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# Symmetry Solutions of a Third-Order Ordinary Differential Equation which Arises from Prandtl Boundary Layer Equations

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**Abstract**

The similarity solution to Prandtl's boundary layer equations for two-dimensional and radial flows with vanishing or constant mainstream velocity gives rise to a third-order ordinary differential equation which depends on a parameter $\alpha$. For special values of $\alpha$ the third-order ordinary differential equation admits a three-dimensional symmetry Lie algebra $L_3$. For solvable $L_3$ the equation is integrated by quadrature. For non-solvable $L_3$ the equation reduces to the Chazy equation. The Chazy equation is reduced to a first-order differential equation in terms of differential invariants which is transformed to a Riccati equation. In general the third-order ordinary differential equation admits a two-dimensional symmetry Lie algebra $L_2$. For $L_2$ the differential equation can only be reduced to a first-order equation. The invariant solutions of the third-order ordinary differential equation are also derived.

## 1 Introduction

Prandtl [14] introduced the concept of a boundary layer in large Reynolds number flows in 1904 and he also showed how the Navier-Stokes equation could be simplified to yield approximate solutions. The similarity solution to Prandtl's boundary layer equation for the stream function for steady two-dimensional and radial flows with vanishing or constant mainstream velocity yields the third-order ordinary differential equation

$$\frac{d^3y}{dx^3} + By\frac{d^2y}{dx^2} + C(\frac{dy}{dx})^2 = 0, \tag{1.1}$$

where

$$B = \begin{cases} 1-\alpha & \text{two-dimensional} \\ 2-\alpha & \text{radial} \end{cases}, \quad C = 2\alpha - 1, \tag{1.2}$$

and $\alpha$ is a constant determined from further conditions. Equation (1.1) arises in the study of steady flows produced by free jets, wall jets and liquid jets (two-dimensional or radial), the flow past a stretching plate and Blasius flow. The numerical solution for a free two-dimensional jet for which $\alpha = 2/3$ was obtained by Schlichting [17] and later an analytic solution was derived by Bickley [3]. In [18], Squire obtained the solution for the free radial jet for which $\alpha = 1$. The solutions for two-dimensional and radial wall jets for which $\alpha = 3/4$ (two-dimensional) and $\alpha = 5/4$ (radial) were obtained in parametric form by Glauert [9]. Riley [15] derived the solution for a radial liquid jet for which $\alpha = 2$. Later, two-dimensional flow past a stretching plate with $\alpha = 0$ was discussed by Crane [7].

The purpose of this paper is to obtain reductions and solutions of the third-order differential equations which arise from Prandtl boundary layer equations for two-dimensional and radial flows with vanishing or constant mainstream velocity using Lie symmetry methods of reduction. The invariant solutions are also derived.

The Lie point symmetry generators of equation (1.1) for general values of $\alpha$ are

$$X_1 = \partial/\partial x, \quad X_2 = x\partial/\partial x - y\partial/\partial y. \tag{1.3}$$

For special values of $\alpha$ three Lie point symmetry generators exist and the third-order ordinary differential equation is solved by the Lie approach as described, for example, by Ibragimov and Nucci [11], Mahomed [12] and Olver [13]. For $B = 0$ the third-order ordinary differential equation (1.1) describes radial and two-dimensional liquid jets and admits a solvable Lie algebra. We solve the equation by the Lie approach ( Ibragimov and Nucci [11], Mahomed [12], Olver [13]). For $\alpha = -1$ (two-dimensional) and $\alpha = -4$ (radial), the third-order ordinary differential equation (1.1) admits a non-solvable Lie algebra and can be reduced to the Chazy equation [4, 5, 6, 8]. Clarkson and Olver [8] expressed the general solution of the Chazy equation as the ratio of two solutions of a hypergeometric equation. We reduce the Chazy equation by the Lie approach using the semi-canonical variables of Ibragimov and Nucci [11]. Another approach is given by Adam and Mahomed [1].

## 2 Mathematical formulation

Prandtl's boundary layer equation for the stream function for an incompressible, steady two-dimensional flow with uniform or vanishing mainstream velocity is [16]

$$\frac{\partial\bar{\psi}}{\partial\bar{y}}\frac{\partial^2\bar{\psi}}{\partial\bar{x}\partial\bar{y}} - \frac{\partial\bar{\psi}}{\partial\bar{x}}\frac{\partial^2\bar{\psi}}{\partial\bar{y}^2} = \nu\frac{\partial^3\bar{\psi}}{\partial\bar{y}^3}, \tag{2.1}$$

where $\nu$ is the kinematic viscosity. Using the classical Lie method of infinitesimal transformations [13] the similarity solution for equation (2.1) is found to be

$$\psi(\bar{x},\bar{y}) = \bar{x}^{1-\alpha}F(\chi),\ \chi = \frac{\bar{y}}{\bar{x}^{\alpha}}. \tag{2.2}$$

The substitution of equation (2.2) in equation (2.1) yields a third-order ordinary differential equation in $F(\chi)$:

$$\nu\frac{d^3F}{d\chi^3} + (1-\alpha)F\frac{d^2F}{d\chi^2} + (2\alpha-1)(\frac{dF}{d\chi})^2 = 0. \tag{2.3}$$

The variable $\chi = \bar{y}/\bar{x}^{\alpha}$ is the similarity variable.

For radial flow, Prandtl's boundary layer equation for uniform or vanishing mainstream velocity is [9]

$$\frac{1}{r}\frac{\partial\psi}{\partial z}\frac{\partial^2\psi}{\partial r\partial z} - \frac{1}{r^2}(\frac{\partial\psi}{\partial z})^2 - \frac{1}{r}\frac{\partial\psi}{\partial r}\frac{\partial^2\psi}{\partial z^2} = \nu\frac{\partial^3\psi}{\partial z^3}. \tag{2.4}$$

The similarity solution of equation (2.4) derived using the Lie method is

$$\psi(r,z) = r^{2-\alpha}F(\chi),\ \chi = \frac{z}{r^{\alpha}}, \tag{2.5}$$

which reduces equation (2.4) to

$$\nu\frac{d^3F}{d\chi^3} + (2-\alpha)F\frac{d^2F}{d\chi^2} + (2\alpha-1)(\frac{dF}{d\chi})^2 = 0. \tag{2.6}$$

Equations (2.3) and (2.6) can be combined to give the following third-order ordinary differential equation:

$$\nu\frac{d^3F}{d\chi^3} + BF\frac{d^2F}{d\chi^2} + C(\frac{dF}{d\chi})^2 = 0, \tag{2.7}$$

where $B$ and $C$ are defined in terms of $\alpha$ by (1.2). The transformation $(\chi, F) \mapsto (x, \nu y)$ reduces equation (2.7) to (1.1).

## 3 Lie point symmetry generators

Equation (1.1) can be written as

$$E(y, y', y'', y''') = 0, \tag{3.1}$$

where

$$E = \frac{d^3y}{dx^3} + By\frac{d^2y}{dx^2} + C(\frac{dy}{dx})^2. \tag{3.2}$$

The Lie point symmetry generators

$$X = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}, \tag{3.3}$$

are obtained from the determining equation [2]

$$X^{[3]}E\mid_{E=0} = 0, \tag{3.4}$$

where $X^{[3]}$ is the third-order prolongation given by

$$X^{[3]} = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta_1\frac{\partial}{\partial y'} + \zeta_2\frac{\partial}{\partial y''} + \zeta_3\frac{\partial}{\partial y'''}, \tag{3.5}$$

where

$$\zeta_1 = D(\eta) - y'D(\xi), \quad \zeta_2 = D(\zeta_1) - y''D(\xi), \quad \zeta_3 = D(\zeta_2) - y'''D(\xi), \tag{3.6}$$

and

$$D = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'} + y'''\frac{\partial}{\partial y''}. \tag{3.7}$$

Equation (3.4) is separated according to derivatives of $y$. The solution depends whether $B = 0$ or $B \neq 0$.

Table 3.1: Lie point symmetries for $B = 0$ and $B \neq 0$.

| Two-dimensional flow | Radial flow | Lie point symmetries for both flows | Lie algebra |
|---|---|---|---|
| $B = 0$ | $B = 0$ | $X_1 = \frac{\partial}{\partial x}$ | $[X_1, X_2] = 0$ |
| $\alpha = 1$ | $\alpha = 2$ | $X_2 = \frac{\partial}{\partial y}$ | $[X_1, X_3] = X_1$ |
| | | $X_3 = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$ | $[X_2, X_3] = -X_2$ |
| $B = 2$ | $B = 6$ | $X_1 = \frac{\partial}{\partial x}$ | $[X_1, X_2] = X_1$ |
| $\alpha = -1$ | $\alpha = -4$ | $X_2 = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$ | $[X_1, X_3] = 2X_2$ |
| | | $X_3 = x^2\frac{\partial}{\partial x} + (\frac{12}{B} - 2xy)\frac{\partial}{\partial y}$ | $[X_2, X_3] = X_3$ |
| $B \neq 0$, $\alpha \neq 1$ | $B \neq 0$, $\alpha \neq 2$ | $X_1 = \frac{\partial}{\partial x}$ | $[X_1, X_2] = X_1$ |
| $B \neq 2$, $\alpha \neq -1$ | $B \neq 6$, $\alpha \neq -4$ | $X_2 = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$ | |

# 4 Symmetry solutions

## 4.1 *Case I: $B = 0$ (two-dimensional and radial* )

For this case, $\alpha = 1$, $C = 1$ for two-dimensional flow and $\alpha = 2$, $C = 3$ for radial flow. Using the transformation $y \to \frac{3}{C}Y$ in equation (1.1), we obtain

$$\frac{d^3Y}{dx^3} + 3(\frac{dY}{dx})^2 = 0. \tag{4.1}$$

Equation (4.1) applies for both radial and two-dimensional liquid jets [15]. Using $y \to \frac{3}{C}Y$ in Table 3.1, the commutators of the Lie point symmetry generators for this case are $[X_1, X_2] = 0$, $[X_1, X_3] = X_1$, $[X_2, X_3] = -X_2$. Thus equation (4.1) admits a solvable Lie algebra $L_3$ and can be solved by the Lie approach as outlined, for example, by Ibragimov and Nucci [11], Mahomed [12] and Olver [13].

Consider the subalgebra $L_2 =< X_1, X_2 >$. A basis of differential invariants of the subalgebra $L_2 = L_2 < X_1, X_2 >$ is

$$s = Y', \quad t = Y'', \tag{4.2}$$

which reduces equation (4.1) to the following first-order ordinary differential equation:

$$\frac{dt}{ds} = -\frac{3s^2}{t}, \tag{4.3}$$

which is in variables separable form. The solution of equation (4.3) is

$$t = \left[2(c_1 - s^3)\right]^{\frac{1}{2}}, \tag{4.4}$$

which can be expressed in the original variables as

$$Y'' = \left[2(c_1 - Y'^3)\right]^{\frac{1}{2}}. \tag{4.5}$$

The solution of equation (4.5) is

$$-\frac{2}{3c_1^{\frac{2}{3}}}(c_1 - s^3)^{\frac{1}{2}} \times {}_2F_1[\frac{1}{2}, \frac{2}{3}; \frac{3}{2}; 1 - \frac{s^3}{c_1}] = \sqrt{2}x + c_2, \tag{4.6}$$

where ${}_2F_1$ is the Hypergeometric function of first kind and $c_1$, $c_2$ are arbitrary constants.

For both the radial and two-dimensional liquid jets the boundary conditions are

$$Y(0) = 0, \quad Y'(0) = 0, \quad Y'(1) = 1, \quad Y''(1) = 0, \tag{4.7}$$

and therefore

$$c_1 = 1, \quad c_2 = -\frac{2}{3}\, {}_2F_1[\frac{1}{2}, \frac{2}{3}; \frac{3}{2}; 1]. \tag{4.8}$$

Equation (4.6) finally yields

$$x = \frac{\sqrt{2}}{3}\left( {}_2F_1[\frac{1}{2}, \frac{2}{3}; \frac{3}{2}; 1] - (1 - s^3)^{\frac{1}{2}} \times {}_2F_1[\frac{1}{2}, \frac{2}{3}; \frac{3}{2}; 1 - s^3]\right), \tag{4.9}$$

which can be used to tabulate the values of $x$ for given values of the parameter $s = Y'$. The scaled velocity profiles for radial and two-dimensional liquid jets are the same and are shown in Figure 1. Figure 1 agrees with the velocity profile of a radial jet given by Riley [15].

## 4.2 *Case II:* $B = 2, \alpha = -1$ (two-dimensional), $B = 6, \alpha = -4$ (radial)

Then $C = -3B/2$ and with this value of $C$ the transformation $y \to -2Y/B$ reduces equation (1.1) to

$$\frac{d^3Y}{dx^3} - 2Y\frac{d^2Y}{dx^2} + 3(\frac{dY}{dx})^2 = 0, \tag{4.10}$$

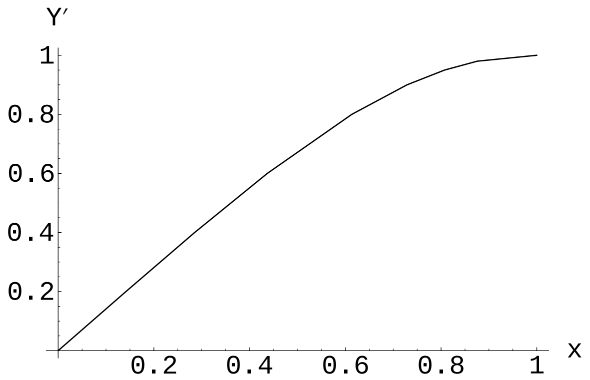


Figure 1: The velocity function $Y'(x)$ for two-dimensional and radial liquid jets.

which is the Chazy equation ([4, 5, 6, 8]). Using $y \to -2Y/B$ in the Lie point symmetries for this case in Table 3.1, we obtain

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = x\frac{\partial}{\partial x} - Y\frac{\partial}{\partial Y}, \quad X_3 = x^2\frac{\partial}{\partial x} - 2(3 + xY)\frac{\partial}{\partial Y}, \tag{4.11}$$

which are the Lie point symmetries of the Chazy equation ([4, 5, 6, 8]). The commutators of the Lie point symmetry generators in (4.11) are

$$[X_1, X_2] = X_1, \quad [X_1, X_3] = 2X_2, \quad [X_2, X_3] = X_3. \tag{4.12}$$

Thus equation (4.10) possesses a non-solvable Lie symmetry algebra $L_3$ and, therefore, cannot be solved by the Lie approach (Ibragimov and Nucci [11], Mahomed [12] and Olver [13]).

Consider the subalgebra $L_2 =< X_1, X_2 >$. A basis of differential invariants of the subalgebra $L_2 = L_2 < X_1, X_2 >$ is

$$s = Y'Y^{-2}, \quad t = Y''Y^{-3}, \tag{4.13}$$

which reduces equation (4.10) to the following first-order ordinary differential equation:

$$\frac{dt}{ds} = \frac{(3s - 2)t + 3s^2}{2s^2 - t}. \tag{4.14}$$

The non-local generator $X_3$ admitted by equation (4.14) in the space $(s, t)$ is [11]

$$X_3 = Y^{-1}[-2(1 - 6s)\frac{\partial}{\partial s} + 6(3t - s)\frac{\partial}{\partial t}]. \tag{4.15}$$

By solving the first-order linear partial differential equation

$$(1 - 6s)\frac{\partial w}{\partial s} - 3(3t - s)\frac{\partial w}{\partial t} = 0, \tag{4.16}$$

we obtain a new variable

$$w = \frac{1 + 9(t - s)}{(1 - 6s)^{\frac{3}{2}}}, \tag{4.17}$$

which transforms the generator (4.15) to its semi-canonical form

$$X_3 = Y^{-1}(1-6s)\frac{\partial}{\partial s}. \tag{4.18}$$

Equation (4.14) in variables $s$ and $w$ becomes

$$\frac{ds}{dw} = \frac{(1-6s)^{\frac{1}{2}}(-1+3s)+(1-6s)w}{9(w^2-1)}. \tag{4.19}$$

By the Vessiot-Guldberg-Lie theorem [10, 11], the generators

$$\Lambda_1 = s(1-6s)^{\frac{1}{2}}\frac{\partial}{\partial s}, \quad \Lambda_2 = (1-6s)^{\frac{1}{2}}\frac{\partial}{\partial s}, \quad \Lambda_3 = (1-6s)\frac{\partial}{\partial s}, \tag{4.20}$$

form a three-dimensional Lie algebra. To convert equation (4.19) to a Riccati equation, consider

$$\bar{\Lambda}_1 = -\frac{1}{3}\Lambda_2, \quad \bar{\Lambda}_2 = -\frac{1}{3}\Lambda_3, \quad \bar{\Lambda}_3 = 2\Lambda_1 - \frac{1}{3}\Lambda_2, \tag{4.21}$$

and define $\phi$

$$\phi = (1-6s)^{\frac{1}{2}}, \tag{4.22}$$

where $\phi$ satisfies

$$\frac{1}{3}(1-6s)\frac{d\phi}{dt} = -\phi. \tag{4.23}$$

Equation (4.19) reduces to a Riccati equation in the variables $(w, \phi)$:

$$\frac{d\phi}{dw} = \frac{1}{6(w^2-1)}\phi^2 - \frac{w}{3(w^2-1)}\phi + \frac{1}{6(w^2-1)}. \tag{4.24}$$

The substitution

$$u = \exp[-\frac{1}{6}(\int \frac{\phi}{w^2-1}dw)], \tag{4.25}$$

reduces the Riccati equation (4.24) to a second order linear differential equation:

$$\frac{d^2u}{dw^2} + \frac{7w}{3(w^2-1)}\frac{du}{dw} - \frac{1}{36(w^2-1)^2}u = 0. \tag{4.26}$$

The solution of equation (4.27) is

$$u = \frac{c_1 P[\frac{1}{6}, w] + c_2 Q[\frac{1}{6}, w]}{(w^2-1)^{\frac{1}{12}}}, \tag{4.27}$$

where $P[\frac{1}{6}, w]$ and $Q[\frac{1}{6}, w]$ are Legendre functions of first and second kind. Thus the reduction of the Chazy equation is

$$Y^2 = \frac{6Y'}{1-\phi^2(w)}, \tag{4.28}$$

where $w$ in the original variables is

$$w = \frac{Y^3 - 9YY' + 9Y''}{(Y^2 - 6Y')^{\frac{3}{2}}}, \tag{4.29}$$

and $\phi$ is given by

$$\phi = \frac{[8w(c_1 P[\frac{1}{6}, w] + c_2 Q[\frac{1}{6}, w]) - 7(c_1 P[\frac{7}{6}, w] + c_2 Q[\frac{7}{6}, w])}{c_1 P[\frac{1}{6}, w] + c_2 Q[\frac{1}{6}, w]}. \tag{4.30}$$

To obtain the reduction of equation (1.1) in $(x, y)$ variables with $B = 2, C = -3$ (two-dimensional) $B = 6, C = -9$ (radial) replace $Y \to \frac{-By}{2}$. For $(x, Y) \to (\chi, \frac{-BF}{2\nu})$ the reduction of equation (2.7) can be recovered.

The approach used here is due to Ibragimov and Nucci [11] and is different from those of Clarkson and Olver [8] and Adam and Mahomed [1].

### 4.3 *Case III: $B \neq 0,\ B \neq 2$ (two-dimensional), $B \neq 0,\ B \neq 6$ (radial)*

From Table 3.1, for this case we have only two generators. Using the transformation $y \to \frac{1}{B}Y$ in equation (1.1), we obtain

$$\frac{d^3Y}{dx^3} + Y\frac{d^2Y}{dx^2} + \frac{C}{B}(\frac{dY}{dx})^2 = 0. \tag{4.31}$$

The Lie point symmetry generators transform to

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = x\frac{\partial}{\partial x} - Y\frac{\partial}{\partial Y}. \tag{4.32}$$

The invariants of $X_1$ are

$$u = Y, \quad v = Y', \tag{4.33}$$

which reduce equation (4.31) to the second-order ordinary differential equation

$$v\frac{d^2v}{du^2} + (\frac{dv}{du})^2 + u\frac{dv}{du} + \frac{C}{B}v = 0. \tag{4.34}$$

The generator

$$X_2 = u\frac{\partial}{\partial u} + 2v\frac{\partial}{\partial v}, \tag{4.35}$$

in $(u, v)$ coordinates is a Lie point symmetry of the reduced equation (4.34). The invariants of the generator $X_2$ are

$$s = vu^{-2}, \quad t = u^{-1}\frac{dv}{du}, \tag{4.36}$$

which reduce the second-order ordinary differential equation (4.34) to the first-order differential equation

$$\frac{dt}{ds} = \frac{t^2 + t + st + \frac{C}{B}s}{s(2s - t)}. \tag{4.37}$$

There are several subcases depending on the value of the ratio $C/B$.

For $C/B = 0$, $\alpha = 1/2$ with $B = 1/2$ (two-dimensional) and $B = 3/2$ (radial). Equation (4.31) reduces to

$$\frac{d^3Y}{dx^3} + Y\frac{d^2Y}{dx^2} = 0, \tag{4.38}$$

which is the Blasius equation [2]. Equation (4.38) in terms of invariants reduces to the following first-order ordinary differential equation:

$$\frac{dt}{ds} = \frac{t^2 + t + st}{s(2s - t)}. \tag{4.39}$$

For $C/B = -1$, $\alpha = 0$ for two-dimensional flows and $\alpha = -1$ for radial flows. Using appropriate boundary conditions, Crane [7] derived an exact solution for a two-dimensional stretching plate with $\alpha = 0$. For $C/B = -1$ equation (4.31) in terms of differential invariants becomes

$$\frac{dt}{ds} = \frac{t^2 + t + st - s}{s(2s - t)}. \tag{4.40}$$

It is of interest to observe that for $C/B = 1$ and $C/B = 2$, the second-order equation (4.34) obtained by using the invariants of $X_1$ becomes exact. The second reduction is not needed to obtain a solution.

For $C/B = 1$, $\alpha = 2/3$ for two-dimensional flows and $\alpha = 1$ for radial flows. Equation (4.34) can be integrated immediately with respect to $u$ to give

$$\frac{dv}{du} = -u + \frac{c_1}{v}, \tag{4.41}$$

where $c_1$ is a constant. Integrating equation (4.41) again with respect to $u$ and expressing the result in terms of the original variables gives the Riccati equation

$$Y' = -\frac{Y^2}{2} + c_1 x + c_2, \tag{4.42}$$

where $c_2$ is a constant. For a free two-dimensional jet [17, 3] the conserved quantity gives $\alpha = 2/3$ and for the free radial jet [18] the conserved quantity gives $\alpha = 1$. In both cases the boundary conditions are

$$Y(0) = 0,\ Y''(0) = 0,\ Y'(\pm\infty) = 0. \tag{4.43}$$

Imposing the boundary condition $Y'(\pm\infty) = 0$ gives

$$c_1 = 0, \quad c_2 = \frac{1}{2}Y^2(\infty), \tag{4.44}$$

and equation (4.42) reduces to

$$Y' = \frac{1}{2}(Y^2(\infty) - Y^2). \tag{4.45}$$

The solution may be completed for the free two-dimensional jet as described by Bickley [3] and for the free radial jet as described by Squire [18].

For $C/B = 2$, $\alpha = 3/4$ for two-dimensional flows and $\alpha = 5/4$ for radial flows. By first multiplying equation (4.34) by $u$, equation (4.34) can be integrated with respect to $u$ to give

$$\frac{dv}{du} - \frac{1}{2u}v = -u + \frac{c_3}{uv}, \tag{4.46}$$

where $c_3$ is a constant. Multiplying equation (4.46) by the integrating factor $u^{-1/2}$, integrating again with respect to $u$ and expressing the result in the original variables gives

$$Y' = -\frac{2Y^2}{3} + (c_3 \int^x Y^{-3/2} dx + c_4)Y^{1/2}, \tag{4.47}$$

where $c_4$ is a constant. Using a conserved quantity, Glauert [9] showed that for a two-dimensional wall jet $\alpha = 3/4$ and for a radial wall jet $\alpha = 5/4$. The boundary conditions in both cases are

$$Y(0) = 0,\ Y'(0) = 0,\ Y'(\infty) = 0. \tag{4.48}$$

Now for a wall jet the stress at the wall is non-zero and finite and therefore $Y''(0)$ is non-zero and finite. Since $Y(0) = 0$ and $Y'(0) = 0$ it follows that

$$Y(x) \sim \frac{1}{2}Y''(0)x^2 \quad \text{as } x \to 0. \tag{4.49}$$

Hence

$$c_3 Y^{1/2} \int^x \frac{1}{Y^{3/2}} dx \sim -\frac{c_3}{Y''(0)}\frac{1}{x}, \quad \text{as } x \to 0. \tag{4.50}$$

Imposing on equation (4.47) the boundary condition at $x = 0$ therefore gives $c_3 = 0$ and imposing $Y'(\infty) = 0$ yields

$$c_4 = \frac{2}{3}Y^{3/2}(\infty). \tag{4.51}$$

Equation (4.47) reduces to

$$Y' = \frac{2}{3}Y^{1/2}(Y^{3/2}(\infty) - Y^{3/2}). \tag{4.52}$$

The solution for two-dimensional and radial wall jets may be completed as described by Glauert [9].

# 5 Invariant solutions

## 5.1 *Case I: $B = 0$ (two-dimensional and radial)*

From Table 3.1 the generator $X$ of the invariant solution, for $y \to 3Y/C$, is

$$X = (c_1 + c_3 x)\frac{\partial}{\partial x} + (\frac{Cc_2}{3} - Yc_3)\frac{\partial}{\partial Y}, \tag{5.1}$$

where $c_1$, $c_2$ and $c_3$ are constants. The invariant solution will depend only on the ratio of the constants. The invariant solution is obtained by solving the following characteristic equation

$$\frac{dx}{(c_1 + c_3 x)} = \frac{dY}{(\frac{Cc_2}{3} - Yc_3)}, \tag{5.2}$$

which yields

$$Y = \frac{1}{c_3}[\frac{C}{3}c_2 + \frac{k}{(c_1 + c_3 x)}], \tag{5.3}$$

provided $c_3 \neq 0$, where $k$ is the constant of integration. To obtain $k$, equation (5.3) is substituted into equation (4.1). This gives $k = 0$ and $k = 2c_3^2$. For $k = 0$, we obtain the constant solution $Y = c_2 C/3c_3$. The invariant solution for $k = 2c_3^2$ is

$$Y = \frac{C}{3}\frac{c_2}{c_3} + \frac{2c_3}{(c_1 + c_3 x)}. \tag{5.4}$$

After using $y \rightarrow 3Y/C$, the invariant solutions for two-dimensional and radial flows can be obtained from equation (5.4) by taking $C = 1$ and $C = 3$, respectively. The trivial constant solution is obtained for $c_3 = 0$.

### 5.2 *Case II: $B = 2, \alpha = -1$ (two-dimensional), $B = 6, \alpha = -4$ (radial)*

In Section 4.2, we saw that equation (1.1) reduces to the Chazy equation (4.10). From equation (4.11), the generator $X$ of the invariant solution is

$$X = (c_1 + c_2 x + c_3 x^2)\frac{\partial}{\partial x} - (c_2 Y + 2c_3(3 + xY))\frac{\partial}{\partial Y}, \tag{5.5}$$

where $c_1$, $c_2$ and $c_3$ are constants. The solution of the characteristic equation gives

$$Y = \frac{k - 6c_3 x}{c_1 + c_2 x + c_3 x^2}. \tag{5.6}$$

By substituting equation (5.6) into equation (4.10) it is found that the constant of integration $k$ satisfies

$$(c_2^2 - 4c_1 c_3)(k^2 + 6kc_2 + 36c_1 c_3) = 0. \tag{5.7}$$

Thus either $c_2^2 - 4c_1c_3 = 0$ and $k$ is arbitrary or $c_2^2 - 4c_1c_3 \neq 0$ and

$$k = 3[-c_2 \pm (c_2^2 - 4c_1 c_3)^{1/2}]. \tag{5.8}$$

If $c_2^2 - 4c_1c_3 < 0$, $k$ is complex and the invariant solution does not exist.

### 5.3 *Case III: $B \neq 0,\ B \neq 2$ (two-dimensional), $B \neq 0,\ B \neq 6$ (radial)*

From Table 3.1, for all real values of $\alpha$, except $\alpha = 1$ and $\alpha = -1$ for two-dimensional flow and $\alpha = 2$ and $\alpha = -4$ for radial flow, the generator $X$ of the invariant solution of equation (1.1) is

$$X = (c_1 + c_2 x)\frac{\partial}{\partial x} - c_2 y \frac{\partial}{\partial y}, \tag{5.9}$$

where $c_1$ and $c_2$ are constants. The non-trivial invariant solution is

$$y = \frac{6c_2}{(2B + C)(c_1 + c_2 x)}, \tag{5.10}$$

where $2B + C$ does not depend on $\alpha$ and is 1 for two-dimensional flow and 3 for radial flow. Thus for Blasius flow we obtain the invariant solution

$$y = \frac{6c_2}{(c_1 + c_2 x)}. \tag{5.11}$$

## 6 Conclusions

For $\alpha = 1$ (two-dimensional) $\alpha = 2$ (radial), equation (1.1) admits three independent Lie point symmetries generating a solvable Lie algebra. We therefore solved equation (1.1) by the Lie approach and obtained the scaled velocity profile for two-dimensional and radial liquid jets. For $\alpha = -1$ (two-dimensional) $\alpha = -4$ (radial), equation (1.1) has three independent Lie point symmetries generating a non-solvable Lie algebra. The Chazy equation was recovered and its reduction was obtained using the semi-canonical variables of Ibragimov and Nucci [11].

For the values of $\alpha$ which correspond to two-dimensional and radial (free or wall) jets, we have given an alternative method of solution which is more systematic. Equation (1.1) has two independent Lie point symmetries. The equation can be integrated because the second-order differential equations obtained using the invariants of $X_1$ are exact and the boundary conditions give the constants of integration special values. For all other real values of $\alpha$, the differential invariants of $X_1$ and $X_2$ can be used to reduce the third-order ordinary differential equation to a first-order ordinary differential equation.

We have also derived the invariant solutions of equation (1.1) which give singular solutions of the third-order ordinary differential equation. Particularly, for the Chazy equation and Blasius equation the invariant solutions have been obtained.

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