

Complete Invariant Characterization of Scalar Linear (1+1) Parabolic Equations

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Abstract

We obtain a complete invariant characterization of scalar linear (1+1) parabolic equations under equivalence transformations for all the four canonical forms. Firstly semi-invariants under changes of independent and dependent variables and the construction of the relevant transformations that relate the two parabolic equations are given. Then necessary and sufficient conditions for a (1+1) parabolic equation, in terms of the coefficients of the equation, to be reducible via local equivalence transformations to the one-dimensional classical heat equation and the Lie canonical equation $u_t = u_{xx} + Au/x^2$, A a nonzero constant, are presented. These invariant conditions provide practical criteria for reduction to the respective canonical equations. Also the construction of the transformation formulas that do the reductions are provided. We further show how one can transform a (1+1) parabolic equation to the third and fourth Lie canonical forms thus providing invariant criteria for a parabolic equation to have two and one nontrivial symmetries as well. Ample examples are given to illustrate the various results.

1 Introduction

The scalar linear (1+1) parabolic partial differential equation (PDE) of one space variable and one time variable

$$u_t = a(t, x)u_{xx} + b(t, x)u_x + c(t, x)u, \quad (1.1)$$

where a , b and c are continuous functions in t and x , arises in several applications and is an essential part of fundamental courses on PDEs. Indeed, the well-known one-dimensional Fokker-Planck (FP) PDE (see, e.g., the books Gardiner [9] and Risken [23])

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}[A(t, x)u] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[B(t, x)u], \quad (1.2)$$

in which u is the probability density and A and B are the coefficients of drift and diffusion, is contained in (1.1). This equation (1.2) models many phenomena (see, e.g., [1, 2, 8, 11,

16, 25]). The FP PDE is also called a forward Kolmogorov equation and in this sense describes the evolution of the transition probability density for diffusion processes. The Kolmogorov [17] equation is

$$u_t + \alpha(t, x)u_{xx} + \beta(t, x)u_x = 0, \quad \alpha(t, x) > 0, \quad (1.3)$$

where u is the transition probability density function of a Markov process.

The Black-Scholes [3] as well as the bond-pricing equations (see, e.g. [27]) are also subsumed in the family (1.1).

Algebraic properties of the FP equation (1.2) were considered before, for example, in [24] and [26]. We point out that sequences of drifts for the FP equation were derived in Bluman [6]. Notwithstanding, necessary and sufficient conditions for mapping by means of point transformations of the Kolmogorov equation (1.3) into the ‘backward’ heat equation $\bar{u}_{\bar{t}} + \bar{u}_{\bar{x}\bar{x}} = 0$ were obtained in Bluman [4]. These conditions were shown to be equivalent to invariance of equation (1.3) under a six-dimensional Lie algebra of point symmetries in addition to the trivial infinite dimensional algebra of superposition operators. Moreover, Bluman and Shtelen [7] provided an extension of the Kolmogorov problem of determining classes of Kolmogorov equations which are transformable to the backward heat equation via nonlocal transformations by embedding (1.3) in an auxiliary system of PDEs. The symmetry transformation approach was employed in Pooe et al [22] to reduce one-factor bond pricing parabolic equations to the classical heat equation. Fundamental solutions of two zero-coupon bond-pricing PDEs were derived. Gazizov and Ibragimov [10], and Goard [12] also used the Lie group approach for the Black-Scholes and bond pricing equations respectively, that admit six nontrivial symmetry operators. In [10], reduction formulas to the heat equation were constructed.

Lie [18] gave the complete group classification of the parabolic PDE (1.1). He provided all the canonical forms of the PDE (1.1) for which (1.1) admits nontrivial point symmetry algebras of dimensions 1, 2, 4 and 6 (apart from the infinite dimensional algebra of trivial point symmetries of the superposition operators).

The parabolic equation (1.1) was further studied by Ovsianikov [20] by utilising the reduction to the fourth Lie canonical form which corresponds to $a = 1$ and $b = 0$ in (1.1). Bluman [5], inter alia, developed a mapping algorithm based on analyzing the symmetry generators of the parabolic PDE (1.1). The equation (1.1) was first reduced to the fourth Lie canonical form for this purpose.

We now turn to the notion of semi-invariants and invariants of the PDE (1.1). Equivalence transformations of the equation (1.1) are transformations that map the family (1.1) into itself. Ibragimov [13] obtained the semi-invariants of equation (1.1) under linear changes in the dependent variable only (a subgroup of the equivalence group). Semi-invariants of PDE (1.1) under transformations of just the independent variables were derived in Johnpillai and Mahomed [21]. These authors also found the joint invariant equation for the family (1.1) under both changes of the independent and dependent variables thereby providing necessary and sufficient conditions for equivalence of the family (1.1) under point transformations to the classical heat PDE. The results obtained were in terms of the coefficients of the equation (1.1) and provides practical criteria for reduction to the heat equation. Further work on joint invariant equations for PDE (1.1) and reducibility to the heat and second Lie canonical form $u_t = u_{xx} + Au/x^2$ were reported briefly in Mahomed and Pooe [19]. Herein, a refinement of the invariant condition for

reduction to the heat PDE was given. However, some unfortunate errors in the transformation formulas have crept into [19] which we remedy here in addition to presenting better statements of the results on reducibility to the heat and second Lie canonical forms. Furthermore we extend the results of [21] and [19] as far as semi-invariants and reducibility criteria for the third and fourth Lie canonical forms. As a matter of fact we present here a complete characterization of the parabolic equation (1.1) in terms of the invariants and its reduction to the four Lie canonical forms. We have also, together with co-workers, done some work for (1+1) linear hyperbolic equations (see [14, 15]).

The outline of this work is as follows. In the next section we present results on the transformation formulas that relate two parabolic PDEs (1.1) which have the same values of the semi-invariants. This provides a complete picture on semi-invariants for the family (1.1) and reductions. Then in Section 3 we provide necessary and sufficient conditions for equivalence of PDE (1.1) to the classical heat PDE in terms of the coefficients of (1.1). The transformation formulas are given too. Section 4 deals with invariant criteria for reduction to the second Lie canonical form and the associated transformations. Section 5 is devoted to the third and fourth Lie canonical forms and invariant conditions for reducibility to these forms. Concluding remarks are made in Section 6.

2 Semi-invariants and related transformations

In this section we present results on the semi-invariants of the subgroup of the group of equivalence transformations that map the parabolic PDE family (1.1) into itself.

It is indeed well-known from Lie [18] that the equivalence transformations of the parabolic PDE (1.1) is an infinite group which comprises linear transformations of the dependent variable given by

$$\bar{u} = \sigma(t, x)u, \quad \sigma \neq 0 \quad (2.1)$$

and invertible transformations of the independent variables

$$\bar{t} = \phi(t), \quad \bar{x} = \psi(t, x), \quad \dot{\phi} \neq 0, \psi_x \neq 0, \quad (2.2)$$

where ϕ , ψ and σ are arbitrary functions, with the stated restrictions for invertibility, and \bar{u} is the new dependent variable. Two parabolic PDEs of the form (1.1) are equivalent to each other if one can be mapped to the other by appropriate combinations of the equivalence transformations (2.1) and (2.2).

We quickly review the situation when equation (1.1) is transformed into the same family via changes of both the independent and dependent variables (2.1) and (2.2). By means of the transformations (2.1) and (2.2), PDE (1.1) becomes (see, e.g. [21])

$$\bar{L}\bar{u} \equiv \bar{u}_{\bar{t}} - \bar{a}(\bar{t}, \bar{x})\bar{u}_{\bar{x}\bar{x}} - \bar{b}(\bar{t}, \bar{x})\bar{u}_{\bar{x}} - \bar{c}(\bar{t}, \bar{x})\bar{u} = 0, \quad (2.3)$$

where

$$\bar{L} = \frac{\partial}{\partial \bar{t}} - \bar{a}(\bar{t}, \bar{x})\frac{\partial^2}{\partial \bar{x}^2} - \bar{b}(\bar{t}, \bar{x})\frac{\partial}{\partial \bar{x}} - \bar{c}(\bar{t}, \bar{x}) \quad (2.4)$$

and the transformed parabolic PDE has coefficients

$$\bar{a} = \frac{a\psi_x^2}{\dot{\phi}},$$

$$\begin{aligned}\bar{b} &= \frac{\psi_x}{\phi} \left(b - 2a \frac{\sigma_x}{\sigma} + a \frac{\psi_{xx}}{\psi_x} - \frac{\psi_t}{\psi_x} \right), \\ \bar{c} &= \frac{1}{\phi} \left(c - a \frac{\sigma_{xx}}{\sigma} - b \frac{\sigma_x}{\sigma} + 2a \frac{\sigma_x^2}{\sigma^2} + \frac{\sigma_t}{\sigma} \right).\end{aligned}\quad (2.5)$$

We are now in a position to study the case when equation (1.1) is mapped into the same family by a subgroup of the equivalence transformations consisting just of changes in the independent variables (2.2). The second-order semi-invariants for $c \neq 0$,

$$g_1 = \frac{ac_x^2}{c^3}, \quad c \neq 0; \quad g_2 = \frac{2}{c^2} (c_t + ca_{xx} - 2cb_x - bc_x) + \frac{a_x^2 c_x}{c^2} - 2 \frac{a_t}{ac} + 2 \frac{ba_x}{ac} - \frac{a_x^2}{ac} \quad (2.6)$$

were determined in [21]. However, the transformations in terms of the coefficients of (1.1) and $\bar{L}u = 0$,

$$\dot{\phi} = \frac{c}{\bar{c}}, \quad \psi_x = \pm \left(\frac{\bar{a}\bar{c}}{a\bar{c}} \right)^{1/2}, \quad \psi_t = \pm b \left(\frac{\bar{a}\bar{c}}{a\bar{c}} \right)^{1/2} \pm a \left[\left(\frac{\bar{a}\bar{c}}{a\bar{c}} \right)^{1/2} \right]_x - \bar{b} \frac{c}{\bar{c}}. \quad (2.7)$$

were not attempted in [21]. These are easily derivable from (2.5) by setting $\sigma = 1$ since we are interested in changes in the independent variables here only.

Note that the semi-invariants (2.6) and the transformations on the independent variables (2.7) are valid for $c \neq 0$. Hence the case $c = 0$ was omitted in [21] (see p 11036 of this paper). This gap is not onerous to fill. In fact we have for $c = 0$ the second-order semi-invariants

$$\begin{aligned}a, \\ h = \frac{1}{2}a_t + \frac{1}{4}a_x^2 - \frac{1}{2}a_x b - \frac{1}{2}a_{xx} + ab_x.\end{aligned}\quad (2.8)$$

The allowable transformations determined from (2.5) with $\sigma = 1$, $c = \bar{c} = 0$ are no more than translations

$$\bar{t} = t + a_1, \quad \bar{x} = x + a_2, \quad (2.9)$$

where a_1 and a_2 are constants. So they are quite restrictive. The point to be made though is that we now have all the semi-invariants under changes of the independent variables (2.2). As a consequence of the foregoing, we have the following theorem.

Theorem 1. (a) The parabolic equation (1.1) for $c \neq 0$ is equivalent to the parabolic PDE $\bar{L}u = 0$ for $\bar{c} \neq 0$, under the transformations of the independent variables (2.2) for which ϕ and ψ satisfy the overdetermined system (2.7) if and only if $g_1 = \bar{g}_1$ and $g_2 = \bar{g}_2$, where g_1 and g_2 are given by (2.6) and

$$\bar{g}_1 = \frac{\bar{a}\bar{c}_x^2}{\bar{c}^3}, \quad \bar{c} \neq 0; \quad \bar{g}_2 = \frac{2}{\bar{c}^2} (\bar{c}_t + \bar{c}\bar{a}_{\bar{x}\bar{x}} - 2\bar{c}\bar{b}_{\bar{x}} - \bar{b}\bar{c}_{\bar{x}}) + \frac{\bar{a}_x^2 \bar{c}_x}{\bar{c}^2} - 2 \frac{\bar{a}_t}{\bar{a}\bar{c}} + 2 \frac{\bar{b}\bar{a}_x}{\bar{a}\bar{c}} - \frac{\bar{a}_x^2}{\bar{a}\bar{c}}. \quad (2.10)$$

(b) The parabolic equation (1.1) for $c = 0$ is equivalent to the transformed PDE $\bar{L}u = 0$ for $\bar{c} = 0$ via the translations (2.9) if and only if $a = \bar{a}$ and $h = \bar{h}$, where h is given in (2.8) and

$$\bar{h} = \frac{1}{2}\bar{a}_t + \frac{1}{4}\bar{a}_x^2 - \frac{1}{2}\bar{a}_x \bar{b} - \frac{1}{2}\bar{a}_{\bar{x}\bar{x}} + \bar{a}\bar{b}_{\bar{x}}. \quad (2.11)$$

We now illustrate the use of Theorem 1 by two examples.

Example 1. The parabolic equation $u_t = tu_{xx} - xu_x/2t + u$ has $c \neq 0$ and $g_1 = 0 = g_2$. This PDE is equivalent to $u_{\bar{t}} = u_{\bar{x}\bar{x}} + u$ which has also $\bar{g}_1 = 0 = \bar{g}_2$. A transformation that does the reduction is easily obtained from system (2.7) and is $\bar{t} = t$, $\bar{x} = t^{-1/2}x$. One merely substitutes in the coefficients of the respective PDEs into equations (2.7) in order to construct it.

Example 2. The PDE $u_t = (t + A)u_{xx} + (x + B)^2u_x$, where A and B are constants, has $c = 0$ and $a = t + A$, $h = 1/2 + 2(t + A)(x + B)$. The target equation $u_{\bar{t}} = \bar{t}u_{\bar{x}\bar{x}} + \bar{x}^2u_{\bar{x}}$ has semi-invariants $\bar{a} = \bar{t}$, $\bar{h} = 1/2 + 2\bar{t}\bar{x}$ which respectively equal a and h of the original equation. Thus the transformation that does the reduction is $\bar{t} = t + A$, $\bar{x} = x + B$. Of course this could have easily been guessed without knowledge of the semi-invariants! However, the explanation is that the semi-invariants have the same values.

We next turn to the case for which the parabolic PDE (1.1) is transformed into itself via the linear transformations of the dependent variables (2.1) only. The semi-invariants were found in [13]. We can write them more conveniently as

$$a, \quad k = \frac{\partial}{\partial x} \left(c - a \left(\frac{b}{2a} \right)_x - \frac{b^2}{4a} \right) + \left(\frac{b}{2a} \right)_t \quad (2.12)$$

in which k is related to the K of [13] by means of the equation $K = -2a^2k$. However, the determination of σ in (2.1) for the transformation in u that relates the two parabolic PDEs (1.1) and $\bar{L}\bar{u} = 0$, with $\bar{t} = t$ and $\bar{x} = x$ in \bar{L} , was not given in [13]. The σ in (2.1) in terms of the coefficients is obtained by solving the system of two equations

$$\frac{\sigma_x}{\sigma} = \frac{b}{2a} - \frac{\bar{b}}{2\bar{a}}, \quad \frac{\sigma_t}{\sigma} = \bar{c} - \bar{a} \left(\frac{\bar{b}}{2\bar{a}} \right)_x - \frac{\bar{b}^2}{4\bar{a}} - c + a \left(\frac{b}{2a} \right)_x + \frac{b^2}{4a}. \quad (2.13)$$

These are deduced from equation (2.5) by setting $\phi = t$ and $\psi = x$. We state the following theorem which captures the results above.

Theorem 2. The parabolic equation (1.1) is equivalent to the parabolic PDE $\bar{L}\bar{u} = 0$, with $\bar{t} = t$ and $\bar{x} = x$ in \bar{L} , under the transformations of the dependent variables (2.1) in which σ satisfies the overdetermined system (2.13) if and only if $a = \bar{a}$ and $k = \bar{k}$, where k is given in equation (2.12) and \bar{k} is

$$k = \frac{\partial}{\partial x} \left(\bar{c} - \bar{a} \left(\frac{\bar{b}}{2\bar{a}} \right)_x - \frac{\bar{b}^2}{4\bar{a}} \right) + \left(\frac{\bar{b}}{2\bar{a}} \right)_t. \quad (2.14)$$

We present an example to illustrate Theorem 2.

Example 3. The parabolic equation $u_t = u_{xx} + xu_x + x^2u/4$ has $a = 1$ and $k = 0$. The equation $\bar{u}_t = \bar{u}_{xx} + \bar{u}$ has $\bar{a} = 1$ and $\bar{k} = 0$ as well. Hence, one can transform the original equation to the target PDE via changes of the dependent variable. This linear transformation is constructable from equations (2.13) and is $\bar{u} = u \exp(3t/2 + x^2/4)$.

3 Reduction to the classical heat PDE

We provide the necessary and sufficient conditions for the parabolic equation (1.1) to be reducible to the classical heat equation under the equivalence group comprising the transformations (2.1) and (2.2). Previously in Johnpillai and Mahomed [21] the practical criterion, in terms of the coefficients of the equation, of reducing the parabolic PDE (1.1) to the classical heat equation was given. A compact form of this criterion was mentioned briefly in Mahomed and Poore [19]. The transformation formulas in [21] though were for the autonomous case as a nontrivial symmetry of the heat equation was utilised to show the existence of the transformations once the invariance condition was satisfied. In [19] there were unfortunate misprints in the formula (10) upon which Theorem 1 of that paper was based. Here we present refined conditions which improve both the results of [21] and [19] which of course is largely inaccessible being the proceedings of a conference.

The canonical forms of the linear parabolic PDE (1.1) in Lie's [18] classification are

$$\begin{aligned}
 u_t &= u_{xx}, \\
 u_t &= u_{xx} + \frac{A}{x^2}u, \quad A \neq 0 \\
 u_t &= u_{xx} + c(x)u, \quad c \neq 0, A/x^2, \\
 u_t &= u_{xx} + c(t, x)u, \quad c \neq 0, A/x^2.
 \end{aligned} \tag{3.1}$$

The heat equation has six nontrivial point symmetries in addition to the infinite number of trivial superposition symmetries. The second canonical form in (3.1) has in general four nontrivial symmetries and the third has two. The last PDE in (3.1) has in general one nontrivial symmetry. The precise conditions under which the third and last canonical PDEs possess more point symmetries are given by Theorems 3 and 4 below. Some of these are stated as restrictions in (3.1). The others are given as Examples 5 and 7 later.

The point transformations that reduce the parabolic equation (1.1) to the Lie canonical form

$$\bar{u}_{\bar{t}} = \bar{u}_{\bar{x}\bar{x}} + \bar{c}(\bar{t}, \bar{x})\bar{u} \tag{3.2}$$

is given by

$$\begin{aligned}
 \bar{t} &= \phi(t), \\
 \bar{x} &= \pm \int [\dot{\phi}a(t, x)^{-1}]^{1/2} dx + \beta(t), \\
 \bar{u} &= \nu(t)|a(t, x)|^{-1/4}u \exp \left\{ \int \frac{b(t, x)}{2a(t, x)} dx - \frac{1}{8} \frac{\ddot{\phi}}{\dot{\phi}} \left(\int \frac{dx}{a(t, x)^{1/2}} \right)^2 \right. \\
 &\quad \left. - \frac{1}{2} \int \frac{1}{a(t, x)^{1/2}} \partial_t \left(\int \frac{dx}{a(t, x)^{1/2}} \right) dx \mp \frac{1}{2} \frac{\dot{\beta}}{\dot{\phi}^{1/2}} \int \frac{dx}{a(t, x)^{1/2}} \right\},
 \end{aligned} \tag{3.3}$$

where $\dot{\phi}$ and a have the same sign, and ϕ , β and ν satisfy

$$\dot{\phi}\bar{c} = J + \partial_t \int \frac{b(t, x)}{2a(t, x)} dx - \frac{1}{2} \int \frac{1}{a(t, x)^{1/2}} \partial_t^2 \left(\int \frac{dx}{a(t, x)^{1/2}} \right) dx$$

$$+f(t) \left(\int \frac{dx}{a(t,x)^{1/2}} \right)^2 + g(t) \left(\int \frac{dx}{a(t,x)^{1/2}} \right) + h(t), \quad (3.4)$$

in which

$$J = c - \frac{b_x}{2} + \frac{ba_x}{2a} + \frac{a_{xx}}{4} - \frac{3}{16} \frac{a_x^2}{a} - \frac{a_t}{2a} - \frac{b^2}{4a} \quad (3.5)$$

and

$$\begin{aligned} f(t) &= \frac{1}{16} \frac{\ddot{\phi}^2}{\dot{\phi}^2} - \frac{1}{8} \left(\frac{\ddot{\phi}}{\dot{\phi}} \right)_t, \\ g(t) &= \pm \frac{1}{4} \frac{\ddot{\phi}}{\dot{\phi}} \frac{\dot{\beta}}{\dot{\phi}^{1/2}} \mp \frac{1}{2} \left(\frac{\dot{\beta}}{\dot{\phi}^{1/2}} \right)_t, \\ h(t) &= \frac{1}{4} \frac{\ddot{\phi}}{\dot{\phi}} + \frac{1}{4} \frac{\dot{\beta}^2}{\dot{\phi}} + \frac{\dot{\nu}}{\nu}. \end{aligned} \quad (3.6)$$

The transformations (3.3) and conditions (3.4) to (3.6) can be deduced by a straightforward albeit tedious computation. This is done by invoking relations (2.5) and equations (3.2).

The invariant condition for the reduction of a parabolic equation (1.1) to the heat PDE was determined in [21] by projecting the generator

$$X = p(t) \frac{\partial}{\partial t} + q(t,x) \frac{\partial}{\partial x} + \mu \frac{\partial}{\partial a} + \nu \frac{\partial}{\partial b} + \dots \quad (3.7)$$

into the space of semi-invariants $a, a_t, a_x, a_{tt}, a_{tx}, a_{xx}$ and K as

$$X = X(a) \frac{\partial}{\partial a} + X(a_t) \frac{\partial}{\partial a_t} + \dots, \quad (3.8)$$

that is

$$X = \mu \frac{\partial}{\partial a} + \mu_t \frac{\partial}{\partial a_t} + \mu_x \frac{\partial}{\partial a_x} + \mu_{tt} \frac{\partial}{\partial a_{tt}} + \mu_{tx} \frac{\partial}{\partial a_{tx}} + \mu_{xx} \frac{\partial}{\partial a_{xx}} + \Gamma \frac{\partial}{\partial K}, \quad (3.9)$$

where

$$\begin{aligned} \mu &= 2aq_x - apt, \\ \mu_t &= (-2a_t)p_t + (-a)p_{tt} + (-a_x)q_t + (2a_t)q_x + (2a)q_{tx}, \\ \mu_x &= (-a_x)p_t + (a_x)q_x + (2a)q_{xx}, \\ \mu_{tt} &= (-3a_{tt})p_t + (-3a_t)p_{tt} + (-a)p_{ttt} + (-2a_{tx})q_t + (2a_{tt})q_x \\ &\quad + (-a_x)q_{tt} + (4a_t)q_{tx} + (2a)q_{ttx}, \\ \mu_{tx} &= (-2a_{tx})p_t + (-a_x)p_{tt} + (-a_{xx})q_t + (a_{tx})q_x \\ &\quad + (a_x)q_{tx} + (2a_t)q_{xx} + (2a)q_{txx}, \\ \mu_{xx} &= (-a_{xx})p_t + (3a_x)q_{xx} + (2a)q_{xxx}, \\ \Gamma &= (-3K)p_t + (aa_{xx} - a_t - a_x^2)q_t + (3K)q_x + aq_{tt} + (aa_x)q_{tx} \\ &\quad + (-2a^2)q_{txx} + (a^2a_x)q_{xxx} + a^3q_{xxxx}. \end{aligned} \quad (3.10)$$

The joint invariant equation

$$\begin{aligned} \lambda \equiv & 4a(2aK_{xx} - 5a_xK_x) - 12K(aa_{xx} - 2a_x^2) + a_x(4aa_{tt} - 9a_x^4) - 12a_t a_x(a_t + 2a_x^2) \\ & + 4a(3a_t + 6a_x^2 - 5aa_{xx})a_{tx} + 2aa_x(16a_t a_{xx} - 12aa_{xx}^2 + 15a_x^2 a_{xx}) - 4a^2 a_{ttx} \\ & - 12a^2 a_x a_{txx} - 4a^2 a_{xxx}(2a_t - 4aa_{xx} + 3a_x^2) + 8a^3 a_{txxx} - 4a^4 a_{xxxx} = 0. \end{aligned} \quad (3.11)$$

was found in [21] by extending the operator X in (3.9) thrice and then solving for its invariants. What transpired was the invariant equation $\lambda = 0$ above.

We re-write (3.11) compactly. Moreover, we provide the general transformations that do the reduction of the parabolic equation (1.1) to the classical heat PDE. We utilise the formulas (3.4) to (3.6) in order to achieve this. Therefore, we have the following theorem.

Theorem 3. The following are equivalent statements:

- (a) the scalar linear (1+1) parabolic PDE (1.1) has six nontrivial point symmetries in addition to the infinite number of superposition symmetries;
- (b) the coefficients of the parabolic equation (1.1) satisfies the invariant equation

$$2L_x + 2M_x - N_x = 0, \quad (3.12)$$

where

$$L = a^{1/2}[a^{1/2}J_x]_x, \quad M = a^{1/2}[a^{1/2}\partial_t(b/2a)]_x, \quad N = a^{1/2}\partial_t^2(1/a^{1/2}), \quad (3.13)$$

and J is given by equation (3.5);

- (c) the linear parabolic equation (1.1) is reducible to the classical heat PDE $\bar{u}_{\bar{t}} = \bar{u}_{\bar{x}\bar{x}}$ via the transformations (3.3) for which ϕ , β and ν are constructed from equations (3.6) with the functions f , g and h constrained by the relation

$$J + \partial_t \int \frac{b}{2a} dx - \frac{1}{2} \int \frac{1}{a^{1/2}} \partial_t^2 \left(\int \frac{dx}{a^{1/2}} \right) dx + f(t) \left(\int \frac{dx}{a^{1/2}} \right)^2 + g(t) \int \frac{dx}{a^{1/2}} + h(t) = 0. \quad (3.14)$$

The following examples illustrates its utility.

Example 4. The constant coefficient parabolic PDE $u_t = au_{xx} + bu_x + cu$ has $J = c - b^2/4a$ which is constant as the coefficients are constant in the PDE. The invariant condition of Theorem 3 is thus satisfied and we get reduction to the heat equation. The relation (3.14) then gives $f = 0$, $g = 0$ and $h = -J$. Two cases arise depending on a positive or negative. For $a > 0$ we get the transformation $\bar{t} = t$, $\bar{x} = a^{-1/2}x$ and $\bar{u} = u \exp[bx/2a + (b^2/4a - c)t]$ and for $a < 0$ we have that $\bar{x} = (-a)^{-1/2}x$ with the remaining transformations in t , $\bar{t} = -t$, and u the same as for $a > 0$.

Example 5. The PDE $u_t = u_{xx} + c(t, x)u$ which is the fourth Lie canonical form has $J = c(t, x)$. Thus $L = c_{xx}$ and the invariant condition (3.12) easily gives that c must be at most quadratic in x of the form $c(t, x) = \alpha(t)x^2 + \beta(t)x + \gamma(t)$. The constraining relation (3.14) then results in $f = -\alpha(t)$, $g = -\beta(t)$ and $h = -\gamma(t)$ which means that the transformations depend upon the solution of a Riccati equation for the transformation in t . The others are then consequences of this.

There are several other examples given in [21] including equations of Finance. In the next section we study the reduction to the second Lie canonical form in a similar vein.

4 Reduction to the second Lie canonical form

We now state the conditions for the parabolic PDE (1.1) to be equivalent to the second Lie canonical equation (3.1b) which has four nontrivial point symmetries. The approach utilised is the same as that given in the Theorem 3 above.

Theorem 4. Necessary and sufficient conditions for the reduction of the scalar linear (1+1) parabolic PDE (1.1) to the second Lie canonical form $\bar{u}_{\bar{t}} = \bar{u}_{\bar{x}\bar{x}} + A\bar{u}/\bar{x}^2$, where A is a nonzero constant, by means of the transformations (3.3) with $\beta = 0$, are that the coefficients of equation (1.1) and ϕ and ν of the transformations (3.3) defined by $f(t)$ and $h(t)$ with $\beta = 0$ in equation (3.6) satisfy the invariant condition, provided that condition (3.12) does not hold,

$$\begin{aligned} & 20L_x + 20M_x - 10N_x + 10[a^{1/2}M_x]_x \int \frac{dx}{a^{1/2}} - 5[a^{1/2}N_x]_x \int \frac{dx}{a^{1/2}} \\ & \quad + 10[a^{1/2}L_x]_x \int \frac{dx}{a^{1/2}} + [a^{1/2}[a^{1/2}L_x]_x]_x \left(\int \frac{dx}{a^{1/2}} \right)^2 \\ & + [a^{1/2}[a^{1/2}M_x]_x]_x \left(\int \frac{dx}{a^{1/2}} \right)^2 - \frac{1}{2}[a^{1/2}[a^{1/2}N_x]_x]_x \left(\int \frac{dx}{a^{1/2}} \right)^2 = 0, \end{aligned} \quad (4.1)$$

in which L , M , N and J are as in (3.13) and (3.5), as well as the constraining relation, with $\beta = 0$ in $h(t)$,

$$A = \left(\int \frac{dx}{a^{1/2}} \right)^2 \left[J + \partial_t \int \frac{b}{2a} dx - \frac{1}{2} \int \frac{1}{a^{1/2}} \partial_t^2 \left(\int \frac{dx}{a^{1/2}} \right) dx + f(t) \left(\int \frac{dx}{a^{1/2}} \right)^2 + h(t) \right]. \quad (4.2)$$

The transformations used in the above conditions (4.1) and (4.2) are more restrictive than those for the heat equation equivalence in Theorem 3 as should be expected.

We look at the two examples.

Example 6. Consider the parabolic equation $u_t = u_{xx} + u_x + (1/4 + 1/x^2)u$. Here we have that $J = 1/x^2$ and thus the condition (4.1) holds. Moreover we have from (4.2) that $A = 1$, $f = 0 = h$. It is easy to then get the transformation $\bar{t} = t$, $\bar{x} = x$ and $\bar{u} = u \exp(x/2)$ that does the reduction to the second Lie form.

Example 7. A more interesting test for this Theorem 4 is again the fourth Lie canonical form and this time reduction to the second Lie form is sought. Here $J = c(t, x)$. The condition on c from (4.1) is $20c_{3x} + 10c_{4x}x + c_{5x}x^2 = 0$ which has solution $c = \alpha(t)x^{-1} + \beta(t)x^{-2} + \gamma(t)x^2 + \omega(t)x + \delta(t)$ for as yet arbitrary functions of α to δ . The constraining relation (4.2) then forces $\alpha = 0$, $\omega = 0$ and $\beta = A$ finally giving $c = Ax^{-2} + \gamma(t)x^2 + \delta(t)$ with obviously $A \neq 0$. The determination of the transformations again requires the solution of a Riccati equation.

These examples amply illustrate Theorem 4. We now proceed to reducibility criteria for the third and fourth Lie forms.

5 Reduction to the third and fourth Lie forms

The criteria for reducibility to the third and fourth Lie canonical forms follow by default from the previous results contained in Theorems 3 and 4. These are simply that the parabolic PDE (1.1) does not satisfy the conditions of Theorems 3 and 4 in order to be transformable to the third and fourth Lie forms which admit at most two nontrivial point symmetries. Precisely we have the following statement that characterizes reduction to the third Lie form.

Theorem 5. The scalar linear (1+1) parabolic equation (1.1) which does not satisfy the conditions of Theorems 3 and 4 is equivalent to the third Lie canonical form $\bar{u}_{\bar{t}} = \bar{u}_{\bar{x}\bar{x}} + \bar{c}(\bar{x})\bar{u}$, via the transformations

$$\begin{aligned}\bar{t} &= \epsilon t + a_1, \quad \epsilon = \pm 1, \quad a_1 = \text{const.} \\ \bar{x} &= \pm \int [\epsilon a(t, x)]^{-1/2} dx, \\ \bar{u} &= \nu_0 |a(t, x)|^{-1/4} u \exp \left\{ \int \frac{b(t, x)}{2a(t, x)} dx \right. \\ &\quad \left. - \frac{1}{2} \int \frac{1}{a(t, x)^{1/2}} \partial_t \left(\int \frac{dx}{a(t, x)^{1/2}} \right) dx \right\},\end{aligned}\tag{5.1}$$

where ν_0 is a constant, if and only if the condition

$$\frac{\partial}{\partial t} \left(J + \partial_t \left(\int \frac{b}{2a} dx \right) - \frac{1}{2} \int \frac{1}{a^{1/2}} \partial_t^2 \left(\int_0^x \frac{dx}{a^{1/2}} \right) dx \right) = 0\tag{5.2}$$

holds. The \bar{c} in the transformed PDE is

$$\epsilon \bar{c} = J + \partial_t \left(\int \frac{b}{2a} dx \right) - \frac{1}{2} \int \frac{1}{a^{1/2}} \partial_t^2 \left(\int \frac{dx}{a^{1/2}} \right) dx.\tag{5.3}$$

If $a > 0$, then $\epsilon = 1$, otherwise $\epsilon = -1$. The transformations in the above theorem are more restrictive than those of the higher symmetry cases as they should be. These transformations and condition follow from (3.3) to (3.6).

Hence if Theorems 3, 4 and 5 are not satisfied, then the parabolic PDE (1.1) is reducible to the fourth Lie form with appropriate \bar{c} . That is we must have $\bar{c}_t \neq 0$ in equation (5.3)

We consider the following examples.

Example 8. The equation $u_t = u_{xx} + xu_x + \alpha x^5 u$, where α is a constant, has $J = \alpha x^5 - 1/2 - x^2/4$ and satisfies the condition (5.2). Therefore this equation can be transformed to the third Lie form with $\bar{c} = \alpha x^5 - 1/2 - x^2/4$. A transformation that does the trick is $\bar{u} = u \exp(x^2/4)$. If α was a function of t , then one would get reduction to the fourth Lie form as $\bar{c}_t \neq 0$ in this case.

Example 9. The PDE $u_t = x^2 u_{xx} + 3xu_x + (1 + \alpha(t)x^5)u$ has $J = \alpha(t)x^5$ and does not satisfy (5.2) unless α is a constant. So in general it is reducible to the fourth Lie form. The transformation to the form $\bar{u}_{\bar{t}} = \bar{u}_{\bar{x}\bar{x}} + \alpha(\bar{t}) \exp(5\bar{x})$ is $\bar{t} = t$, $\bar{x} = \ln x$, $\bar{u} = xu$.

6 Concluding remarks

Lie in 1881 [18] completely classified scalar linear (1+1) parabolic equations according to their point symmetry groups. Lie derived the four canonical forms, viz. the classical heat equation which admits six nontrivial point symmetries (apart from the infinite superposition symmetries), the second Lie form having four nontrivial symmetries, the third one possessing two symmetries and the fourth Lie form with one symmetry. Subsequently, algebraic properties and reductions of parabolic equations to the classical heat equation have been studied by many authors (see [4, 5, 7, 10, 12, 22, 24, 26]). Invariant criteria too were investigated for such equations with reduction to the heat equation [21]. The second Lie canonical form was considered for invariants under equivalence transformations briefly in [19]. We have presented refined criteria for reduction to the first two Lie canonical forms. Moreover we derived conditions for reductions to the third and last Lie forms. Thus we have obtained a complete description of scalar linear (1+1) parabolic equations in terms of semi-invariants and invariants under equivalence transformations. These provide practical criteria, in terms of coefficients of the parabolic equation, for reduction to simpler form.

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