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Group Invariant Solution for a Two-Dimensional Turbulent Free Jet described by Eddy Viscosity

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Abstract

The group invariant solution for the stream function and the effective viscosity of a two-dimensional turbulent free jet are derived. Prandtl's hypothesis is not imposed. When the eddy viscosity is constant across the jet it is found that the mean velocity profile is the same as that of a laminar jet in agreement with Görtler (1942). When the eddy viscosity decreases across the jet it is found that the jet is narrower due to the decrease in the effective viscosity of the mean flow.

1 Introduction

In many practical situations a free jet will be turbulent. In a turbulent flow the flucuations which are superimposed on the mean flow cause an apparent increase in the viscosity of the mean flow by factors of one hundred, one thousand, ten thousand or even more [14]. The increased apparent viscosity of the mean flow is central to all theories of turbulent flow [14].

In order to model turbulence, Boussinesq [2] introduced the concept of eddy viscosity. In analogy with the coefficient of viscosity in laminar flow the eddy viscosity in turbulent flow relates the Reynolds stress to the spatial gradient of the mean velocity. The Reynolds stresses are due to turbulent fluctuations and are the time averaged values of the quadratic terms in the velocity fluctuations. The eddy viscosity is not a physical property of the fluid like the kinematic viscosity. It is a function of position and time and is dependent on the flow under consideration [15].

The simplest assumption to make on the eddy viscosity in a free boundary layer flow is the hypothesis due to Prandtl [13] that the eddy viscosity is constant across the layer and is proportional to the product of the maximum mean velocity and the width of the layer [6]. Görtler [7] has shown that if the Prandtl hypothesis is satisfied then for a two-dimensional free jet the velocity profile of turbulent and laminar jets are the same. We will not assume the Prandtl hypothesis.

Group theoretical methods are increasingly being applied to investigate turbulence. Unal [16] derived an equivalence transformation of the Navier-Stokes equation by considering the viscosity as a coordinate of the frame of the equation. Ibragimov and Unal [8] introduced the concept of the Kolmogorov invariant and determined the subgroup of the equivalence group for which the Kolmogorov invariant is the first order differential invariant. Gandarias et al [5] investigated the evolution of turbulent bursts by applying Lie group analysis and established the existence of decaying bounded travelling wave solutions. Bruzon et al [3] derived symmetry reductions of the system of two coupled parabolic partial differential equations which model the evolution of turbulent bursts. Oberlack [10] presented a unified approach for symmetries in plane parallel turbulent shear flows while Oberlack and Khujadze [11] used the Lie symmetry method to derive new scaling laws for a zero pressure gradient turbulent boundary layer flow.

In this paper we will investigate a two-dimensional turbulent free jet in which the eddy viscosity can vary across the breadth of the jet as well as in the direction of the jet. The jet is described by Prandtl's two-dimensional boundary layer equation for the stream function with eddy viscosity and vanishing mainstream velocity. The Lie point symmetries of the third order partial differential equation for the stream function are derived and the general form of the group invariant solution for the stream function and the eddy viscosity is obtained. Analytical solutions are derived for the case in which the eddy viscosity varies only in the direction of the jet. When the eddy viscosity varies across the breadth of the jet the problem is reduced to a second order ordinary differential equation which is solved numerically.

The group invariant solution for a two-dimensional turbulent free jet is compared with the group invariant solution for a laminar jet derived by Mason [9].

2 Two-dimensional turbulent free jet

Consider a two-dimensional turbulent free jet. The jet emerges from a long narrow orifice in a wall into a viscous incompressible fluid at rest which consists of the same fluid as the jet. The x-axis is in the direction of the jet, the y-axis is perpendicular to the jet. The origin of the coordinate system, $x = 0,\ y = 0$, is at the orifice. The mean flow is symmetric about the line $y = 0$. Prandtl's two-dimensional boundary layer equations for a turbulent jet are [6]

$$\bar{v}_x \frac{\partial \bar{v}_x}{\partial x} + \bar{v}_y \frac{\partial \bar{v}_x}{\partial y} = \frac{\partial}{\partial y}\left(E(x,y)\, \frac{\partial \bar{v}_x}{\partial y} \right), \qquad (2.1)$$

$$\frac{\partial \bar{v}_x}{\partial x} + \frac{\partial \bar{v}_y}{\partial y} = 0, \qquad (2.2)$$

which are subject to the boundary conditions

$$y = 0 : \qquad \bar{v}_y = 0, \qquad \frac{\partial \bar{v}_x}{\partial y} = 0, \tag{2.3}$$

$$y = \pm\infty : \qquad \bar{v}_x = 0, \qquad \frac{\partial \bar{v}_x}{\partial y} = 0, \tag{2.4}$$

where $\bar{v}_x(x, y)$ and $\bar{v}_y(x, y)$ are the x and y components of the mean fluid velocity. The mean values are taken over a sufficiently long time interval for them to be independent of time. In (2.1), $E(x, y)$ is the sum of the kinematic viscosity ν and the kinematic eddy viscosity $\nu_T(x, y)$ [12, 14] :

$$E(x, y) = \nu + \nu_T(x, y). \tag{2.5}$$

In general, $\nu_T(x, y)$ far outweighs the kinematic viscosity ν. We will refer to $E(x, y)$ as the effective viscosity [12]. The eddy viscosity depends on the coordinate y across the jet as well as on the coordinate x in the direction of the jet but the functional form of $E(x, y)$ is not specified at this stage.

It can be shown by integrating (2.1) with respect to y from $y = -\infty$ to $y = +\infty$, using (2.2) and the boundary conditions (2.4) and assuming that $E(x, y)$ remains finite as $y \to \pm\infty$, that

$$J = \rho \int_{-\infty}^{\infty} \bar{v}_x^2(x, y) dy = \text{ constant independent of } x, \tag{2.6}$$

where ρ is the density of the incompressible fluid. The conserved quantity J is the total mean flux in the x-direction of the x-component of the momentum of the mean flow.

The stream function $\psi(x, y)$ is defined in terms of the mean fluid velocity by

$$\bar{v}_x(x, y) = \frac{\partial \psi}{\partial y}, \qquad \bar{v}_y(x, y) = -\frac{\partial \psi}{\partial x}. \tag{2.7}$$

The continuity equation (2.2), which holds for the mean flow, is identically satisfied.

The problem is to solve the third order partial differential equation

$$\frac{\partial \psi}{\partial y} \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial y^2} = \frac{\partial}{\partial y} \left(E(x, y) \frac{\partial^2 \psi}{\partial y^2} \right) \tag{2.8}$$

for $\psi(x, y)$ subject to the boundary conditions

$$y = 0 : \qquad \frac{\partial \psi}{\partial x} = 0, \qquad \frac{\partial^2 \psi}{\partial y^2} = 0, \tag{2.9}$$

$$y = \pm\infty : \qquad \frac{\partial \psi}{\partial y} = 0, \qquad \frac{\partial^2 \psi}{\partial y^2} = 0, \tag{2.10}$$

and to the condition

$$J = 2\rho \int_0^{\infty} \left(\frac{\partial \psi}{\partial y}(x, y) \right)^2 dy = \text{ constant independent of } x. \tag{2.11}$$

We will derive the group invariant solution for the stream function $\psi(x, y)$ for the mean flow and determine the most general form for the effective viscosity $E(x, y)$ consistent with the group invariant solution.

3 Symmetry generators and partial differential equation for eddy viscosity

The Lie point symmetry generators

$$X = \xi^1 (x, y, \psi) \frac{\partial}{\partial x} + \xi^2 (x, y, \psi) \frac{\partial}{\partial y} + \eta (x, y, \psi) \frac{\partial}{\partial \psi} \tag{3.1}$$

of (2.8) are obtained by solving the determining equation

$$X^{[3]} F \Big|_{F=0} = 0 \tag{3.2}$$

where

$$F = \psi_y \psi_{xy} - \left(\psi_x + \frac{\partial E}{\partial y}(x, y) \right) \psi_{yy} - E(x, y) \psi_{yyy}, \tag{3.3}$$

$$X^{[3]} = X + \zeta_1 \frac{\partial}{\partial \psi_x} + \zeta_2 \frac{\partial}{\partial \psi_y} + \zeta_{12} \frac{\partial}{\partial \psi_{xy}} + \zeta_{22} \frac{\partial}{\partial \psi_{yy}} + \zeta_{222} \frac{\partial}{\partial \psi_{yyy}} \tag{3.4}$$

and

$$\zeta_i = D_i (\eta) - \psi_s D_i (\xi^s), \tag{3.5}$$

$$\zeta_{ij} = D_j (\zeta_i) - \psi_{is} D_j (\xi^s), \tag{3.6}$$

$$\zeta_{ijk} = D_k (\zeta_{ij}) - \psi_{ijs} D_k (\xi^s). \tag{3.7}$$

Subscripts denote partial derivatives and there is summation over an upper and lower repeated index. The total derivatives, D_1 and D_2, are defined by

$$D_1 = D_x = \frac{\partial}{\partial x} + \psi_x \frac{\partial}{\partial \psi} + \psi_{xx} \frac{\partial}{\partial \psi_x} + \psi_{yx} \frac{\partial}{\partial \psi_y} + \cdots, \tag{3.8}$$

$$D_2 = D_y = \frac{\partial}{\partial y} + \psi_y \frac{\partial}{\partial \psi} + \psi_{xy} \frac{\partial}{\partial \psi_x} + \psi_{yy} \frac{\partial}{\partial \psi_y} + \cdots. \tag{3.9}$$

It is found that

$$X = A(x) \frac{\partial}{\partial x} + (c_3 y + k(x)) \frac{\partial}{\partial y} + (c_1 \psi + c_2) \frac{\partial}{\partial \psi} \tag{3.10}$$

where $A(x)$ and $k(x)$ are arbitrary functions of x and c_1, c_2, c_3 are constants, provided $E(x, y)$ satisfies the first order linear partial differential equation

$$A(x) \frac{\partial E}{\partial x} + (c_3 y + k(x)) \frac{\partial E}{\partial y} = \left(c_1 + c_3 - \frac{dA}{dx} \right) E. \tag{3.11}$$

For a two-dimensional laminar jet, $E(x, y) = \nu =$ constant. From (3.11) it follows that for a laminar jet,

$$\frac{dA}{dx} = c_1 + c_3 \tag{3.12}$$

and therefore

$$A(x) = (c_1 + c_3)\, x + c_4, \tag{3.13}$$

where c_4 is a constant. The Lie point symmetry (3.10) reduces to

$$X = ((c_1 + c_3)\, x + c_4)\, \frac{\partial}{\partial x} + (c_3 y + k(x))\, \frac{\partial}{\partial y} + (c_1 \psi + c_2)\, \frac{\partial}{\partial \psi}. \tag{3.14}$$

Turbulence can therefore change the coefficient $\xi^1(x)$ in the Lie point symmetry.

4 Group invariant solution

The group invariant solution $\psi = \Phi(x, y)$ of the partial differential equation (2.8) satisfies

$$X\,(\psi - \Phi(x, y))\Big|_{\psi=\Phi} = 0, \tag{4.1}$$

where X is given by (3.10). Substituting (3.10) into (4.1) yields the first order linear partial differential equation for $\Phi(x, y)$,

$$A(x)\frac{\partial \Phi}{\partial x} + (c_3 y + k(x))\, \frac{\partial \Phi}{\partial y} = c_1 \Phi + c_2. \tag{4.2}$$

It will be assumed that $A(x) \neq 0$ and $c_1 \neq 0$. The general solution of (4.2) is readily derived and setting $\psi = \Phi(x, y)$ we obtain

$$\psi = \exp\,(c_1 B(x))\, F(\xi) - \frac{c_2}{c_1}, \tag{4.3}$$

where F is an arbitrary function,

$$\xi = \frac{y}{\exp\,(c_3 B(x))} - D(x), \tag{4.4}$$

and

$$D(x) = \int^x \frac{k(x)}{A(x)} \exp\,(-c_3 B(x))\, dx, \tag{4.5}$$

$$B(x) = \int^x \frac{dx}{A(x)}, \qquad \frac{dB}{dx} = \frac{1}{A(x)}. \tag{4.6}$$

The general solution of (3.11) for the effective viscosity is

$$E(x, y) = \frac{1}{A(x)} \exp\,[(c_1 + c_3)\, B(x)]\, G(\xi), \tag{4.7}$$

where G is an arbitrary function. The group invariant solutions (4.3) and (4.7) for $\psi(x, y)$ and $E(x, y)$ depend on the same similarity variable ξ because the left hand sides of equations (4.2) and (3.11) have the same form.

When (4.3) and (4.7) are substituted into (2.8) the partial differential equation reduces to the ordinary differential equation

$$\frac{d}{d\xi}\left(G(\xi)\frac{d^2F}{d\xi^2}\right) + c_1\frac{d}{d\xi}\left(F\frac{dF}{d\xi}\right) + (c_3 - 2c_1)\left(\frac{dF}{d\xi}\right)^2 = 0 \tag{4.8}$$

for $F(\xi)$ and $G(\xi)$. Equation (4.8) is independent of $D(x)$ and c_2 which therefore may be chosen suitably. The choice of $D(x)$ determines the origin of ξ. We choose $k(x) = 0$ in (3.10) and therefore $D(x) = 0$ so that $\xi = 0$ when $y = 0$. We also choose $c_2 = 0$ because the additive constant c_2/c_1 in the stream function (4.3) does not contribute to the mean velocity field.

The final condition on the constants is obtained from the conserved quantity J given by (2.11). We substitute (4.3) into (2.11) and make the change of variable from y to ξ at fixed x. Equation (2.11) becomes

$$J = 2\rho \exp\left[(2c_1 - c_3)\, B(x)\right] \int_0^\infty \left(\frac{dF}{d\xi}\right)^2 d\xi. \tag{4.9}$$

Since $B(x) \neq 0, J$ is a constant independent of x provided

$$c_3 = 2c_1. \tag{4.10}$$

In order to simplify the notation let

$$H(x) = \exp\left(c_1 B(x)\right), \qquad \hat{G}(\xi) = \frac{1}{c_1}G(\xi), \qquad \hat{X} = \frac{1}{c_1}X \tag{4.11}$$

and suppress the circumflex on G and X. Then, expressed in terms of $H(x)$,

$$\xi = \frac{y}{H^2(x)}, \tag{4.12}$$

$$\psi(x,y) = H(x)\, F(\xi), \tag{4.13}$$

$$E(x,y) = H'(x)\, H^2(x)\, G(\xi), \tag{4.14}$$

$$\bar{v}_x(x,y) = \frac{\partial\psi}{\partial y} = \frac{1}{H(x)}\,\frac{dF}{d\xi}, \tag{4.15}$$

$$\bar{v}_y(x,y) = -\frac{\partial\psi}{\partial x} = H'(x)\left(2\xi\frac{dF}{d\xi} - F(\xi)\right), \tag{4.16}$$

where $F(\xi)$ satisfies the ordinary differential equation

$$\frac{d}{d\xi}\left(G(\xi)\frac{d^2F}{d\xi^2}\right) + \frac{d}{d\xi}\left(F\frac{dF}{d\xi}\right) = 0, \tag{4.17}$$

subject to the boundary conditions

$$F(0) = 0, \qquad F''(0) = 0, \qquad F'(\pm\infty) = 0 \tag{4.18}$$

and to the condition

$$J = 2\rho \int_0^\infty \left(\frac{dF}{d\xi}\right)^2 d\xi, \tag{4.19}$$

where J is given constant. The Lie point symmetry which generates the group invariant solution is

$$X = \frac{H(x)}{H'(x)}\,\frac{\partial}{\partial x} + 2y\,\frac{\partial}{\partial y} + \psi\frac{\partial}{\partial \psi}. \tag{4.20}$$

The boundary conditions (4.18) are readily derived from (2.3) and (2.4) using (4.15) and (4.16). Since $c_3 = 2c_1$, the differential equation (4.8) can be integrated once immmediately.

A boundary condition on $H(x)$ will be required. This is obtained as follows. The long narrow orifice in the wall from which the jet emerges is assumed to be infinitely thin. Since the volume of the mean flow and the mean momentum are finite it is necessary to assume that the mean fluid velocity at the orifice is infinite [14]. Since

$$\bar{v}_x(x,0) = \frac{F'(0)}{H(x)} \tag{4.21}$$

it is therefore necessary to assume that

$$H(0) = 0. \tag{4.22}$$

Finally, consider Prandtl's hypothesis [6,13]. We will not adopt Prandtl's hypothesis but it gives a reference against which to compare results. Prandtl's hypothesis applied to a two-dimensional free jet states that the eddy viscosity is constant across the jet and is proportional to the product of the maximum mean velocity and the width of the jet. Since in general $\nu_T \gg \nu$, we will neglect ν compared with ν_T and impose Prandtl's hypothesis on the effective viscosity $E(x,y)$. For $E(x,y)$ to be constant across the jet it must be independent of y and therefore $G(\xi)$ must be a constant. From (4.12), the width of the jet is proportional to $H^2(x)$ and from (4.15) the maximum mean velocity, which occurs at $y = 0$, is proportional to $1/H(x)$. Thus, from Prandtl's hypothesis, $E(x,y)$ must be proportional to $H(x)$ and therefore from (4.14)

$$H\frac{dH}{dx} = \alpha, \qquad H(0) = 0, \tag{4.23}$$

where α is a constant. Thus

$$H(x) = (2\alpha x)^{\frac{1}{2}} \tag{4.24}$$

and therefore from (4.14)

$$E(x) = E_0 x^{\frac{1}{2}}, \tag{4.25}$$

where E_0 is a constant. Equation (4.25) gives the form of the effective viscosity if Prandtl's hypothesis is valid.

A group invariant solution for the two-dimensional turbulent free jet may be derived as follows. The functions $H(x)$ and $G(x)$ are either obtained from further information given on $E(x,y)$ or are prescribed subject to the condition $H(0)=0$. The function $F(\xi)$ is found by solving the third order differential equation (4.17) subject to the boundary conditions (4.18). The conserved quantity (4.19) is used to obtain the remaining constant of integration in terms of J. The stream function and the x and y components of the mean velocity are obtained from (4.13), (4.15) and (4.16). The Lie point symmetry which generates the group invariant solution is given by (4.20). Unlike a laminar jet it may not be a scaling symmetry.

5 Eddy viscosity a function of x

Consider first an effective viscosity of the form $E=E(x)$. Then from (4.14),

$$\frac{E(x)}{H'(x)H^2(x)} = G(\xi) \tag{5.1}$$

and by the technique of separation of variables, each side of (5.1) must equal a constant which we take as unity. Hence

$$G(\xi) = 1 \tag{5.2}$$

and $H(x)$ satisfies the first order ordinary differential equation

$$H^2(x)\frac{dH}{dx} = E(x), \qquad H(0) = 0. \tag{5.3}$$

Equation (4.17) becomes

$$\frac{d^3F}{d\xi^3} + \frac{d}{d\xi}\left(F\frac{dF}{d\xi}\right) = 0. \tag{5.4}$$

The solution of (5.4) subject to the boundary conditions (4.18) and the integral condition (4.19) is [1,9]

$$F(\xi) = \left(\frac{3J}{2\rho}\right)^{\frac{1}{3}} \tanh\left(\frac{1}{2}\left(\frac{3J}{2\rho}\right)^{\frac{1}{3}}\xi\right), \tag{5.5}$$

where ξ is given by (4.12). Thus the scaled mean velocity profile

$$\frac{\bar{v}_x(x,y)}{\bar{v}_x(x,0)} = \mathrm{sech}^2\left[\frac{1}{2}\left(\frac{3J}{2\rho}\right)^{\frac{1}{3}}\xi\right] \tag{5.6}$$

plotted against ξ is the same for all turbulent jets when the effective viscosity is independent of y and is the same as that for a two-dimensional laminar jet for which $E=\nu$. A similar result was established by Görtler [7] who showed on the basis of Prandtl's hypothesis that the mean velocity profile of a turbulent jet is the same as that of a laminar jet [6].

An important special case of $E=E(x)$ is the power law [6]

$$E(x) = E_0 x^{\beta} \tag{5.7}$$

where E_0 is a constant and $\beta > -1$ to satisfy $H(0) = 0$. When Prandtl's hypothesis applies, $\beta = \dfrac{1}{2}$. The solution of (5.3) with (5.7) is

$$H(x) = \left(\frac{3E_0}{1+\beta}\right)^{\frac{1}{3}} x^{\frac{1}{3}(1+\beta)}. \tag{5.8}$$

From (4.12), (4.13) and (5.5) the stream function is

$$\psi(x,y) = \left(\frac{9E_0 J}{2(1+\beta)\rho}\right)^{\frac{1}{3}} x^{\frac{1}{3}(1+\beta)} \tanh\left[\frac{1}{2}\left(\frac{J(1+\beta)^2}{6\rho E_0^2}\right)^{\frac{1}{3}} \frac{y}{x^{\frac{2}{3}(1+\beta)}}\right] \tag{5.9}$$

and hence from (2.7),

$$\bar{v}_x(x,y) = \frac{1}{2}\left(\frac{3(1+\beta)J^2}{4E_0\rho^2}\right)^{\frac{1}{3}} \frac{1}{x^{\frac{1}{3}(1+\beta)}} \operatorname{sech}^2\left[\frac{1}{2}\left(\frac{J(1+\beta)^2}{6\rho E_0^2}\right)^{\frac{1}{3}} \frac{y}{x^{\frac{2}{3}(1+\beta)}}\right]. \tag{5.10}$$

From (4.20) and (5.8) the solution is generated by the Lie point symmetry

$$X = \frac{3x}{(1+\beta)}\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} + \psi\frac{\partial}{\partial \psi}, \tag{5.11}$$

which is a scaling symmetry. The solution with a power law effective viscosity can therefore be derived using a scaling transformation as described by Glauert [6] and Dresner [4]. For a laminar jet, $E_0 = \nu$ and $\beta = 0$. Equations (5.8) and (5.11) reduce to

$$H(x) = (3\nu x)^{\frac{1}{3}} \tag{5.12}$$

and

$$X = 3x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} + \psi\frac{\partial}{\partial \psi}. \tag{5.13}$$

When the eddy viscosity is a power law,

$$E(x) = \nu + \nu_{T_0} x^{\beta}, \qquad \beta > -1 \tag{5.14}$$

where ν_{T_0} is a constant. The effective viscosity (5.14) allows us to follow the gradual development from a laminar jet to a turbulent jet as ν_{T_0} increases from zero. The solution of (5.3) with (5.14) is

$$H(x) = (3\nu x)^{\frac{1}{3}}\left[1 + \frac{1}{(1+\beta)}\frac{\nu_{T_0}}{\nu}x^{\beta}\right]^{\frac{1}{3}}, \tag{5.15}$$

which from (4.12), (4.15) and (5.5) gives the mean velocity profile

$$\bar{v}_x(x,y) = \frac{1}{2}\left(\frac{3J^2}{4\nu\rho^2}\right)^{\frac{1}{3}} \frac{1}{\left[x + \dfrac{1}{(1+\beta)}\dfrac{\nu_{T_0}}{\nu}x^{1+\beta}\right]^{\frac{1}{3}}} \operatorname{sech}^2\left[\frac{1}{2}\left(\frac{J}{6\rho\nu^2}\right)^{\frac{1}{3}} \frac{y}{\left(x + \dfrac{1}{(1+\beta)}\dfrac{\nu_{T_0}}{\nu}x^{1+\beta}\right)^{\frac{2}{3}}}\right]$$

(5.16)

From (4.20) and (5.15) the solution is generated by the Lie point symmetry

$$X = 3x\left(\frac{1+\dfrac{1}{(1+\beta)}\dfrac{\nu_{T_0}}{\nu}x^{\beta}}{1+\dfrac{\nu_{T_0}}{\nu}x^{\beta}}\right)\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} + \psi\frac{\partial}{\partial \psi}, \tag{5.17}$$

which is not a scaling symmetry. The solution (5.16) therefore cannot be derived by a scaling transformation. Results for a laminar jet are obtained by setting $\nu_{T_0} = 0$.

6 Eddy viscosity a function of x and y

The general form for the effective viscosity in a group invariant solution is given by (4.14). It can depend on y through the similarity variable ξ. The function $G(\xi)$ may be chosen but it must be an even function of ξ and be bounded as ξ tends to infinity.

In order to investigate the effect of the dependence of the eddy viscosity on y consider

$$E(x,y) = \nu + \nu_{T_0}K(\xi), \tag{6.1}$$

where $K(\xi)$ has still to be specified. Equation (4.14) becomes

$$H'(x)H^2(x)G(\xi) = \nu\left(1 + \frac{\nu_{T_0}}{\nu}K(\xi)\right). \tag{6.2}$$

We can set

$$H^2(x)\frac{dH}{dx} = \nu, \qquad H(0) = 0, \tag{6.3}$$

and

$$G(\xi) = 1 + \frac{\nu_{T_0}}{\nu}\,K(\xi). \tag{6.4}$$

Then $H(\xi)$ is given by (5.12) as for a laminar jet and from (4.12) the similarity variable is

$$\xi = \frac{y}{(3\nu x)^{\frac{2}{3}}}. \tag{6.5}$$

We choose

$$G(\xi) = 1 + \frac{\nu_{T_0}}{\nu}\exp\left(-\xi^2\right), \qquad G(\xi) = 1 + \frac{\nu_{T_0}}{\nu\left(1+\xi^2\right)}. \tag{6.6}$$

The corresponding effective viscosities are

$$E(x,y) = \nu + \nu_{T_0}\exp\left(-\xi^2\right), \qquad E(x,y) = \nu + \frac{\nu_{T_0}}{1+\xi^2}. \tag{6.7}$$

Since $G(\xi)$ is now specified, $F(\xi)$ can be obtained from the differential equation (4.17). Integrating (4.17) once with respect to ξ, imposing the boundary conditions (4.18) at $\xi = 0$ and assuming that $F(0)F'(0) = 0$ gives

$$G(\xi)\frac{d^2F}{d\xi^2} + F\frac{dF}{d\xi} = 0. \tag{6.8}$$

Equation (6.8) is solved numerically subject to (4.18) and (4.19) using a shooting-point, bisection method. Initial guesses, one high and one low, are made for $F'(0)$. For each of these guesses, MAPLE's dsolve command is used to integrate forward to some pre-chosen large time point. The integral constraint, equation (4.19), is then tested. The respective deviations from exactness are used to modify one of the initial guesses as in the classical bisection method and the whole procedure is iterated until satisfactory convergence is achieved. For each choice of $G(\xi)$ in (6.6) the mean velocity profile

$$\frac{dF}{d\xi} = (3\nu x)^{\frac{1}{3}}\bar{v}_x(x,y) \tag{6.9}$$

is calculated. Since $H(\xi)$ is given by (5.12) the solutions are generated by the Lie point symmetry (5.13) as for a laminar jet.

For comparison we also consider two solutions with eddy viscosity constant across the jet. Both solutions are obtained from (5.16). Firstly, for a laminar jet, $\nu_{T_0} = 0$ and (5.16) reduces to

$$(3\nu x)^{\frac{1}{3}}\bar{v}_x(x,y) = \frac{1}{2}\left(\frac{3J}{2\rho}\right)^{\frac{2}{3}}\operatorname{sech}^2\left[\frac{1}{2}\left(\frac{3J}{2\rho}\right)^{\frac{1}{3}}\xi\right], \tag{6.10}$$

where ξ is given by (6.5). Secondly, adopting the Prandtl hypothesis, we set $\beta = \dfrac{1}{2}$ and rewrite (5.16) in terms of the similarity variable (6.5). This gives

$$(3\nu x)^{\frac{1}{3}}\,\bar{v}_x(x,y) = \frac{1}{2}\left(\frac{3J}{2\rho}\right)^{\frac{2}{3}}\frac{1}{\left(1+\dfrac{2}{3}\dfrac{\nu_{T_0}}{\nu}x^{\frac{1}{2}}\right)^{\frac{1}{3}}}\operatorname{sech}^2\left[\frac{1}{2}\left(\frac{3J}{2\rho}\right)^{\frac{1}{3}}\frac{\xi}{\left(1+\dfrac{2}{3}\dfrac{\nu_{T_0}}{\nu}x^{\frac{1}{2}}\right)^{\frac{2}{3}}}\right]. \tag{6.11}$$

The effective viscosities of the solutions (6.10) and (6.11) are, from (5.14),

$$E = \nu, \qquad E = \nu + \nu_{T_0}x^{\frac{1}{2}}. \tag{6.12}$$

In Figure 1 the scaled velocity $(3\nu x)^{\frac{1}{3}}\bar{v}_x(x,y)$ is plotted against ξ for the four jets with effective viscosity given in (6.7) and (6.12). The ratio ν_{T_0}/ν describes the apparent increase in the viscosity of the mean flow. For illustrative purposes we took $\nu_{T_0}/\nu = 100$. From Figure 1 we see that the maximum velocities of the turbulent jets are less than that of the laminar jet but the widths of the turbulent jets are greater than that of the laminar jet. This is due to the increase in the apparent viscosity of the mean flow as described by the effective viscosity. Diffusion in the turbulent jets is therefore greater than in the laminar jet and hence the turbulent jets are wider than the laminar jet. We also see from

Table 1. Group invariant solution for special cases of the effective viscosity $E(x, y)$.

	$E(x,y)$	$H(x)$	$G(\xi)$	$\xi = \frac{y}{H^2(x)}$	$F(\xi)$	$\bar{v}_x(x,y) = \frac{1}{H(x)}\frac{dF}{d\xi}$
Laminar	ν	$(3\nu x)^{\frac{1}{3}}$	1	$\frac{y}{(3\nu x)^{\frac{2}{3}}}$	$\left(\frac{3J}{2\rho}\right)^{\frac{1}{3}} \cdot \tanh\left[\frac{1}{2}\left(\frac{3J}{2\rho}\right)^{\frac{1}{3}}\xi\right]$	$\frac{1}{2}\left(\frac{3J}{2\rho}\right)^{\frac{2}{3}}\frac{1}{H(x)} \cdot \operatorname{sech}^2\left[\frac{1}{2}\left(\frac{3J}{2\rho}\right)^{\frac{1}{3}}\xi\right]$
Power law $\beta > -1$	$E_0 x^{\beta}$	$\left(\frac{3E_0}{1+\beta}\right)^{\frac{1}{3}} x^{\frac{1}{3}(1+\beta)}$	1	$\left(\frac{1+\beta}{3E_0}\right)^{\frac{2}{3}} \frac{y}{x^{\frac{2}{3}(1+\beta)}}$	$\left(\frac{3J}{2\rho}\right)^{\frac{1}{3}} \cdot \tanh\left[\frac{1}{2}\left(\frac{3J}{2\rho}\right)^{\frac{1}{3}}\xi\right]$	$\frac{1}{2}\left(\frac{3J}{2\rho}\right)^{\frac{2}{3}}\frac{1}{H(x)} \cdot \operatorname{sech}^2\left[\frac{1}{2}\left(\frac{3J}{2\rho}\right)^{\frac{1}{3}}\xi\right]$
Prandtl's hypothesis $\beta = \frac{1}{2}$	$E_0 x^{\frac{1}{2}}$	$(2E_0)^{\frac{1}{3}} x^{\frac{1}{2}}$	1	$\frac{1}{(2E_0)^{\frac{2}{3}}}\frac{y}{x}$	$\left(\frac{3J}{2\rho}\right)^{\frac{1}{3}} \cdot \tanh\left[\frac{1}{2}\left(\frac{3J}{2\rho}\right)^{\frac{1}{3}}\xi\right]$	$\frac{1}{2}\left(\frac{3J}{2\rho}\right)^{\frac{2}{3}}\frac{1}{H(x)} \cdot \operatorname{sech}^2\left[\frac{1}{2}\left(\frac{3J}{2\rho}\right)^{\frac{1}{3}}\xi\right]$
Power law with ν not neglected $\beta > -1$	$\nu + \nu_{T_0} x^{\beta}$	$(3\nu x)^{\frac{1}{3}} \cdot \left[1 + \frac{1}{(1+\beta)}\frac{\nu_{T_0}}{\nu} x^{\beta}\right]^{\frac{1}{3}}$	1	$\frac{y}{(3\nu x)^{\frac{2}{3}}\left(1+\frac{1}{(1+\beta)}\frac{\nu_{T_0}}{\nu}x^{\beta}\right)^{\frac{2}{3}}}$	$\left(\frac{3J}{2\rho}\right)^{\frac{1}{3}} \cdot \tanh\left[\frac{1}{2}\left(\frac{3J}{2\rho}\right)^{\frac{1}{3}}\xi\right]$	$\frac{1}{2}\left(\frac{3J}{2\rho}\right)^{\frac{2}{3}}\frac{1}{H(x)} \cdot \operatorname{sech}^2\left[\frac{1}{2}\left(\frac{3J}{2\rho}\right)^{\frac{1}{3}}\xi\right]$
Exponential decrease of eddy viscosity across the jet	$\nu + \nu_{T_0} \exp\left(-\xi^2\right)$	$(3\nu x)^{\frac{1}{3}}$	$1 + \frac{\nu_{T_0}}{\nu} \exp\left(-\xi^2\right)$	$\frac{y}{(3\nu x)^{\frac{2}{3}}}$	Numerical solution	Numerical solution
Algebraic decrease of eddy viscosity across the jet	$\nu + \frac{\nu_{T_0}}{1+\xi^2}$	$(3\nu x)^{\frac{1}{3}}$	$1 + \frac{\nu_{T_0}}{\nu(1+\xi^2)}$	$\frac{y}{(3\nu x)^{\frac{2}{3}}}$	Numerical solution	Numerical solution

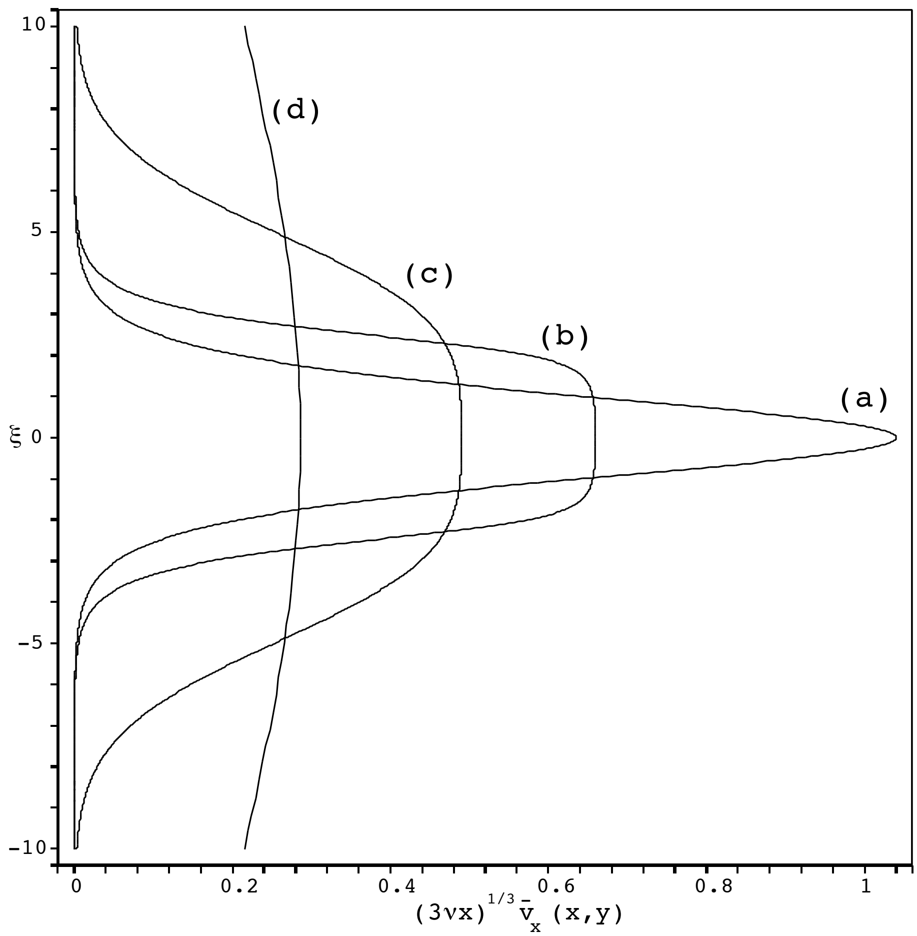

Figure 1: Scaled mean velocity profile $(3\nu x)^{\frac{1}{3}}\bar{v}_x(x,y)$ plotted against $\xi = \dfrac{y}{(3\nu x)^{\frac{2}{3}}}$ for effective viscosities (a) $E = \nu$, (b) $E = \nu + \nu_{T_0}\exp\left(-\xi^2\right)$, (c) $E = \nu + \dfrac{\nu_{T_0}}{1+\xi^2}$ and (d) $E = \nu + \nu_{T_0}x^{\frac{1}{2}}$, where $\nu_{T_0}/\nu = 100$, $J/2\rho = 1$ and $x = 0.5$.

Figure 1 that when the eddy viscosity is not constant across the jet but decreases then the width of the jet also decreases. When the eddy viscosity decreases exponentially across the jet, the jet is narrower than when it decreases more slowly algebraically. This can also be explained in terms of a reduction in the diffusion of the mean flow.

The effective viscosities which have been considered in this paper are summarised in Table 1.

7 Conclusions

We have derived the general form for the group invariant solution for the stream function and effective viscosity for a two-dimensional turbulent free jet. The eddy viscosity can vary across the jet through dependence on the similarity variable as well as in the direction of the jet. When Prandtl's hypothesis is imposed and the kinematic viscosity can be neglected, the eddy viscosity is proportional to the square root of the distance from the orifice measured in the direction of the jet.

When the eddy viscosity is constant across the jet the mean velocity profile, $\bar{v}_x(x, y)/\bar{v}_x(x, 0)$, plotted against the similarity variable ξ is the same for all turbulent jets and is also the same as the velocity profile of a laminar jet. This agrees with the observation of Görtler [7]. When the eddy viscosity decrease across the jet the width of the jet decreases due to the decrease in the diffusion of the mean flow. The exponential decrease in the eddy viscosity across the jet gave a narrower jet than the slower algebraic decrease in eddy viscosity.

The Lie point symmetry which generates the group invariant solution for a two-dimensional laminar free jet with infinite mean velocity at the orifice is given by (5.13). It is of interest that the group invariant solution for turbulent jets with $H'(x)H^2(x)$ constant are also generated by (5.13) and for these jets the eddy viscosity varies across the breadth of the jet as well as in the direction of the jet. We see from (4.20) that in the Lie point symmetry only the coefficient $\xi^1(x)$ can be changed by the turbulence. When the effective viscosity is a power law the Lie point symmetry is of the form (5.11). When the kinematic viscosity cannot be negleted and the eddy viscosity is a power law the Lie point symmetry is given by (5.17). This solution was the only solution not generated by a scaling transformation.

The majority of interesting practical turbulent flows are three-dimensional phenomena. The simplest extension of this analysis to three dimensions would be to a turbulent free jet with axisymmetric mean flow and described by eddy viscosity.

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