries', [5]). The advantage of using this method is that, in general, the prolongation formulae required are of an order *less* than the order of the system of partial differential equations in question. In the usual determination of symmetries, the order of prolongation is equal to that of the system; these coefficients are usually very tedious to calculate. In particular, second-order systems will reduce to first-order potential systems so that no prolongation coefficients are required for determining possible potential symmetries.

The original method requires that the differential forms constructed should form the basis of a differential ideal. Possibly this would also work for the usual method of constructing symmetries; set up an ideal of differential equations, and then, instead of ensuring that the action of a symmetry on the equations is zero on solutions of the equation, ensure that the action of the symmetry on the equations leads to another equation in the ideal.

In this section, we will be using the reverse of this idea; in other words, rather than ensuring that the Lie derivative of the forms stays within an ideal, we ensure that the Lie derivative of the forms is zero when the forms themselves are zero. There are some advantages to this method, one of which being that it is easy to extend the method to approximate symmetries.

This idea will be used to calculate potential symmetries and approximate symmetries.

2 Potential Symmetries

Consider Burgers' equation which has an associated auxilliary system which we write here for convenience:

$$\begin{aligned} v_x &= 2u \\ v_t &= 2u_x - u^2. \end{aligned} \tag{2.1}$$

We introduce the 2-forms

$$\begin{aligned} \alpha &= \mathrm{d}v\mathrm{d}t - 2u\mathrm{d}x\mathrm{d}t \\ &= v_x\mathrm{d}x\mathrm{d}t - 2u\mathrm{d}x\mathrm{d}t, \\ \beta &= \mathrm{d}v\mathrm{d}x + 2\mathrm{d}u\mathrm{d}t + u^2\mathrm{d}t\mathrm{d}x \\ &= v_t\mathrm{d}t\mathrm{d}x - 2u_x\mathrm{d}t\mathrm{d}x + u^2\mathrm{d}t\mathrm{d}x, \end{aligned}$$

which return the system (2.1) when annulled. To calculate a symmetry

$$X = \tau\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \phi\frac{\partial}{\partial u} + \eta\frac{\partial}{\partial v}$$

of (2.1), we calculate the Lie derivatives of these forms. First,

$$\begin{aligned} \mathcal{L}_X\alpha &= X\rfloor\mathrm{d}\alpha + \mathrm{d}(X\rfloor\alpha) \\ &= X\rfloor(-2\mathrm{d}u\mathrm{d}x\mathrm{d}t) + \mathrm{d}(\eta\mathrm{d}t - \tau\mathrm{d}v - 2u\xi\mathrm{d}t + 2u\tau\mathrm{d}x) \\ &= (2\phi - \eta_x + 2u\xi_x + 2u\tau_t)\mathrm{d}t\mathrm{d}x + (2u\xi_u - \eta_u)\mathrm{d}t\mathrm{d}u + (2u\xi_v - \eta_v - \tau_t)\mathrm{d}t\mathrm{d}v \\ &\quad -2u\tau_u\mathrm{d}x\mathrm{d}u + (-\tau_x - 2u\tau_v)\mathrm{d}x\mathrm{d}v - \tau_u\mathrm{d}u\mathrm{d}v. \end{aligned}$$

When $\alpha = \beta = 0$, we have $\mathrm{d}t\mathrm{d}v = 2u\mathrm{d}t\mathrm{d}x$ and $\mathrm{d}x\mathrm{d}v = u^2\mathrm{d}t\mathrm{d}x - 2\mathrm{d}t\mathrm{d}u$, so that

$$\begin{aligned} \mathcal{L}_X\alpha|_{\alpha=\beta=0} &= (2\phi - \eta_x + 2u\xi_x + 2u\tau_t - u^2\tau_x - 2u^3\tau_v + 4u^2\xi_v - 2u\eta_v - 2u\tau_t)\mathrm{d}t\mathrm{d}x \\ &\quad +(2u\xi_u - \eta_u + 2\tau_x + 4u\tau_v)\mathrm{d}t\mathrm{d}u - 2u\tau_u\mathrm{d}x\mathrm{d}u - \tau_u\mathrm{d}u\mathrm{d}v, \end{aligned}$$

and we may now split the coefficients of $\mathrm{d}t\mathrm{d}x$, $\mathrm{d}t\mathrm{d}u$, etc, to obtain

$$\begin{aligned} \mathrm{d}t\mathrm{d}x &: 2\phi - \eta_x + 2u\xi_x + 2u\tau_t \\ &\qquad -u^2\tau_x - 2u^3\tau_v + 4u^2\xi_v - 2u\eta_v - 2u\tau_t = 0 && (2.2) \\ \mathrm{d}t\mathrm{d}u &: 2u\xi_u - \eta_u + 2\tau_x + 4u\tau_v = 0 && (2.3) \\ \mathrm{d}u\mathrm{d}v &: \tau_u = 0 && (2.4) \\ \mathrm{d}x\mathrm{d}u &: \text{the same as } \mathrm{d}u\mathrm{d}v \end{aligned}$$

Next,

$$\begin{aligned} \mathcal{L}_X\beta &= X\rfloor(\mathrm{d}\beta) + \mathrm{d}(X\rfloor\beta) \\ &= X\rfloor(2u\mathrm{d}u\mathrm{d}t\mathrm{d}x) + \mathrm{d}(\eta\mathrm{d}x - \xi\mathrm{d}v + 2\phi\mathrm{d}t - 2\tau\mathrm{d}u + u^2\tau\mathrm{d}x - u^2\xi\mathrm{d}t) \\ &= (2u\phi + \eta_t - 2\phi_x + u^2\tau_t + u^2\xi_x)\mathrm{d}t\mathrm{d}x + (u^2\xi_u - 2\phi_u - 2\tau_t)\mathrm{d}t\mathrm{d}u \\ &\quad +(u^2\xi_v - \xi_t - 2\phi_v)\mathrm{d}t\mathrm{d}v + (-u^2\tau_u - \eta_u - 2\tau_x)\mathrm{d}x\mathrm{d}u \\ &\quad +(-\eta_v - \xi_x - u^2\tau_v)\mathrm{d}x\mathrm{d}v + (2\tau_v - \xi_u)\mathrm{d}u\mathrm{d}v. \end{aligned}$$

When $\alpha = \beta = 0$, we obtain

$$\begin{aligned}\mathcal{L}_X\beta|_{\alpha=\beta=0} &= (2u\phi + \eta_t - 2\phi_x - u^2\tau_t + u^2\xi_x + 2u^3\xi_v - 2u\xi_t - 4u\phi_v \\ &\quad -u^2\eta_v - u^2\xi_x - u^4\tau_v)\mathrm{d}t\mathrm{d}x \\ &\quad +(u^2\xi_u - 2\phi_u - 2\tau_t + 2\eta_v + 2\xi_x + 2u^2\tau_v)\mathrm{d}t\mathrm{d}u \\ &\quad -(\eta_u + 2\tau_x + u^2\tau_u)\mathrm{d}x\mathrm{d}u + (2\tau_v - \xi_u)\mathrm{d}u\mathrm{d}v,\end{aligned}$$

which may be split into

$$\begin{aligned}2u\phi + \eta_t - 2\phi_x - u^2\tau_t + u^2\xi_x + 2u^3\xi_v & \\ -2u\xi_t - 4u\phi_v - u^2\eta_v - u^2\xi_x - u^4\tau_v &= 0 \quad (2.5)\\ u^2\xi_u - 2\phi_u - 2\tau_t + 2\eta_v + 2\xi_x + 2u^2\tau_v &= 0 \quad (2.6)\\ \eta_u + 2\tau_x + u^2\tau_u &= 0 \quad (2.7)\\ 2\tau_v - \xi_u &= 0. \quad (2.8)\end{aligned}$$

Straight away we see from (2.4) that $\tau = \tau(t,x,v)$, so that (2.7) becomes $\eta_u = -2\tau_x$, which combined with (2.8) means that (2.3) can be written

$$6u\tau_v + 4\tau_x = 0.$$

Separating coefficients of u gives $\tau_v = \tau_x = 0$, that is

$$\tau = \tau(t).$$

Thus (2.7) and (2.8) tell us that $\eta = \eta(t,x,v)$ and $\xi = \xi(t,x,v)$. We may now rewrite (2.6) as

$$\phi_u = \xi_x + \eta_v - \tau_t$$

and therefore

$$\phi = u(\xi_x + \eta_v - \tau_t) + A(t,x,v),$$

where A is an unknown function. We can now write (2.5) in terms of functions all independent of u:

$$\begin{aligned}u^2(2\xi_x + \eta_v - 3\tau_t) + 2uA + \eta_t - 2u(\xi_{xx} + \eta_{vx}) - 2A_x & \\ +2u^3\xi_v - 2u\xi_t - 4u^2(\xi_{vx} + \eta_{vv}) - 4uA_v - u^2\xi_x - u^4\tau_v &= 0,\end{aligned}$$

which may be split by powers of u into

$$\begin{array}{ll}(1) & \eta_t - 2A_x = 0 \\ (u) & 2A - 2\xi_{xx} - 2\eta_{vx} - 2\xi_t - 4A_v = 0 \\ (u^2) & 2\xi_x + \eta_v - 3\tau_t - 4\xi_{vx} - 4\eta_{vv} = 0 \\ (u^3) & \xi_v = 0 \\ (u^4) & \tau_v = 0.\end{array} \quad (2.9)$$

We see that (2.9e) tells us nothing new and (2.9d) gives us $\xi = \xi(t,x)$. Next, (2.9c) can be simplified:

$$2\xi_x + \eta_v - 4\eta_{vv} - 3\tau_t = 0$$

and therefore

$$\eta_v - \frac{1}{4}\eta = \frac{v}{4}(2\xi_x - 3\tau_t) - \frac{1}{4}B(t,x),$$

where $B(t,x)$ is an arbitrary function. We now have a linear differential equation with integrating factor $e^{\frac{-v}{4}}$, so

$$\eta = (v+4)(3\tau_t - 2\xi_x) + B(t,x) + e^{\frac{v}{4}}C(t,x),$$

where $C(t,x)$ is another arbitrary function. We now turn our attention to (2.9b), which may be written independently of η as

$$A_v - \frac{1}{2}A = \frac{1}{2}\xi_{xx} - \frac{1}{2}\xi_t - \frac{1}{8}e^{\frac{v}{4}}C_x,$$

which is again a linear differential equation with integrating factor $e^{-\frac{1}{2}v}$, so that

$$A = \xi_t - \xi_{xx} + \frac{1}{2}e^{\frac{v}{4}}C_x + e^{\frac{v}{2}}E(t,x),$$

where $E(t,x)$ is an arbitrary function. Last, (2.9a) can be written

$$\begin{aligned}(v+4)(3\tau_{tt} - 2\xi_{xt}) + B_t + e^{\frac{v}{4}}C_t & \\ -2\left[\xi_{tx} - \xi_{xxx} + \frac{1}{2}e^{\frac{v}{4}}C_{xx} + e^{\frac{v}{2}}E_x\right] &= 0,\end{aligned}$$

and may be split according to the coefficients of v, $e^{\frac{v}{4}}$ and $e^{\frac{v}{2}}$ to give

$$(1): \quad 12\tau_{tt} - 6\xi_{xt} + B_t + 2\xi_{xxx} = 0, \tag{2.10}$$
$$v: \quad 3\tau_{tt} - 2\xi_{xt} = 0, \tag{2.11}$$
$$e^{\frac{v}{4}}: \quad C_t - C_{xx} = 0, \tag{2.12}$$
$$e^{\frac{v}{2}}: \quad E_x = 0. \tag{2.13}$$

By virtue of (2.11), (2.10) can be rewritten

$$B_t - 2\xi_{tx} + 2\xi_{xxx} = 0. \tag{2.14}$$

We may now write (2.2) as

$$(v+4)2\xi_x - B_x + 4u\xi_x - 2u\tau_t + 2\xi_t - 2\xi_{xx} + 2e^{\frac{v}{2}}E = 0,$$

which can be split according to coefficients of u, v and $e^{\frac{v}{2}}$ as follows.

$$(1): \quad 8\xi_x - B_x + 2\xi_t - 2\xi_{xx} = 0, \tag{2.15}$$
$$u: \quad 4\xi_x - 2\tau_t = 0, \tag{2.16}$$
$$v: \quad 2\xi_x = 0, \tag{2.17}$$
$$e^{\frac{v}{2}}: \quad E = 0. \tag{2.18}$$

From (2.17) we see that $\xi = \xi(t)$, so that (2.16) becomes $\tau_t = 0$, that is τ is a constant. We now see that (2.14) becomes $B_t = 0$, that is $B = B(x)$, and (2.15) is then $B_x = 2\xi_t$. This means that $\xi_{tt}=0$, so that $\xi_t = 1/2B_x = k_2$, say. Integrating further gives $B = 2k_2x + k_3$, and we then have

$$\begin{aligned}
\tau &= k_1, \\
\xi &= k_2 t + k_4, \\
\phi &= \frac{1}{2}e^{\frac{v}{4}}\left(\frac{u}{2}C + C_x\right) + k_2, \\
\eta &= 2k_2x + k_3 + e^{\frac{v}{4}}C,
\end{aligned}$$

where C is any solution of the equation $C_t = C_{xx}$. Thus we have the symmetry generators

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial t}, \\
X_2 &= t\frac{\partial}{\partial x} + \frac{\partial}{\partial u} + 2x\frac{\partial}{\partial v}, \\
X_3 &= \frac{\partial}{\partial v}, \\
X_4 &= \frac{\partial}{\partial x}, \\
X_\infty &= e^{\frac{v}{4}}\left(2C_x + uC\right)\frac{\partial}{\partial u} + 4e^{\frac{v}{4}}C\frac{\partial}{\partial v}.
\end{aligned}$$

Remarks.

1. The symmetry X_∞ is the only "genuine" potential symmetry of Burgers' equation, as it is the only potential symmetry for which one or more of ξ, τ and ϕ depend on the auxilliary variable v.
2. The auxilliary system (2.1) is only first order, so it is not necessary to calculate any prolongation coefficients.

3 Approximate symmetries

The method may be extended to calculate approximate symmetries as well. We carry out the following adaptation.

Let

$$E(x^i, u^\alpha, u^\alpha_{(1)}, \ldots) + \epsilon F(x^i, u^\alpha, u^\alpha_{(1)}, \ldots) = 0 \tag{3.1}$$

be a perturbed equation, where $E = 0$ is the associated unperturbed equation. An approximate symmetry of (3.1) is a vector field X such that

$$X(E + \epsilon F)|_{E+\epsilon F=0} = O(\epsilon^2).$$

Now the perturbed equation will give rise to differential forms $\gamma_j = \alpha_j + \epsilon\beta_j$, where the α^j are forms arising from the unperturbed equation $E = 0$. We shall refer to the γ_j collectively as I, and the α_j as I_0. The phrase $I = 0$ should be taken to mean that for each γ_j, we have $\gamma_j = 0$, and similarly for $I_0 = 0$. The condition that X be an approximate symmetry of (3.1) can now be rewritten as the system

$$\mathcal{L}_X\gamma_j|_{I=0} = O(\epsilon^2).$$

How do we find such a symmetry? We adapt the algorithm of Baikov *et al* ([6], [7]).

1. Find a symmetry X_0 of the unperturbed equation.
2. Let
$$h_j = \frac{1}{\epsilon}\mathcal{L}_{X_0}\gamma_j|_{I=0}.$$
3. Find a vector field X_1 such that
$$\mathcal{L}_{X_1}\alpha_j|_{I_0=0} + h_j = 0.$$
4. The vector field $X = X_0+\epsilon X_1$ is then an approximate symmetry of (3.1). The following calculation shows why.

$$\epsilon\mathcal{L}_{X_1}\alpha_j|_{I_0=0} + \mathcal{L}_{X_0}\gamma_j|_{I=0} = 0$$

which implies that
$$\mathcal{L}_{\epsilon X_1}\alpha_j|_{I_0=0} + \mathcal{L}_{X_0}\gamma_j|_{I=0} = 0.$$
Now
$$\mathcal{L}_{\epsilon X_1}\epsilon\beta_j = \epsilon^2\mathcal{L}_{X_1}\beta = O(\epsilon^2),$$
so that
$$\mathcal{L}_{\epsilon X_1}\alpha_j|_{I_0=0} + \mathcal{L}_{\epsilon X_1}\epsilon\beta_j|_{I=0} + \mathcal{L}_{X_0}\gamma_j|_{I=0} = O(\epsilon^2).$$
Next, we observe that
$$\mathcal{L}_{\epsilon X_1}\alpha_j|_{I=0} - \mathcal{L}_{\epsilon X_1}\alpha_j|_{I_0=0} = O(\epsilon^2),$$
so we have that
$$\mathcal{L}_{\epsilon X_1}\gamma_j|_{I=0} + \mathcal{L}_{X_0}\gamma_j|_{I=0} = O(\epsilon^2),$$
that is
$$\mathcal{L}_{(X_0+\epsilon X_1)}\gamma_j|_{I=0} = O(\epsilon^2),$$
and so $X = X_0 + \epsilon X_1$ is an approximate symmetry of (3.1).

We demonstrate the algorithm above using a perturbed wave equation

$$u_{tt} - e^{2x}u_{xx} + \epsilon F(t, x, u, u_t, u_x) = 0, \tag{3.2}$$

the unperturbed version of which is $u_{tt} - e^{2x}u_{xx} = 0$. For the unperturbed equation, we can introduce new variables $w = u_t$ and $z = u_x$, and use the forms $\alpha = \mathrm{d}u - z\mathrm{d}x - w\mathrm{d}t$ and $\beta = \mathrm{d}w\mathrm{d}x + e^{2x}\mathrm{d}z\mathrm{d}t$, which give rise to, among others, the symmetry

$$X_0 = \frac{\partial}{\partial x} - t\frac{\partial}{\partial t} + \frac{u}{2}\frac{\partial}{\partial u} \quad + \frac{3}{2}w\frac{\partial}{\partial w} + \frac{z}{2}\frac{\partial}{\partial z}.$$

For the perturbed equation (3.2), we continue to use the 1-form $\alpha = \mathrm{d}u - w\mathrm{d}t - z\mathrm{d}x$, which gives $w = u_t$ and $z = u_x$ when annulled, but $\beta = \mathrm{d}w\mathrm{d}x + e^{2x}\mathrm{d}z\mathrm{d}t$ describes the *unperturbed* equation, so we introduce

$$\gamma \;=\; \beta + \epsilon F\mathrm{d}t\mathrm{d}x,$$

which gives $u_{tt} - e^{2x}u_{xx} + \epsilon F = 0$ when annulled. Using the symmetry X_0 for the algorithm described above, we calculate
$$h_1 = \frac{1}{\epsilon}\mathcal{L}_{X_0}\alpha|_{I=0} = 0.$$

Thus, for α, we must find a symmetry X_1 such that

$$\mathcal{L}_{X_1}\alpha|_{I_0=0} + h_1 = 0$$

which means

$$\mathcal{L}_{X_1}\alpha|_{I_0=0} = 0,$$

which is no different to the unperturbed case. We end up finding that X_1 must have the usual prolongation coefficients, although we note that, as before, they need only be calculated to first order.

Next,

$$\begin{aligned} h_2 &= \frac{1}{\epsilon}\mathcal{L}_{X_0}\gamma|_{I=0} \\ &= \left(F_x - tF_t + \frac{u}{2}F_u + \frac{3}{2}wF_w + \frac{1}{2}zF_z - \frac{5}{2}F\right)\mathrm{d}t\mathrm{d}x. \end{aligned}$$

The next step in our algorithm is to find X_1 (which we will call Y to avoid confusion with subscripts), such that

$$\mathcal{L}_Y\beta|_{\alpha=\beta=0} + h_2 = 0, \tag{3.3}$$

where we recall that $\beta = \gamma$ when $\epsilon = 0$. Now

$$\begin{aligned} &\mathcal{L}_Y\beta|_{\alpha=\beta=0} + h_2 \\ &\quad = \{Y\rfloor\mathrm{d}\beta + \mathrm{d}(Y\rfloor\beta)\}\,|_{\alpha=\beta=0} + h_2 \\ &\quad = \left\{Y\rfloor 2e^{2x}\mathrm{d}x\mathrm{d}z\mathrm{d}t + \mathrm{d}(Y^w\mathrm{d}x - \xi\mathrm{d}w + e^{2x}Y^z\mathrm{d}t - e^{2x}\tau\mathrm{d}z)\right\}|_{\alpha=\beta=0} + h_2 \\ &\quad = \big(Y^w_t - e^{2x}Y^z_x - ze^{2x}Y^z_u + wY^w_u \\ &\qquad\quad + F_x - tF_t + \frac{u}{2}F_u + \frac{3}{2}wF_w + \frac{1}{2}zF_z - \frac{5}{2}F\bigg)\mathrm{d}t\mathrm{d}x \\ &\qquad + \left(-2\xi_t - e^{2x}Y^z_w - w\xi_u\right)\mathrm{d}t\mathrm{d}w \\ &\qquad + e^{2x}\left(-\xi - Y^z_z - \tau_t - w\tau_u + Y^w_w + \xi_x + z\xi_u\right)\mathrm{d}t\mathrm{d}z \\ &\qquad + \left(-Y^w_z - e^{2x}\tau_x - ze^{2x}\tau_u\right)\mathrm{d}x\mathrm{d}z. \end{aligned}$$

Thus (3.3) implies that

$$\begin{aligned} Y^w_t - e^{2x}Y^z_x - ze^{2x}Y^z_u + wY^w_u \qquad & \\ + F_x - tF_t + \tfrac{u}{2}F_u + \tfrac{3}{2}wF_w + \tfrac{1}{2}zF_z - \tfrac{5}{2}F &= 0 \\ \xi_t + e^{2x}Y^z_w + w\xi_u &= 0 \\ 2\xi + Y^z_z + \tau_t + w\tau_u - Y^w_w - \xi_x - z\xi_u &= 0 \\ Y^w_z + e^{2x}\tau_x + ze^{2x}\tau_u &= 0, \end{aligned}$$

which is exactly the same set of determining equations that the method of Baikov *et al* [6, 7] gives, so from here on, the calculations are identical.

For further discussions on potential symmetries, the reader may refer to [8, 9].

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