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Transformation Groups Applied to Two-Dimensional Boundary Value Problems in Fluid Mechanics

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Abstract

The boundary value problems for the two-dimensional, steady, irrotational flow of a frictionless, incompressible fluid past a wedge and a circular cylinder are considered. It is shown that by considering first the invariance of the boundary condition we are able to obtain a transformation group that can be used to solve each boundary value problem.

1 Introduction

In theoretical fluid mechanics the problem of determining the two-dimensional, steady, irrotational flow of a frictionless, incompressible fluid (an ideal fluid) past a body B can be solved by using conformal transformations (see, for example, [5, 7]). The basic idea behind conformal transformations is very simple: find an invertible transformation, T say, that will transform the body B into the body B^* given that the flow past the body B^* is known; then, using the inverse transformation T^{-1} the flow past the body B can be found. By using conformal transformations we are able to determine the flow past the body B without actually having to solve the two-dimensional Laplace equation which models the problem. It does nonetheless provide us with a solution of Laplace's equation subject to some boundary condition.

It was discovered in the 19th century, mainly through the work of the Norwegian mathematician Sophus Lie, that the majority of ad hoc methods used to solve differential equations could be explained and deduced by means of the theory of transformation groups [6]. Transformation groups at present are mainly used to solve differential equations while ignoring boundary conditions. Although it is known how to use transformation groups to solve boundary value problems (see, for example, [3]), there is an infinite number of transformation groups that leave Laplace's equation invariant. This means that one has to resort to guessing the transformation group that leaves the boundary value problem invariant, thus making the usual method unsuitable to treat this problem. In this article we show how to overcome this difficulty by considering the invariance of the boundary condition before the invariance of Laplace's equation. This is a reversal of the usual method.

The outline for the rest of the article is as follows. In section 2 we give a brief overview of some of the concepts and basic equations in fluid mechanics. In section 3 we justify our belief that the problem of determining the two-dimensional, steady, irrotational motion of an ideal fluid past a body can be solved via the theory of transformation groups. Using this approach we obtain known solutions for the flow past a wedge and a circular cylinder. The article ends with some conclusions.

2 The two-dimensional steady irrotational flow of an ideal fluid

In the absence of any external forces (such as gravity) the steady flow of an ideal fluid past a body is governed by the continuity equation,

$$\underline{\nabla} \cdot \underline{v} = 0, \tag{2.1}$$

and Euler's equation,

$$\rho \left(\underline{v} \cdot \underline{\nabla}\right) \underline{v} + \underline{\nabla} p = \underline{0}, \tag{2.2}$$

where ρ is the density of the fluid, $p\left(\underline{x}\right)$ is the pressure at a point $\underline{x}$ in the fluid and $\underline{v}\left(\underline{x}\right)$ the velocity of a fluid particle at the point $\underline{x}$ (see, for example, [1, 4, 8]).

2.1 Streamlines

It is common practice in fluid mechanics to picture the flow of a fluid by drawing streamlines. A streamline, given by the equation $\psi\left(\underline{x}\right) = c$, c a constant, is a fictitious line drawn in the fluid whose tangent is everywhere parallel to $\underline{v}$ [1, 4, 8]. Following [8], let $\underline{e}^1$ be a unit vector tangent to the streamline drawn in the direction of $\underline{v}$, let $\underline{e}^2 = -\underline{\nabla}\psi / \left|\underline{\nabla}\psi\right|$ be a unit vector normal to the streamline drawn in the direction towards which $\psi\left(\underline{x}\right)$ decreases and let $\underline{e}^3$ be a constant unit vector perpendicular to the plane of motion such that $\underline{e}^1$, $\underline{e}^2$ and $\underline{e}^3$ form a right-hand system of base vectors; then

$$\underline{v} = \underline{e}^3 \times \underline{\nabla}\psi, \tag{2.3}$$

where $\underline{\nabla} = \sum_{i=1}^{3} \underline{e}^i \left(\partial / \partial x^i\right)$. Using the vector identity,

$$\underline{\nabla} \times \left(\underline{e}^3 \times \underline{\nabla}\psi\right) = \left(\underline{\nabla} \cdot \underline{\nabla}\right) \underline{e}^3 - \left(\underline{e}^3 \cdot \underline{\nabla}\right) \underline{\nabla}\psi - \underline{\nabla}\psi \left(\underline{\nabla} \cdot \underline{e}^3\right) + \underline{e}^3 \left(\underline{\nabla} \cdot \underline{\nabla}\psi\right),$$

we have that

$$\underline{\nabla} \times \underline{v} = -\left(\underline{e}^3 \cdot \underline{\nabla}\right) \underline{\nabla}\psi + \underline{e}^3 \nabla^2 \psi,$$

where $\underline{\nabla} \cdot \underline{\nabla}\psi \equiv \nabla^2\psi$. However, $\left(\underline{e}^3 \cdot \underline{\nabla}\right) \underline{\nabla}\psi = \underline{0}$ because there is no variation of $\underline{\nabla}\psi$ perpendicular to the plane of motion since the motion is taken to be two-dimensional. Consequently

$$\omega = \nabla^2 \psi, \tag{2.4}$$

where ω is the magnitude of the vorticity $\underline{\omega} = \underline{\nabla} \times \underline{v}$.

2.2 Equipotential lines

If the vorticity is everywhere equal to zero then the flow is said to be irrotational and $\underline{v}$ can be written as the gradient of a scalar function $\phi(\underline{x})$, that is,

$$\underline{v} = \underline{\nabla}\phi. \tag{2.5}$$

Substituting for $\underline{v}$ in the continuity equation (2.1) we have that

$$\nabla^2\phi = 0. \tag{2.6}$$

Furthermore, if we imagine the surface of the body to be a rigid impermeable wall and the fluid next to this wall can slip past it, then the problem of determining the steady irrotational flow of an ideal fluid past the body is reduced to solving Laplace's equation (2.6) for the velocity potential $\phi(\underline{x})$ subject to the boundary condition,

$$\underline{n} \cdot \underline{\nabla}\phi = 0 \text{ along the surface of the body,} \tag{2.7}$$

where $\underline{n}$ is a unit vector normal to the surface of the body in the outward direction. Since the motion is irrotational it follows from equation (2.4) that an alternative approach to determining the flow past the body would be to solve Laplace's equation for the stream function $\psi(\underline{x})$,

$$\nabla^2\psi = 0, \tag{2.8}$$

subject to the boundary condition,

$$\psi(\underline{x}) = 0 \text{ along the surface of the body.} \tag{2.9}$$

We do not, however, have to solve the boundary value problem (2.6)-(2.7) and the boundary value problem (2.8)-(2.9) seperately to determine the velocity potential and the stream function because, by equation (2.3), the stream function and the velocity potential are related by means of the equation

$$\underline{\nabla}\phi = \underline{e}^3 \times \underline{\nabla}\psi. \tag{2.10}$$

For this reason we shall consider the problem of solving the boundary value problem (2.6)-(2.7) for the velocity potential instead of solving the boundary value problem (2.8)-(2.9) for the stream function.

An important consequence of equation (2.10) is that equipotential lines and the streamlines intersect at right angles. It was this observation that first led us to believe that we could solve either boundary value problem using transformation groups.

3 Continuous one-parameter transformation groups

Let the problem of determining the flow past a body be posed in a rectangular cartesian coordinate system so that, as is usual, $\underline{x} = (x, y)$. Then, if $F(x, y) = 0$ is the equation of the surface of the body, the unit outward normal vector $\underline{n}$ is given by

$$\underline{n} = \frac{\underline{\nabla}F}{|\underline{\nabla}F|} = \frac{(F_x, F_y)}{\sqrt{F_x^2 + F_y^2}}$$

where subscripts denote differentiation. Hence to determine the velocity potential ϕ for the flow past a body we have to solve Laplace's equation,

$$\phi_{xx} + \phi_{yy} = 0, \tag{3.1}$$

subject to the boundary condition,

$$F_x\phi_x + F_y\phi_y = 0 \text{ when } F = 0. \tag{3.2}$$

The possibilty of using a continuous one-parameter transformation group to solve the boundary value problem (3.1)-(3.2) can be justified as follows. The graph of the solution of the boundary value problem is a surface, the contours of which are just equipotential lines by another name. This fact, combined with the fact that equipotential lines and streamlines intersect at right angles, implies that as we move along a streamline in the direction of $\underline{v}$ we are continuously moving from one solution of the boundary value problem to another solution of the boundary value problem. It is, however, a property of transformation groups, in our case a continuous one-parameter transformation group, G say, given by

$$\left.\begin{array}{rcl} x^* &=& x^*(x, y, \phi; \epsilon) \\ y^* &=& y^*(x, y, \phi; \epsilon) \\ \phi^* &=& \phi^*(x, y, \phi; \epsilon) \end{array}\right\}, \tag{3.3}$$

to transform solutions to solutions [2, 9]. The streamlines therefore can be taken to be the projection on the xy-plane of the path curves of the group G.

The infinitesimal transformations corresponding to (3.3),

$$\left.\begin{array}{rcl} x^* &=& x + \epsilon X(x, y, \phi) + \mathcal{O}\left(\epsilon^2\right) \\ y^* &=& y + \epsilon Y(x, y, \phi) + \mathcal{O}\left(\epsilon^2\right) \\ \phi^* &=& \phi + \epsilon \Phi(x, y, \phi) + \mathcal{O}\left(\epsilon^2\right) \end{array}\right\}, \tag{3.4}$$

define a vector (X, Y, Φ) that is tangent to the path curves of the group G. The projection of this vector on the xy-plane is thus a vector parallel to $\underline{v}$, that is, $\underline{v} \times (X, Y, 0) = \underline{0}$. Since we are taking the surface of the body to be a rigid impermeable wall (given by the equation $F(x, y) = 0$) and that the fluid next to this wall can slip past it, we have that, along the surface of the body, $\underline{v}$ must be parallel to the unit tangent vector

$$\underline{t} = \frac{(F_y, -F_x, 0)}{\sqrt{F_x^2 + F_y^2}},$$

from which it follows that

$$(F_y, -F_x, 0) \times (X, Y, 0) = \underline{0}. \tag{3.5}$$

A solution of equation (3.5) is $X = F_y$ and $Y = F_x$. Hence

$$(X, Y, \Phi) = (F_y, -F_x, \Phi) \;\text{ when } F = 0. \tag{3.6}$$

If Φ can be found then we will have the infinitesimal form of the transfomation group G that leaves the boundary value problem (3.1)-(3.2) invariant. We can then apply the invariance principle [6] and seek the solution of the boundary value problem among functions that are invariant under the transformation group G.

The solution $\phi = \mathcal{F}(x, y)$ of the boundary value problem (3.1)-(3.2) is said to be invariant under the transformation group G if

$$\phi^* = \mathcal{F}(x^*, y^*) \text{ when } \phi = \mathcal{F}(x, y). \tag{3.7}$$

The infinitesimal version of the invariance condition (3.7) is

$$\phi + \epsilon\Phi = \mathcal{F}(x + \epsilon X, y + \epsilon Y) + \mathcal{O}\left(\epsilon^2\right) \text{ when } \phi = \mathcal{F}(x, y). \tag{3.8}$$

Expanding and equating $\mathcal{O}(\epsilon)$ terms we have that

$$X\phi_x + Y\phi_y = \Phi. \tag{3.9}$$

3.1 Flow past a wedge

Consider the problem of determining the flow past a wedge placed in a uniform flow such that the streamline pattern is symmetric with respect to the x-axis. Let the surface of the wedge be given by the equation $F = y - mx = 0$, m a positive constant. To determine the velocity potential ϕ for the flow past the wedge we have to solve Laplace's equation,

$$\phi_{xx} + \phi_{yy} = 0, \tag{3.10}$$

subject to the boundary condition,

$$-y\phi_x + x\phi_y = 0 \text{ when } y = mx. \tag{3.11}$$

From (3.6)

$$\begin{aligned} (X, Y, \Phi) &= (1, m, \Phi) \text{ when } y - mx = 0 \\ &= \frac{1}{x}(x, y, x\Phi). \end{aligned}$$

Let $X = x$, $Y = y$ and redefine $x\Phi$ to be Φ ($X = 1$ and $Y = y/x$ will not leave Laplace's equation invariant). Then we should expect the boundary value problem (3.10)-(3.11) to be invariant under the transformation group G given infinitesimally by

$$\left.\begin{aligned} x^* &= x + \epsilon x + \mathcal{O}\left(\epsilon^2\right) \\ y^* &= y + \epsilon y + \mathcal{O}\left(\epsilon^2\right) \\ \phi^* &= \phi + \epsilon\Phi(x, y, \phi) + \mathcal{O}\left(\epsilon^2\right) \end{aligned}\right\}. \tag{3.12}$$

It follows from the infinitesimal transformations (3.12) that the derivatives ϕ_{xx} and ϕ_{yy} transform as (see, for example, [3, 6])

$$\phi^*_{x^*x^*} = \phi_{xx} + \epsilon\left(\Phi_{xx} + 2\phi_x\Phi_{x\phi} + \phi_{xx}\Phi_\phi + \phi_x^2\Phi_{\phi\phi} - 2\phi_{xx}\right) + \mathcal{O}\left(\epsilon^2\right)$$

and

$$\phi^*_{y^*y^*} = \phi_{yy} + \epsilon \left(\Phi_{yy} + 2\phi_y \Phi_{y\phi} + \phi_{yy}\Phi_\phi + \phi_y^2 \Phi_{\phi\phi} - 2\phi_{yy} \right) + \mathcal{O}\left(\epsilon^2\right).$$

Then the condition for Laplace's equation (3.10) to be invariant under the transformation group G, namely,

$$\phi^*_{x^*x^*} + \phi^*_{y^*y^*} = 0 \text{ when } \phi_{xx} + \phi_{yy} = 0, \tag{3.13}$$

implies that

$$\Phi_{xx} + \Phi_{yy} + 2\phi_x \Phi_{x\phi} + 2\phi_y \Phi_{y\phi} + \left(\phi_x^2 + \phi_y^2\right) \Phi_{\phi\phi} = 0. \tag{3.14}$$

Since Φ does not depend on the derivatives of ϕ, equation (3.14) decomposes into a system of partial differential equations for Φ,

$$\Phi_{\phi\phi} = 0, \ \Phi_{x\phi} = 0, \ \Phi_{y\phi} = 0, \text{ and } \Phi_{xx} + \Phi_{yy} = 0,$$

from which we obtain

$$\Phi(x, y, \phi) = a\phi + A(x, y),$$

where a is a constant and $A_{xx} + A_{yy} = 0$. If A is set equal to be zero then substituting for $X = x$, $Y = y$ and $\Phi = a\phi$ in equation (3.9) we have that for a group invariant solution

$$x\phi_x + y\phi_y = a\phi. \tag{3.15}$$

Solving the partial differential equation (3.15) the solution of the boundary value problem (3.10)-(3.11) must be of the form

$$\phi(x, y) = \left(x^2 + y^2\right)^{a/2} f(w), \tag{3.16}$$

where $w = \arctan(y/x)$ (it is easier to solve the resulting ordinary differential equation if we take $w = \arctan(y/x)$ instead of $w = y/x$). Substituting the form of the solution (3.16) into Laplace's equation (3.10) gives the following ordinary differential equation for $f(w)$:

$$f_{ww} + a^2 f = 0. \tag{3.17}$$

The solution of the ordinary differential equation (3.17) yields

$$\phi(x, y) = \left(x^2 + y^2\right)^{a/2} \left[k_1 \cos(aw) + k_2 \sin(aw)\right], \tag{3.18}$$

where k_1 and k_2 are arbitrary constants. It follows from the boundary condition (3.11) that $k_2 = k_1 \tan(a\theta)$ where $\theta = \arctan m$. Hence

$$\phi(x, y) = \frac{k_1 \left(x^2 + y^2\right)^{a/2}}{\cos(a\theta)} \cos\left[a(w - \theta)\right]. \tag{3.19}$$

To determine the corresponding stream function we note from equation (2.10) that

$$\phi_x = \psi_y \text{ and } \phi_y = -\psi_x.$$

Choosing to differentiate the velocity potential (3.19) with respect to x and then integrating with respect to y by parts and setting the additive constant of integration equal to zero we obtain the following stream function:

$$\psi(x,y)=\frac{k_1\left(x^2+y^2\right)^{a/2}}{\cos(a\theta)}\sin\left[a\left(w-\theta\right)\right]. \tag{3.20}$$

Taking $a=\pi/(\pi-\theta)$ gives us the known solution for the flow past a wedge [8]. Sketched in Figure 1 are some equipotential lines and streamlines for the flow past a wedge calculated using equations (3.18) and (3.20).

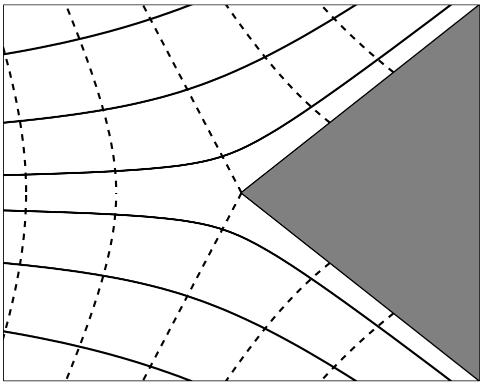

Figure 1: Equipotential lines (- -) and streamlines (—) for the flow past a wedge.

3.2 Flow past a circular cylinder

Consider next the problem of determining the flow past a circular cylinder with unit radius placed in a uniform stream such that the streamline pattern is symmetric with respect to the x-axis. Let the surface of the cylinder be given by the equation $F=x^2+y^2-1=0$ so that we have to solve Laplace's equation,

$$\phi_{xx}+\phi_{yy}=0, \tag{3.21}$$

subject to the boundary condition,

$$x\phi_x+y\phi_y=0 \text{ when } x^2+y^2=1. \tag{3.22}$$

From (3.6)

$$\begin{aligned}(X,Y,\Phi) &= (2y,-2x,\Phi) \text{ when } x^2+y^2=1\\ &= 2\left(y,-x,\frac{\Phi}{2}\right).\end{aligned}$$

Let $X=y$, $Y=-x$ and redefine $\Phi/2$ to be Φ. Then we should expect the boundary value problem (3.21)-(3.22) to be invariant under the transformation group G given infinitesimally by

$$\left.\begin{aligned}x^* &= x+\epsilon y+\mathcal{O}\left(\epsilon^2\right)\\ y^* &= y-\epsilon x+\mathcal{O}\left(\epsilon^2\right)\\ \phi^* &= \phi+\epsilon\Phi(x,y,\phi)+\mathcal{O}\left(\epsilon^2\right)\end{aligned}\right\}. \tag{3.23}$$

It follows from the infinitesimal transformations (3.23) that the derivatives ϕ_{xx} and ϕ_{yy} transform as

$$\phi^*_{x^*x^*} = \phi_{xx} + \epsilon\left(\Phi_{xx} + 2\phi_x\Phi_{x\phi} + \phi_{xx}\Phi_\phi + \phi_x^2\Phi_{\phi\phi} + 2\phi_{xy}\right) + \mathcal{O}\left(\epsilon^2\right)$$

and

$$\phi^*_{y^*y^*} = \phi_{yy} + \epsilon\left(\Phi_{yy} + 2\phi_y\Phi_{y\phi} + \phi_{yy}\Phi_\phi + \phi_y^2\Phi_{\phi\phi} - 2\phi_{xy}\right) + \mathcal{O}\left(\epsilon^2\right).$$

For this example, however, the condition that Laplace's equation (3.21) be invariant under the transformation group G, namely,

$$\phi^*_{x^*x^*} + \phi^*_{y^*y^*} = 0 \text{ when } \phi_{xx} + \phi_{yy} = 0,$$

does not give us the known solution for the flow past a circular cylinder. Following [2], let us consider instead the following invariance condition:

$$\phi^*_{x^*x^*} + \phi^*_{y^*y^*} = 0 \text{ when } \begin{cases} \phi_{xx} + \phi_{yy} &= 0, \\ y\phi_x - x\phi_y &= \Phi. \end{cases} \tag{3.24}$$

The extra condition is obtained from equation (3.9) setting $X = y$ and $Y = -x$. The invariance condition (3.24) reduces to

$$\Phi_{xx} + \Phi_{yy} + 2\phi_x\Phi_{x\phi} + 2\phi_y\Phi_{y\phi} + \left(\phi_x^2 + \phi_y^2\right)\Phi_{\phi\phi} = 0 \text{ when } y\phi_x - x\phi_y = \Phi. \tag{3.25}$$

Introduce a function $\Omega(x, y, \phi)$ such that

$$2\Phi_{x\phi} = y\Omega \text{ and } 2\Phi_{y\phi} = -x\Omega. \tag{3.26}$$

Since Φ does not depend on the derivatives of ϕ it follows from equations (3.25) and (3.26) that

$$\Phi_{xx} + \Phi_{yy} = -\Phi\Omega \text{ and } \Phi_{\phi\phi} = 0. \tag{3.27}$$

Without the function Ω the invariance condition (3.25) cannot be solved for Φ. The solution of the system of partial differential equations (3.26) and (3.27) is

$$\Phi(x, y, \phi) = \phi B(w) + D(x, y),$$

where $w = \arctan(y/x)$,

$$B_{ww} - 2BB_w = 0 \tag{3.28}$$

and

$$D_{xx} + D_{yy} = -D\Omega.$$

Integration of the ordinary differential equation (3.28) with respect to w gives

$$B_w - B^2 = c, \;\; c \text{ an arbitrary constant.}$$

There are three cases to consider. Let $c = c_1^2 > 0$, then $B = c_1 \tan(c_1 w + c_2)$ and, if we set $D = 0$, $\Phi = c_1 \phi \tan(c_1 w + c_2)$ so that for a group invariant solution we must solve the partial differential equation

$$y\phi_x - x\phi_y = c_1 \phi \tan(c_1 w + c_2), \tag{3.29}$$

where c_1 and c_2 are arbitrary constants. Thus the solution of the boundary value problem (3.21)-(3.22) must be of the form

$$\phi(x, y) = \cos(c_1 w + c_2) f(u), \tag{3.30}$$

where $u = x^2 + y^2$. The substitution of the form of the solution (3.30) into Laplace's equation (3.21) gives the following differential equation for $f(u)$:

$$4u^2 f_{uu} + 4u f_u - c_1^2 f = 0. \tag{3.31}$$

The solution of the ordinary differential equation (3.31) yields

$$\phi(x, y) = k_2 \cos(c_1 w + c_2) \left[k_1 u^{c_1/2} + u^{-c_1/2}\right], \tag{3.32}$$

where k_1 and k_2 are arbitrary constants. From the boundary condition (3.22) it follows that $k_1 = 1$. As in the previous example, if we differentiate (3.32) with respect to x, integrate with respect to y by parts and set the additive constant of integration equal to zero we have that

$$\psi(x, y) = k_2 \sin(c_1 w + c_2) \left[u^{c_1/2} - u^{-c_1/2}\right]. \tag{3.33}$$

Taking $c_1 = 1$ and $c_2 = 0$ gives us the known solution for the flow past a circular cylinder [8]. Sketched in Figure 2 are some equipotential lines and streamlines for the flow past a circular cylinder using equations (3.32) and (3.33).

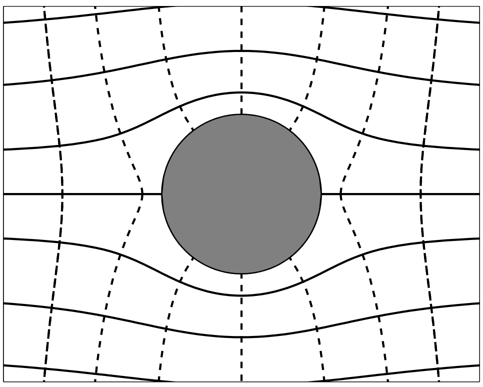

Figure 2: Equipotential lines (- -) and streamlines (—) for the flow past a cylinder.

In the second case, $c = 0$, we find that

$$\phi(x, y) = k_2 w \text{ and } \psi(x, y) = -\frac{k_2}{2} \ln u,$$

which represents circular flow around the cylinder. In the third case, $c = -c_1^2 < 0$, we find that

$$\phi(x) = k_2 \cos\left(n\pi - \frac{c_1}{2}\ln u\right)\left[c_2 \exp(c_1 w) - \exp(-c_1 w)\right]$$

and

$$\psi(x) = k_2 \sin\left(n\pi - \frac{c_1}{2}\ln u\right)\left[c_2 \exp(c_1 w) + \exp(-c_1 w)\right]$$

for which we are unable to offer a physical interpretation.

4 Conclusions

The conventional approach to solving a boundary value problem with one dependent variable and two independent variables using continuous one-parameter transformation groups consists of three steps. In the first step we determine all the transformation groups that leave the differential equation invariant. In the second step we determine which of these transformation groups also leaves the boundary condition invariant. In the last step the transformation group that leaves the differential equation and the boundary condition invariant is used to determine the group invariant solution of the boundary value problem. If we were to follow this approach to solve the boundary value problem for the two-dimensional, steady, irrotational flow of an ideal fluid past a wedge and a circular cylinder then it is difficult to proceed beyond the first step. This is because there are an infinite number of transformation groups that leave Laplace's equation (3.1) invariant, the only restriction being that the infinitesimal form of the transformation group,

$$\left.\begin{array}{rcl} x^* & = & x + \epsilon X(x,y) + \mathcal{O}\left(\epsilon^2\right) \\ y^* & = & y + \epsilon Y(x,y) + \mathcal{O}\left(\epsilon^2\right) \\ \phi^* & = & \phi + \epsilon \Phi(x,y,\phi) + \mathcal{O}\left(\epsilon^2\right) \end{array}\right\}, \tag{4.1}$$

satisfy the following constraints:

$$X_x = Y_y \text{ and } X_y = -Y_x \tag{4.2}$$

and

$$\Phi = \alpha\phi + \beta(x,y), \tag{4.3}$$

where α is a constant and $\beta_{xx} + \beta_{yy} = 0$. The second step therefore requires us to guess X, Y and Φ subject to the contraints (4.2) and (4.3) such that the transformation group (4.1) leaves the boundary condition (3.2) invariant. In this article we showed how this problem may be overcome by interchanging the first two steps. In the first step of our approach we determine X and Y from the boundary condition (3.2) using (3.6) such that X and Y satisfy the constraints (4.2). In the second step we determine Φ such that the transformation group leaves Laplace's equation (3.1) invariant. This is done using either the classical method (as in the case of the wedge) or the non-classical method (as in the case of the circular cylinder). We are then guaranteed that the transformation group thus found will leave the boundary value problem invariant and can proceed to step three.

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