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Exact Time Dependent Solutions to (1+1) Fokker-Planck Equation via Linearizing Transformations to the Itô Equations

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Abstract

It is shown that invertible linearizing transformations of the one-dimensional Itô stochastic differential equations cast the associated Fokker-Planck equation to the heat equation. This leads to the time-dependent exact solutions to the Fokker-Planck equations via inverse transformations. To obtain the linearizing transformations of the Itô stochastic differential equation we have extended Gard's theorem to the time-dependent case.

1 Introduction

The Fokker-Planck equation plays a prominent role in stochastic modelling in science [6]. Relevant previous work [1, 2, 3, 7, 8, 9] considered the group theoretical approach to reduce the Fokker-Planck equation to the heat equation. In [3], it is shown that the Fokker-Planck equation must admit a six dimensional Lie point symmetry algebra to be reducible to the heat equation. In [8, 9], semi-invariants are used as a criterion of reducibility of the Fokker-Planck equation to the heat equation. Our treatment of the reducibility problem differs from the above since we do not consider the group theoretical approach.

We consider the one-dimensional, nonlinear Itô stochastic differential equation (SDE) of the form

$$dX_t = f(X_t, t)dt + g(X_t, t)dW(t), \qquad t \geq 0, \tag{1.1}$$

where $X_t(\omega)$ is the Itô process ($\omega \in \Omega$, where (Ω, U, P) is the probability space) and $dW(t)$ is the infinitesimal increment of the Wiener process [10]. We assume that $f(X_t, t)$ and

$g(X_t, t)$ are sufficiently smooth functions. The Fokker-Planck equation associated with the Itô SDE is

$$\frac{\partial p}{\partial t} + \frac{\partial (pf)}{\partial X} - \frac{1}{2}\frac{\partial^2 (g^2 p)}{\partial X^2} = 0. \tag{1.2}$$

Here $p(X, t)$ is the probability density function. We look for invertible transformations of the form

$$Y = h(X, t), \quad \tau = \phi(t), \quad \frac{\partial h}{\partial X} \neq 0, \tag{1.3}$$

that enables us to cast (1.1) into the linear SDE

$$dY_\tau = E(\tau) d\overline{W}. \tag{1.4}$$

The Fokker-Planck equation associated with the linear SDE (1.4) is

$$\frac{\partial \pi}{\partial \tau} - \frac{1}{2}E^2 \frac{\partial^2 \pi}{\partial Y^2} = 0. \tag{1.5}$$

This is just a variable coefficient heat equation. Solution to the Fokker-Planck equation (1.2) can be found from

$$p = J\pi,$$

where J is the Jacobian of the transformation (1.3). It is clear that not all nonlinear Itô SDEs (1.1) are linearizable. Therefore, we first determine the linearizability conditions in Section 2. This leads us to an extension of Gard's theorem [5]. In Section 3, we then seek further transformations in order to reduce the linear SDEs into the form (1.5). Finally, we summarize the results of the paper in Section 4.

2 Linearization of nonautonomous Itô SDE

The linearization of the Itô SDE (1.1) is the main focus. Here we provide linearization conditions for SDE (1.1). We seek transformations (1.3) which transforms the nonlinear Itô SDE given in (1.1) into a linear equation of the type

$$dY_\tau = (a(\tau)Y_\tau + b(\tau))d\tau + (c(\tau)Y_\tau + e(\tau))d\overline{W}. \tag{2.1}$$

The Itô lemma [10] (see also [10] for the random time change formula for Wiener processes) for $h(X, t)$ yields

$$dY = \frac{1}{\dot{\phi}}\left[\frac{\partial h(X,t)}{\partial t} + f(X,t)\frac{\partial h(X,t)}{\partial X} + \frac{1}{2}g^2(X,t)\frac{\partial^2 h(X,t)}{\partial X^2}\right] d\tau + \frac{1}{\sqrt{\dot{\phi}}}\, g(X,t)\frac{\partial h(X,t)}{\partial X} d\overline{W}. \tag{2.2}$$

From equations (2.1) and (2.2) we obtain

$$\frac{1}{\dot{\phi}}\left[\frac{\partial h(X,t)}{\partial t} + f(X,t)\frac{\partial h(X,t)}{\partial X} + \frac{1}{2}g^2(X,t)\frac{\partial^2 h(X,t)}{\partial X^2}\right] = a(\tau)h(X,t) + b(\tau) \tag{2.3}$$

and

$$\frac{1}{\sqrt{\dot{\phi}}}\left[g(X,t)\frac{\partial h(X,t)}{\partial X}\right] = c(\tau)h(X,t) + e(\tau). \tag{2.4}$$

Equation (2.4) has two distinct solutions for $c(\tau) = 0$ for all values of τ and $c(\tau) \neq 0$ for values of τ in some interval I. Notice that the equivalence transformation for this allows us to transform $c(\tau)$ to the constant c. Therefore, we only need to consider two cases.

Case 1 $c(\tau) = 0$ for all values of τ.
In this case the solution of equation (2.4) is

$$h(X,t) = e(\phi(t))\sqrt{\dot{\phi}}\int \frac{1}{g(X,t)}dX + H(t). \tag{2.5}$$

Substitution of this h into equation (2.3) yields

$$\int \frac{\partial}{\partial t}\left(\frac{1}{g}\right)dX + \frac{f}{g} - \frac{1}{2}\frac{\partial g}{\partial X} = \left(a\dot{\phi} - \frac{\left(e\sqrt{\dot{\phi}}\right)^{\cdot}}{e\sqrt{\dot{\phi}}}\right)\int \frac{dX}{g} + \left(aH - \frac{\dot{H}}{\dot{\phi}} + b\right)\frac{\sqrt{\dot{\phi}}}{e}, \tag{2.6}$$

where the overhead dot denotes differentiation with respect to t. Now differentiation of equation (2.6) with respect to X gives

$$\frac{\partial}{\partial t}\left(\frac{1}{g}\right) + \frac{\partial K}{\partial X} = \left(a\dot{\phi} - \frac{\left(e\sqrt{\dot{\phi}}\right)^{\cdot}}{e\sqrt{\dot{\phi}}}\right)\frac{1}{g}, \quad \text{where} \quad K = \frac{f}{g} - \frac{1}{2}\frac{\partial g}{\partial X}. \tag{2.7}$$

Multiplying both sides of equation (2.7) with $g(X,t)$ results in

$$g\,L = a\dot{\phi} - \frac{\left(e\sqrt{\dot{\phi}}\right)^{\cdot}}{e\sqrt{\dot{\phi}}}, \quad \text{where} \quad L = \frac{\partial}{\partial t}\left(\frac{1}{g}\right) + \frac{\partial}{\partial X}\left(\frac{f}{g} - \frac{1}{2}\frac{\partial g}{\partial X}\right), \tag{2.8}$$

and differentiating (2.8) with respect to X gives

$$\left(g\,L\right)_X = 0. \tag{2.9}$$

Case 2 $c(\tau) \neq 0$ for values of τ in some interval I.
In this case the solution of equation (2.4) is given by

$$h(X,t) = -\frac{e(\phi(t))}{c(\phi(t))} + H(t)\exp\left(c(\phi(t))\sqrt{\dot{\phi}}\int \frac{1}{g(X,t)}dX\right). \tag{2.10}$$

Substitution of h into equation (2.3) yields

$$\left[\dot{H} + H\left(c\sqrt{\dot{\phi}}\right)^{\cdot}\int \frac{dX}{g} + Hc\sqrt{\dot{\phi}}\left(\int \left(\frac{1}{g}\right)_t dX + \frac{f}{g} - \frac{1}{2}\frac{\partial g}{\partial X}\right) + \frac{1}{2}c^2\dot{\phi}H\right]\exp\left(c\sqrt{\dot{\phi}}\int \frac{dX}{g}\right)$$

$$= \left(\frac{e}{c}\right)^{\cdot} - \frac{ae}{c} + aH \exp\left(c\sqrt{\dot{\phi}}\int\frac{dX}{g}\right) + b. \tag{2.11}$$

The rearrangement of the terms of the above equation (2.11) gives

$$\left[\dot{H}+H\left(c\sqrt{\dot{\phi}}\right)^{\cdot}\int\frac{dX}{g}+Hc\sqrt{\dot{\phi}}\left(\int\left(\frac{1}{g}\right)_t dX+\frac{f}{g}-\frac{1}{2}\frac{\partial g}{\partial X}\right)+\frac{1}{2}c^2\dot{\phi}H-aH\right]\exp\left(c\sqrt{\dot{\phi}}\int\frac{dX}{g}\right)$$

$$= \left(\frac{e}{c}\right)^{\cdot} - \frac{ae}{c} + b. \tag{2.12}$$

By setting

$$T = \dot{H}+H\left(c\sqrt{\dot{\phi}}\right)^{\cdot}\int\frac{dX}{g}+Hc\sqrt{\dot{\phi}}\left(\int\left(\frac{1}{g}\right)_t dX+\frac{f}{g}-\frac{1}{2}\frac{\partial g}{\partial X}\right)+\frac{1}{2}c^2\dot{\phi}H-aH, \tag{2.13}$$

we can write equation (2.12) as

$$T\exp\left(c\sqrt{\dot{\phi}}\int\frac{dX}{g}\right) = \left(\frac{e}{c}\right)^{\cdot} - \frac{ae}{c} + b. \tag{2.14}$$

Differentiation of equation (2.14) with respect to X results in

$$\left(\frac{\partial T}{\partial X} + \frac{c\sqrt{\dot{\phi}}}{g}T\right)\exp\left(c\sqrt{\dot{\phi}}\int\frac{dX}{g}\right) = 0, \tag{2.15}$$

where

$$\frac{\partial T}{\partial X} = \frac{H\left(c\sqrt{\dot{\phi}}\right)^{\cdot}}{g} + H\,c\sqrt{\dot{\phi}}L. \tag{2.16}$$

Thus from equation (2.15) we have

$$\frac{\partial T}{\partial X} + \frac{c\sqrt{\dot{\phi}}}{g}T = 0$$

and making use of equation (2.16) we obtain

$$\frac{H\left(c\sqrt{\dot{\phi}}\right)^{\cdot}}{c\sqrt{\dot{\phi}}} + HgL + T = 0. \tag{2.17}$$

Differentiation of equation (2.17) with respect to X yields

$$H(gL)_X + T_X = 0$$

and using equation (2.16) gives

$$H(gL)_X + \frac{H\left(c\sqrt{\dot{\phi}}\right)^{\cdot}}{g} + H\,c\sqrt{\dot{\phi}}L = 0,$$

or

$$g(gL)_X + c\sqrt{\dot{\phi}}\,gL + \left(c\sqrt{\dot{\phi}}\right)^{\cdot} = 0. \tag{2.18}$$

Now differentiation of equation (2.18) with respect to X results in

$$(g(gL)_X)_X + c\sqrt{\dot{\phi}}\;(gL)_X = 0,$$

or

$$c\sqrt{\dot{\phi}} = -\frac{(g(gL)_X)_X}{(gL)_X}$$

and differentiation with respect to X yields

$$\left(\frac{(g(gL)_X)_X}{(gL)_X}\right)_X = 0. \tag{2.19}$$

Note that when

$$(gL)_X = 0,$$

the transformation

$$Y = e(\tau)\sqrt{\dot{\phi}}\int\frac{dX}{g} + H, \quad \tau = \phi(t),$$

transforms equation (1.1) to

$$dY_\tau = (a(\tau)Y_\tau + b(\tau))d\tau + e(\tau)dW_\tau.$$

The map $\phi(t)$ will be determined from equation (2.8) and when

$$\left(\frac{(g(gL)_X)_X}{(gL)_X}\right)_X = 0,$$

the transformation

$$Y = -\frac{e}{c} + H\exp\left(c\sqrt{\dot{\phi}}\int\frac{dX}{g}\right), \quad \tau = \phi(t),$$

transforms equation (1.1) to

$$dY_\tau = (a(\tau)Y_\tau + b(\tau))d\tau + (c(\tau)Y_\tau + e(\tau))dW_\tau,$$

where

$$c(\tau)\sqrt{\dot{\phi}} = -\frac{(g(gL)_X)_X}{(gL)_X}.$$

Thus we have proved the following theorem.

Theorem 1. *The Itô stochastic ordinary differential equation*

$$dX_t = f(X_t, t)dt + g(X_t, t)dW_t,$$

is linearizable if and only if the condition (2.9) or condition (2.19) is satisfied.

Remark. When $g_t = 0$, this result reduces to Gard's linearization theorem.

3 The Fokker-Planck equation for the linearized Itô SDEs

We assume that the Itô SDE corresponding to the Fokker-Planck equation is linearizable to

$$dY_t = (a(t)Y_t + b(t))dt + (c(t)Y_t + e(t))dW_t \tag{3.1}$$

and consider twelve cases to obtain the time-dependent solution to the Fokker-Planck equation.

Case 1 $e(t) \neq 0$ and the other coefficients are zero.

This case may arise when the condition (2.9) is satisfied. The Fokker-Planck equation becomes

$$(P_2)_t = \frac{e^2(t)}{2}\,(P_2)_{YY}\,.$$

Its fundamental solution is

$$P_2(Y,t) = \frac{1}{\sqrt{2\pi\tau}}\exp\left(-\frac{Y^2}{2\tau}\right),\ \tau = \int e^2(t)dt.$$

The solution to the linearizable equation can be obtained from

$$P_1(X,t) = JP_2\left(Y,\tau\right),\ J = \frac{\partial Y}{\partial X} = \frac{e(t)}{g(X,t)},\ Y = e(t)\int \frac{1}{g(X,t)}dX.$$

This leads to the time-dependent solution of the form

$$P_1(X,t) = \frac{e(t)}{g(X,t)\left(2\pi\int e^2(t)dt\right)^{1/2}}\exp\left(-\frac{\left(e(t)\int \frac{1}{g(X,t)}dX\right)^2}{2\int e^2(t)dt}\right).$$

Case 2 $b(t) = 0$ and $c(t) = 0$.

We may have this case when the condition (2.9) is satisfied. The linear transformation

$$Z = k(t)Y, \quad k(t) = \exp\left(-\int a\left(t\right)dt\right),$$

leads to

$$dZ_t = k(t)e(t)dW_t$$

and the corresponding Fokker-Planck equation is of diffusion type having time-dependent coefficient. Following the same procedure as in Case 1 we obtain the solution to the Fokker-Planck equation corresponding to the linearizable Itô SDE as

$$P_1(X,t) = \frac{e(t)k(t)}{g(X,t)\left(2\pi\int e^2(t)k^2(t)dt\right)^{1/2}}\exp\left(-\frac{k^2(t)\left(e(t)\int \frac{1}{g(X,t)}dX\right)^2}{2\int e^2(t)k^2(t)dt}\right).$$

Case 3 $a(t) = 0$ and $c(t) = 0$.

When the condition (2.9) is satisfied one might arrive at this case. The linear transformation

$$Z = Y + l(t), \quad l(t) = -\int b\,(t)\,dt$$

gives

$$dZ_t = e(t)dW_t$$

and in this case the associated Fokker-Planck equation is again a diffusion equation with time-dependent coefficient. Employing the same methodology as in Case 1 above, we have

$$P_1(X,t) = \frac{e(t)}{g(X,t)\left(2\pi\int e^2(t)dt\right)^{1/2}} \exp\left(-\frac{\left(e(t)\int\frac{1}{g(X,t)}dX + l(t)\right)^2}{2\int e^2(t)dt}\right).$$

Case 4 $c(t) = 0$.

This case may arise when the condition (2.9) is satisfied. The linear transformation

$$Z = k(t)Y + l(t), \quad k(t) = \exp\left(-\int a\,(t)\,dt\right), \; l(t) = -\int b\,(t)\,k(t)dt,$$

yields

$$dZ_t = e(t)k(t)dW_t$$

and using the previous method of solution one arrives at

$$P_1(X,t) = \frac{k(t)}{g(X,t)\left(2\pi\int e^2(t)k^2(t)dt\right)^{1/2}} \exp\left(-\frac{\left(k(t)e(t)\int\frac{1}{g(X,t)}dX + l(t)\right)^2}{2\int e^2(t)k^2(t)dt}\right).$$

Case 5 $c(t) \neq 0$ and all other coefficients are zero.

This situation may arise when the condition (2.19) is satisfied. The nonlinear transformation $Z = \ln Y$ leads to

$$dZ_t = -c^2(t)dt + c(t)dW_t$$

and the corresponding Fokker-Planck equation is

$$(P_3)_\tau = \frac{1}{2}\,(P_3)_{ZZ} + (P_3)_Z\,, \; \tau = \int c^2(t)dt.$$

The transformation

$$P_3 = Z^{-1/2}\exp\left(\frac{t}{4}\right)P_4\,(U,\tau)\,, \; U = \ln Z,$$

results in

$$(P_4)_\tau = \frac{1}{2}\,(P_4)_{UU}\,.$$

Using the fundamental solution of the heat equation and performing computations as in Case 1 one obtains

$$P_1(x,t) = \frac{1}{g(X,t)\left(2\pi\int c^2(t)dt\right)^{1/2}} \exp\left(\frac{t}{4} - \frac{\left(\ln\left(c(t)\int \frac{1}{g(X,t)}dX\right)\right)^2}{2\int c^2(t)dt}\right).$$

Case 6 $b(t) = 0$ and $e(t) = 0$.

We may arrive at this case when the condition (2.19) is satisfied. The nonlinear transformation

$$Z = \ln Y + l(t),\ l(t) = \int \left(\frac{c^2(t)}{2} - a(t)\right) dt,$$

gives

$$dZ_t = c(t)dW_t.$$

Again utilizing the fundamental solution of the heat equation and following the similar procedure as in Case 1 we obtain

$$P_1(x,t) = \frac{c(t)\int \frac{1}{g(X,t)}dX}{\left(2\pi\int c^2(t)dt\right)^{1/2}} \exp\left(-\frac{\left(\ln\left(c(t)\int \frac{1}{g(X,t)}dX\right) + l(t)\right)^2}{2\int c^2(t)dt}\right).$$

Case 7 $a(t) = 0$ and $e(t) = 0$.

This case may arise when the condition (2.19) is satisfied. The linear transformation

$$Z = Y + l(t),\ l(t) = -\int b(t)dt,$$

reduces to

$$dZ_t = \left(c(t)Z - c(t)l(t)\right) dW_t.$$

Its corresponding Fokker-Planck equation is

$$(P_3)_t = \frac{1}{2}\left(\left(c(t)Z - c(t)l(t)\right)^2 P_3\right)_{ZZ}. \tag{3.2}$$

Introducing

$$U = c(t)Z - c(t)l(t) \text{ and } \tau = \int ce^2(t)dt$$

into (3.2) we obtain

$$(P_4)_\tau = \frac{1}{2}U^2 (P_4)_{UU} + 2U(P_4)_U + P_4. \tag{3.3}$$

Letting

$$P_4 = U^{-3/2}\exp\left(\frac{t}{8}\right) P_5(V,\tau),\ V = \ln U,$$

one obtains

$$(P_5)_\tau = \frac{1}{2}\,(P_5)_{VV}$$

and

$$\begin{aligned} P_1\,(x,t) &= S(X,t)\exp\left(-\frac{\left(\ln\left(c(t)\exp\left(c(t)\int\frac{1}{g(X,t)}dX\right)\right)\right)^2}{2\int c^2(t)dt} - \frac{t}{8} + c(t)\int\frac{1}{g(X,t)}dX\right), \\ S(X,t) &= \frac{c(t)\left(c(t)\exp\left(c(t)\int\frac{1}{g(X,t)}dX\right)\right)^{-3/2}}{g(X,t)\left(2\pi\int c^2(t)dt\right)^{1/2}}. \end{aligned}$$

Case 8 $e(t) = 0$.

This situation can arise when the condition (2.19) is satisfied. The linear transformation

$$Z = k(t)Y + l(t), \quad k(t) = \exp\left(-\int a\,(t)\,dt\right),\ l(t) = -\int b\,(t)\,k(t)dt,$$

gives

$$dZ_t = ((c(t)Z - c(t)l(t)))\,dW_t$$

and following the steps as in Case 7 one obtains

$$\begin{aligned} P_1\,(x,t) &= J(X,t)\exp\left(-\frac{\left(\ln\left(c(t)k(t)\exp\left(c(t)\int\frac{1}{g(X,t)}dX\right)\right)\right)^2}{2\int c^2(t)dt} - \frac{t}{8} + c(t)\int\frac{1}{g}\,dX\right), \\ J(X,t) &= \frac{c(t)\left(c(t)k(t)\exp\left(c(t)\int\frac{1}{g(X,t)}dX\right)\right)^{-3/2}}{g(X,t)\left(2\pi\int c^2(t)dt\right)^{1/2}}. \end{aligned}$$

Case 9 $a(t) = 0$ and $b(t) = 0$.

We may arrive at this situation when the condition (2.19) is satisfied. The Fokker-Planck equation here is

$$(P_2)_t = \frac{1}{2}\left((c(t)Y + e(t))^2\,P_2\right)_{YY}$$

and the corresponding expressions for P_1 and G are

$$\begin{aligned} P_1\,(x,t) &= G(X,t)\exp\left(-\frac{\left(\ln\left(c(t)\exp\left(c(t)\int\frac{1}{g(X,t)}dX\right) + e(t)\right)\right)^2}{2\int c^2(t)dt} - \frac{t}{8} + c(t)\int\frac{1}{g}\,dX\right), \\ G(X,t) &= \frac{c(t)\left(c(t)\exp\left(c(t)\int\frac{1}{g(X,t)}dX\right) + e(t)\right)^{-3/2}}{g(X,t)\left(2\pi\int c^2(t)dt\right)^{1/2}}. \end{aligned}$$

Case 10 $b(t) = 0$

This case may arise when the condition (2.19) is satisfied. The linear transformation

$$Z = k(t)Y, \quad k(t) = \exp\left(-\int a\,(t)\,dt\right),$$

leads to

$$dZ_t = ((c(t)Z + e(t)k(t)))\,dW_t$$

and in this case

$$P_1\,(x,t) \;=\; H(X,t)\exp\left(-\frac{\left(\ln\left(c(t)k(t)\exp\left(c(t)\int\frac{1}{g}\,dX\right)+ke\right)\right)^2}{2\int c^2(t)dt} - \frac{t}{8} + c(t)\int\frac{1}{g}\,dX\right),$$

$$H(X,t) \;=\; \frac{c(t)\left(c(t)k(t)\exp\left(c(t)\int\frac{1}{g(X,t)}dX\right)+ke\right)^{-3/2}}{g(X,t)\left(2\pi\int c^2(t)dt\right)^{1/2}}.$$

Case 11 $a(t) = 0$

We may arrive at this case when the condition (2.19) is satisfied. The linear transformation

$$Z = Y + l(t),\ \ l(t) = -\int b(t)dt,$$

yields the following expressions

$$dZ_t = (c(t)Z + e(t) - c(t)l(t))\,dW_t,$$

$$P_1\,(x,t) \;=\; R(X,t)\exp\left(-\frac{\left(\ln\left(c(t)\exp\left(c(t)\int\frac{1}{g(X,t)}dX\right)+e\right)\right)^2}{2\int c^2(t)dt} - \frac{t}{8} + c(t)\int\frac{1}{g}\,dX\right),$$

$$R(X,t) \;=\; \frac{c^2(t)\left(c(t)\exp\left(c(t)\int\frac{1}{g(X,t)}dX\right)+e\right)^{-3/2}}{g(X,t)\left(2\pi\int c^2(t)dt\right)^{1/2}}.$$

Case 12 None of the coefficients are zero.

This situation may arise when the condition (2.19) is satisfied. The linear transformation

$$Z = k(t)Y + l(t), \quad k(t) = \exp\left(-\int a\,(t)\,dt\right),\ l(t) = -\int b\,(t)\,k(t)dt,$$

helps us in obtaining the following

$$dZ_t = ((c(t)Z + e(t)k(t) - c(t)l(t)))\,dW_t,$$

$$P_1\,(x,t) = F(X,t)\exp\left(-\frac{\left(\ln\left(c(t)k(t)\exp\left(c(t)\int\frac{1}{g(X,t)}dX\right)+ek\right)\right)^2}{2\int c^2(t)dt} - \frac{t}{8} + c(t)\int\frac{1}{g}\,dX\right),$$

$$F(X,t) = \frac{kc^2(t)\left(c(t)k(t)\exp\left(c(t)\int\frac{1}{g(X,t)}dX\right)+ek\right)^{-3/2}}{g(X,t)\left(2\pi\int c^2(t)dt\right)^{1/2}}.$$

4 Conclusions

We have shown that when the Itô stochastic differential equation is linearizable the corresponding Fokker-Planck equation reduces to the heat equation. Two linearization conditions were derived. Twelve distinct classes of time dependent solutions to (1+1) Fokker-Planck equations have been obtained.

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References

[1] Bluman G W, Similarity solutions of the one-dimensional Fokker-Planck equation, *Int. J. Non-linear Mech.* **6** (1971), 143-153.

[2] Bluman G W and Cole J D, Similarity methods for differential equation, Springer-Verlag, New York, 1974.

[3] Bluman G W, On the transformation of diffusion processes into Wiener processes, *SIAM J. Appl. Math.* **39** (1980), 238-247.

[4] Cox J C, Ingersoll J E and Ross S A, A theory of the term structure of interest rates, *Econometrica* **53** (1985), 385-408.

[5] Gard T C, Introduction to stochastic differential equations, Dekker, 1988.

[6] Gardiner C W, Handbook of stochastic methods for physics, chemistry and natural science, (Springer Series in Synergetics), Springer-Verlag, New York, 1985.

[7] Ibragimov N H, Laplace type invariants for parabolic equations, *Nonlinear Dynamics* **28** (2002), 122-133.

[8] Johnpillai I K and Mahomed F M, Singular invariant equation for the (1+1) Fokker-Planck equation, *J. Phys. A: Math. Gen.* **34** (2001), 11033-11051.

[9] Mahomed F M, Invariant characterization of scalar linear (1+1) parabolic equations, *International J. Nonlinear Math. Phys.* This volume.

[10] Oksendal B, Stochastic differential equations: An introduction with applications, sixth edition, Springer-Verlag, Berlin, 2003.