Journal of Nonlinear Mathematical Physics Volume 15, Supplement 1 (2008), 60–68 ARTICLE

On the Origins of Symmetries of Partial Differential Equations: the Example of the Korteweg–de Vries Equation

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Abstract

Type II hidden symmetries of partial differential equations (PDE) are extra symmetries in addition to the inherited symmetries of the differential equations which arise when the number of independent and dependent variables is reduced by a Lie point symmetry. (Type I hidden symmetries arise in the *increase* of number of variables.) Unlike the case of ordinary differential equations, these symmetries do not arise from contact symmetries or nonlocal symmetries. In fact, we have previously shown that they are symmetries of other differential equations. However, in determining the origin of these symmetries we show that finding the origin of *any* symmetry of a PDE is a non-trivial exercise. The example of the Korteweg–de Vries equation is used to illustrate this point.

1 Introduction

Lie point symmetries provide a useful route to finding particular solutions of PDEs [5]. These, group–invariant, solutions are invariably physically important and include the 'soliton' solutions for evolution equations. However, in this approach, the symmetries of each subsequent equation are important to determine the group to which the final solution belongs.

It has been discovered, that, in a similar manner to that of ordinary differential equations ODEs, PDEs also exhibit the phenomenon of "hidden symmetries" [6, 1]. These are symmetries that arise unexpectedly in the reduction (or increase) of order of an ODE. In the case of PDEs, the symmetries arise in the reduction (or increase) of the number of variables. Unlike the ODE case where these symmetries have their origin in contact or nonlocal symmetries, the origin of hidden symmetries in the PDE case is a point symmetry of another equation. However, determining the "master PDE" which gives rise to these symmetries is a difficult task, though we have made some progress [2, 3, 4].

It is the purpose of this paper to indicate the difficulties of resolving the problem of the determination of the origin of hidden symmetries for PDEs. In the case of ODEs, it is a fairly simple matter to reverse the transformations to obtain the original symmetries. However, in the case of PDEs, the problem is complicated by the fact that parts of symmetries can "disappear" after defining new reduction variables. As a result, obtaining these symmetries by reversing the transformations is no longer a straighforward task. (Note that this is different from merely obtaining equations invariant under a particular symmetry - that is indeed a straightforward task.)

We further illustrate that some preconceptions from ODEs need to be reviewed before undertaking this search. We start by depicting the phenomenon of hidden symmetries using the 2–D Burgers' equation. Thereafter, we indicate the route to find solutions of the Korteweg–de Vries equation which are invariant under translation of the independent variables and show that it is a non trivial task to obtain the original symmetries (even when we know what they are!). Finally we show why a more systematic approach needs further development.

2 Hidden symmetries

The two-dimensional Burgers' equation is

$$u_t + uu_z = u_{xx} + u_{zz}, \qquad u = u(x, z, t). \tag{2.1}$$

The Lie group generators of (2.1) are

$$\begin{aligned} U_1 &= \frac{\partial}{\partial x} \\ U_2 &= \frac{\partial}{\partial z} \\ U_3 &= \frac{\partial}{\partial t} \\ U_4 &= t\frac{\partial}{\partial z} + \frac{\partial}{\partial u} \\ U_5 &= 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + z\frac{\partial}{\partial z} - u\frac{\partial}{\partial u}. \end{aligned} \tag{2.2}$$

We reduce the number of variables of the two-dimensional Burgers' equation by the transformation

$$u = w(t, \rho), \qquad \rho = z - \frac{x}{a} \tag{2.3}$$

found from the symmetry $U_a = aU_1 + U_2$. The reduced PDE is the one-dimensional Burgers' equation

$$w_t + ww_\rho = \frac{1 + a^2}{a^2} w_{\rho\rho}. \tag{2.4}$$

The symmetries of (2.4) are

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial\rho}\\
X_2 &= \frac{\partial}{\partial t}\\
X_3 &= t\frac{\partial}{\partial\rho}+\frac{\partial}{\partial w}\\
X_4 &= 2t\frac{\partial}{\partial t}+\rho\frac{\partial}{\partial\rho}-w\frac{\partial}{\partial w}\\
X_5 &= t^2\frac{\partial}{\partial t}+t\rho\frac{\partial}{\partial\rho}+(\rho-tw)\frac{\partial}{\partial w}.
\end{aligned} \tag{2.5}$$

The symmetries $X_j, j = 1, \dots, 4$ are inherited symmetries of the two-dimensional Burgers' equation but X_5 is a Type II hidden symmetry. We have previously shown [3] that X_5 (along with X_1–X_4) is inherited from the equation

$$u_t + uu_z = \frac{1+a^2}{a^2}u_{zz}, \qquad u = u(x,z,t) \tag{2.6}$$

which reduces to (2.4) via an obvious transformation.

3 The Korteweg–de Vries equation

The Korteweg–de Vries (KdV) equation

$$u_t + u_{xxx} + uu_x = 0 \tag{3.1}$$

has the four Lie point symmetries [8]

$$X_1 = \partial_x \tag{3.2}$$
$$X_2 = \partial_t \tag{3.3}$$
$$X_3 = t\partial_x + \partial_u \tag{3.4}$$
$$X_4 = x\partial_x + 3t\partial_t - 2u\partial_u. \tag{3.5}$$

In order to obtain the so–called travelling wave solutions, we take the following combination of symmetry operators:

$$V_1 = X_2 + cX_1 = \partial_t + c\partial_x. \tag{3.6}$$

This combination defines

$$y = x - ct \qquad v = u \tag{3.7}$$

as reduction variables for the PDE. The reduced equation is

$$v_{yyy} + vv_y - cv_y = 0 \tag{3.8}$$

	V_1	X_2	X_3	X_4
V_1	0	0	$(V_1 - X_2)/c$	V_1
X_2	0	0	$(V_1 - X_2)/c$	$3X_2$
X_3	$-(V_1 - X_2)/c$	$-(V_1 - X_2)/c$	0	$-2X_3$
X_4	$-V_1$	$-3X_2$	$2X_3$	0

Table 1: Commutation relations for symmetries of the KdV equation

which has

$$
\begin{aligned}
G_1 &= \partial_y && (3.9)\\
G_2 &= y\partial_y + 2(c-v)\partial_v && (3.10)
\end{aligned}
$$

as the only symmetries and clearly, no Type II hidden symmetries exist in this case. In order to find solutions to the KdV equation invariant under scalings of the independent variables, we need to solve (3.8) [11].

However, that is not our goal here. We wish to determine the fate of the symmetries (3.3)–(3.6) (where (3.6) has taken the place of (3.2)) and the origins of (3.9)–(3.10). If we apply the original symmetry operators (3.3)–(3.6) to the transformation (3.7) we obtain

$$
\begin{aligned}
V_1 &= \frac{\partial y}{\partial t}\partial_y + \frac{\partial v}{\partial t}\partial_v + c\left(\frac{\partial y}{\partial x}\partial_y + \frac{\partial v}{\partial x}\partial_v\right)\\
&= 0 && (3.11)\\
X_2 &\longrightarrow \frac{\partial y}{\partial t}\partial_y + \frac{\partial v}{\partial t}\partial_v\\
&= -c\partial_y && (3.12)\\
X_3 &\longrightarrow t\left(\frac{\partial y}{\partial x}\partial_y + \frac{\partial v}{\partial x}\partial_v\right) + \frac{\partial y}{\partial u}\partial_y + \frac{\partial v}{\partial u}\partial_v\\
&= t\partial_y + \partial_v && (3.13)\\
X_4 &\longrightarrow x\left(\frac{\partial y}{\partial x}\partial_y + \frac{\partial v}{\partial x}\partial_v\right) + 3t\left(\frac{\partial y}{\partial t}\partial_y + \frac{\partial v}{\partial t}\partial_v\right)\\
&\quad - 2u\left(\frac{\partial y}{\partial u}\partial_y + \frac{\partial v}{\partial u}\partial_v\right)\\
&= x\partial_y - 3ct\partial_y - 2u\partial_v\\
&= (y-2ct)\partial_y - 2v\partial_v && (3.14)
\end{aligned}
$$

then the origins of the symmetries of (3.8) are easily seen as

$$
\begin{aligned}
G_1 &\longleftarrow -X_2/c && (3.15)\\
G_2 &\longleftarrow 2cX_3 + X_4. && (3.16)
\end{aligned}
$$

If we look at Table 1 though, the fate of the original symmetries is surprising. We will return to this later.

In trying to determine the origin of hidden symmetries, we need to investigate the form of the resulting symmetries in the original variables. Once we have obtained this information, we are then in a position to determine the form of the equation that produced them. We utilise the same approach here to see if, 'working backwards' we can still obtain the results (3.15)–(3.16), or indeed (3.3)–(3.6).

Firstly, since the variables are defined as in (3.7), we must start off with a symmetry of the form

$$U_0 = \partial_t + c\partial_x \tag{3.17}$$

(which is V_1). However, our first bit of ambiguity immediately imposes itself as we could also have

$$U_0^1 = f(x,t,u)(\partial_t + c\partial_x) \tag{3.18}$$

but we will adopt Occam's razor and settle for (3.17).

In order to determine the origins of the symmetries (3.9)–(3.10) we start with

$$U = \xi\partial_x + \tau\partial_t + \eta\partial_u \tag{3.19}$$

which becomes $(\xi - c\tau)\partial_y + \eta\partial_v$ in the new variables. A comparison with G_1 ensures that

$$\xi - c\tau = 1 \qquad \eta = 0 \tag{3.20}$$

and so, in the original variables we must have

$$U_1 = (1 + c\tau^1)\partial_x + \tau^1\partial_t \tag{3.21}$$

Since the Lie brackets relations must close (as the symmetries form a Lie algebra) we require

$$\begin{aligned} [U_0, U_1] &= c(c\tau_x^1 + \tau_t^1)\partial_x + (c\tau_x^1 + \tau_t^1)\partial_t \\ &= k_0U_0 + k_1U_1 \\ &= (ck_0 + k_1(1 + c\tau^1))\partial_x + (k_0 + k_1\tau^1)\partial_t \end{aligned} \tag{3.22}$$

which implies that

$$\tau^1 = k_0t + q(x - ct, u) \tag{3.23}$$

and

$$k_1 = 0. \tag{3.24}$$

Thus we have

$$U_0 = c\partial_x + \partial_t \tag{3.25}$$

$$U_1 = (1 + c(k_0t + q(x - ct, u)))\partial_x + (k_0t + q(x - ct, u))\partial_t \tag{3.26}$$

with

$$[U_0, U_1] = k_0U_0. \tag{3.27}$$

The result (3.24) is not surprising from our expectations from reductions of ODEs as (3.27) is necessary for U_1 to be a symmetry of the new equation.

In the case of G_2 the coefficient functions in (3.19) must take the form

$$\xi - c\tau = y = x - ct \qquad \eta = 2(c - v) = 2(c - u) \tag{3.28}$$

and so

$$U_2 = (x - ct + c\tau^2)\partial_x + \tau^2\partial_t + 2(c - u)\partial_u. \tag{3.29}$$

Now requiring

$$[U_0, U_2] = k_3 U_0 \tag{3.30}$$

implies that

$$\tau^2 = k_3 t + r(x - ct, u). \tag{3.31}$$

Thus far, we have obtained

$$\begin{aligned} U_0 &= c\partial_x + \partial_t & (3.32)\\ U_1 &= (1 + c(k_0 t + q(x - ct, u)))\partial_x + (k_0 t + q(x - ct, u))\partial_t & (3.33)\\ U_2 &= (x - ct - c(k_3 t + r(x - ct, u)))\partial_x + (k_3 t + r(x - ct, u))\partial_t + 2(c - u)\partial_u & (3.34) \end{aligned}$$

which is far more general than the symmetries (3.3)–(3.6) even taking the simplified form (3.17) over the more general form (3.18). Thus it is clear that the symmetries G_1 and G_2 could have originated from more general symmetries than those obtainable from the KdV equation.

4 A Cautionary example

Let us take the model equation

$$u_{xxx} + u(u_t + cu_x) + u_x u_{xx} = 0 \tag{4.1}$$

with symmetries

$$\begin{aligned} Y_1 &= \partial_t & (4.2)\\ Y_2 &= \partial_x & (4.3)\\ Y_3 &= 3t\partial_t + (x + 2ct)\partial_x. & (4.4) \end{aligned}$$

Reduction via

$$y = x - ct \qquad w = u$$

yields

$$w_{yyy} + w_y w_{yy} = 0 \tag{4.5}$$

with

$$\begin{aligned} Z_1 &= \partial_y & (4.6)\\ Z_2 &= y\partial_y & (4.7)\\ Z_3 &= \partial_w & (4.8)\end{aligned}$$

as symmetries. In the above, Z_3 is a Type II hidden symmetry.

Here, instead of proceeding via the complicated route in the previous section to obtain the origins of the $Z_i, i = 1, \ldots, 3$ symmetries we adopt a more 'systematic' approach. We observe that the only nonzero Lie bracket relationship is

$$[Z_1, Z_2] = Z_1 \tag{4.9}$$

which is the Lie algebra $A_{2,1} \oplus A_1$ (We use the classification scheme of Mubarakzyanov [9, 10] as explained by Patera *et al* [12, 13].). An obvious choice for the 4–D Lie algebra from whence this algebra arose is $A_{2,1} \oplus 2A_1$ in three variables.

If we examine the tables in [14] we have the four choices

$$\partial_y, \quad y\partial_y, \quad \partial_w, \quad \partial_t \tag{4.10}$$

$$\partial_y, \quad y\partial_y + t\partial_t, \quad \partial_w, \quad t\partial_y \tag{4.11}$$

$$\partial_y, \quad y\partial_y + \phi(t)\partial_w, \quad \partial_w, \quad t\partial_w \tag{4.12}$$

$$\partial_y, \quad y\partial_y + w\partial_w + t\partial_t, \quad w\partial_y, \quad t\partial_y. \tag{4.13}$$

However, none of (4.11)–(4.13) work, and (4.10) corresponds to the obvious assumption of a second independent variable in (4.5).

The reason for this behaviour is that, unlike in ODEs, where

$$[Z_1, Z_2] = 0 \tag{4.14}$$

implies that reduction via either Z_1 or Z_2 means that the other symmetry (extended) is a symmetry of the reduced in equation, the situation for PDEs is different [7]. If we take the simple example of

$$\begin{aligned} W_1 &= \partial_x & (4.15)\\ W_2 &= f(t,u)\partial_x & (4.16)\end{aligned}$$

with

$$[W_1, W_2] = 0 \tag{4.17}$$

we see that W_1 defines

$$p = t \qquad q = u \tag{4.18}$$

as reduction variables. This means that

$$Z_2 \longrightarrow f(p,q)\partial_x \tag{4.19}$$

which will have no relevance for the reduced equation! Thus due care must be taken in examining the Lie bracket relationships of symmetries of PDEs. The information obtained from them is not sufficient to draw conclusions about the fate of the symmetries of PDEs (This is also true for the 2–D non-commuting Lie algebra.). The full form of the symmetry must be analysed.

5 Discussion

While the origin and fate of symmetries of ODEs has been well established, the same cannot be said for PDEs. In section 3 we showed that the origin of symmetries of reductions of the KdV equation were far more general that expected. In section 4 it was indicated that why it was difficult to determine the origin of symmetries (hidden or otherwise) of PDEs. A 'systematic' approach, as followed in that section was not able to provide us with any useful results (beyond the obvious). Indeed, this does caution us in the approach followed for the KdV equation. There we assumed that the Lie bracket relations could provide us with useful usable information. Even there though, the results were far more general and only knowledge of the symmetries of the KdV equation allows us to recover them. Due caution must be taken when working with symmetries of PDEs.

Acknowledgments. *KSG thanks the University of KwaZulu–Natal and the National Research Foundation of South African for ongoing support.*

References

[1] Abraham-Shrauner B,Type II hidden symmetries of some partial differential equations, *1005th AMS Meeting*, Newark, Delaware (2005), 22, 37.

[2] Abraham-Shrauner B, Govinder K S and Arrigo D, Type II hidden symmetries of the linear 2-D and 3-D wave equations, *J. Phys. A: Math. Gen.* **39** (2006), 5739–5747.

[3] Abraham-Shrauner B and Govinder K S, Provenance of Type II hidden symmetries from nonlinear partial differential equations, *J. Nonlin. Math. Phys.* **13** 4 (2006), 612–622.

[4] Abraham-Shrauner B and Govinder K S, Master Partial Differential Equations for a Type II Hidden Symmetry, *J Math Anal Appl* (2008), http://dx.doi.org/10.1016/j.jmaa.2008.01.074

[5] Clark P A, Nonclassical symmetry reductions of the Boussinesq equation, *Chaos, Solitons and Fractals* **5** (1995), 2261–2301.

[6] Coggeshall S V, Abraham-Shrauner B and Knapp C, Hidden symmetries of partial differential equations, *Proceedings of IMACS World Congress*, Ed. Ames W F **1** (1994), 102–107.

[7] Govinder K S, Lie Subalgebras, Reduction of Order and Group Invariant Solutions, *J. Math. Anal. Appl.* **258** (2001), 720–732.

[8] Head A K, LIE, a PC program for Lie analysis of differential equations, *Comp. Phys. Comm.* **77** (1993), 241–248; *Comp. Phys. Comm.* **96** (1996), 311–313.

[9] Mubarakzyanov G M, On solvable Lie algebras, *Isz. Vysshikh Uchebn Zavendenii Mat.* **32** (1963), 114–123.

[10] Mubarakzyanov G M, Classification of real structures of five–dimensional Lie algebras, *Isz. Vysshikh Uchebn Zavendenii Mat.* **34** (1963), 99–106.

[11] Olver P J, Applications of Lie groups to differential equations (Springer, 1986).

[12] Patera J, Sharp R T, Winternitz P and Zassenhaus H, Subgroups of the Poincaré group and the invariants, *J. Math. Phys.* **17** (1976), 977–985.

[13] Patera J, Sharp R T, Winternitz P and Zassenhaus H, Invariants of real low dimension Lie algebras, *J. Math. Phys.* **17** (1976), 986–994.

[14] Popovych R O, Boyko V M, Nesterrenko M O and Lutfullin M W, Realization of real low-dimensional Lie algebras, *J. Phys. A: Math. Gen.* **36** (2003), 7337–7360.