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Partial Noether Operators and First Integrals for a System with two Degrees of Freedom

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Abstract

We construct all partial Noether operators corresponding to a partial Lagrangian for a system with two degrees of freedom. Then all the first integrals are obtained explicitly by utilizing a Noether-like theorem with the help of the partial Noether operators. We show how the first integrals can be constructed for the system without the need of a variational principle although the Lagrangian $L = y'^2/2 + z'^2/2 - v(y, z)$ does exist for the system. Our objective is twofold: one is to see the effectiveness of the partial Noether approach and the other to determine all the first integrals of the system under study which have not been reported before. Thus, we deduce a complete classification of the potentials $v(y, z)$ for which first integrals exist. This can give rise to further studies on systems which are not Hamiltonian via partial Noether operators and the construction of first integrals from a partial Lagrangian viewpoint.

1 Introduction

The notion of partial Noether operators and partial Lagrangians are important in the construction of first integrals for ordinary differential equations that in general do not admit a Lagrangian. Most of the equations that arise in applications do not have a Lagrangian, e.g. in $y'' = y^2 + z^2,\ z'' = y$, no variational problem exists (Douglas [3]). Similarly for $y'' = y^2 + z^2,\ z'' = 0$, the family is non-extremal. So the question is how one can construct first integrals for equations without a variational principle. The objective of this paper is to classify all the partial Noether operators and to construct all first integrals for a system with two degrees of freedom which is Hamiltonian in order to see the effectiveness of using the partial Noether approach. Moreover, we obtain all the first integrals of the Hamiltonian system under consideration. These have not been obtained before in [6].

Hamiltonian systems frequently arise in classical mechanics, in non-linear oscillations and non-linear dynamics [2, 7, 12]. In classical mechanics Hamiltonian systems appear as physical systems. There are some important papers dealing with Hamiltonian systems. The point symmetries of a Hamiltonian system for two degrees of freedom were obtained by

Damianou and Sophocleous [4]. The point symmetry properties of a Lagrangian system for two degrees of freedom were also considered by Sen [16]. Symmetry group classification of a three dimensional Hamiltonian system was investigated by Damianou and Sophocleous [5]. Classification of Noether symmetries for Lagrangian systems with three degrees of freedom were also attempted (Damianou and Sophocleous [6]). Herein, the two degrees of freedom system Noether symmetries are reported.

Noether's theorem [13] provides the relationship between symmetries and the conserved quantities for Euler-Lagrange differential equations once their Noether symmetries are known and is indeed a powerful method to construct conservation laws. However, there are some direct methods ([1, 8, 11, 15, 17, 18]) as well. In [9, 10] a Noether-like theorem is invoked, which gives the first integrals for the differential equations without the use of a Lagrangian. In [14], the authors showed that the first integrals corresponding to the Noether and partial Noether operators of a variable coefficient linear system of two equations are the same. The difference occurs in the guage terms.

In this paper, we obtain all the partial Noether operators and first integrals for the two dimensional system that has been studied before via a standard Lagrangian in [6]. The previous works [6] did not present the first integrals for the Lagrangian system of two degrees of freedom $L = \frac{1}{2}y'^2 + \frac{1}{2}z'^2 - v(y,z)$. Here we take an alternative viewpoint. We first obtain the partial Noether operators and then construct the corresponding first integrals of such a system. This, hopefully, will give rise to further studies in the classification of the partial Noether operators for more general systems of two, three and four degrees of freedom and the construction of first integrals from a partial Lagrangian viewpoint.

Suppose that a particle is moving in the (y,z) plane with potential $v(y,z)$. The Hamiltonian system in two dimensions is given by

$$H(p_1, p_2, x, y) = \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 + v(y,z), \tag{1.1}$$

where $p_1 = \partial L/\partial y' = y',\ p_2 = \partial L/\partial z' = z'$.

In Newton's form one has

$$\begin{cases} y'' + v_y = 0, \\ z'' + v_z = 0, \end{cases} \tag{1.2}$$

corresponding to Lagrangian

$$L = \frac{1}{2}y'^2 + \frac{1}{2}z'^2 - v(y,z). \tag{1.3}$$

We start with the known definition [14].

Definition. Let

$$X = \xi(x,u)\frac{\partial}{\partial x} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha}. \tag{1.4}$$

be a generator in (x,u) space, where $u = (u^1, u^2) = (y,z)$, be the dependent variable and x be the independent variable. The operator X is said to be a partial Noether operator corresponding to a partial Lagrangian $L(x, u, u')$ of

$$u'' = M(x, u, u'), \tag{1.5}$$

if it can be calculated from

$$X^{[1]}L + (D_x\xi)L = (\eta^\alpha - \xi u_x^\alpha)\frac{\delta L}{\delta u^\alpha} + D_x(B), \tag{1.6}$$

for some function $B(x, u)$. In the above equation D_x is the total derivative operator with respect to x given as

$$D_x = \frac{\partial}{\partial x} + u_x^\alpha \frac{\partial}{\partial u^\alpha} + u_{xx}^\alpha \frac{\partial}{\partial u_x^\alpha} +, ...,. \tag{1.7}$$

The summation convention is adopted for repeated indices and the derivative of u^α with respect to x is defined as

$$u_x^\alpha = u_1^\alpha = D_x(u^\alpha), \ \alpha = 1, 2. \tag{1.8}$$

Now we recall the Noether-like theorem [10].

Theorem. If X in (1.4) is a partial Noether operator corresponding to a partial Lagrangian $L(x, u, u')$ of (1.5), then a first integral of (1.5) associated with X can be determined from

$$I = B - [\xi L + (\eta^\alpha - \xi u_x^\alpha)\frac{\partial L}{\partial u_x^\alpha}]. \tag{1.9}$$

2 Partial Noether operators of (1.2)

The operator X in (1.4) is a partial Noether operator for the system (1.2) with respect to the partial Lagrangian

$$L = \frac{1}{2}y'^2 + \frac{1}{2}z'^2 \tag{2.1}$$

if and only if it satisfies the system

$$\xi_y = 0, \ \xi_z = 0, \tag{2.2}$$

$$\eta_y^1 - \frac{1}{2}\xi_x = 0, \ \eta_z^2 - \frac{1}{2}\xi_x = 0, \ \eta_z^1 + \eta_y^2 = 0, \tag{2.3}$$

$$\eta_x^1 = -\xi v_y + B_y, \tag{2.4}$$

$$\eta_x^2 = -\xi v_z + B_z, \tag{2.5}$$

$$\eta^1 v_y + \eta^2 v_z + B_x = 0. \tag{2.6}$$

From equations (2.2) and (2.3), we find that

$$\xi = \alpha(x), \tag{2.7}$$

$$\eta^1 = \frac{1}{2}\alpha' y - C(x)z + S(x), \tag{2.8}$$

$$\eta^2 = \frac{1}{2}\alpha' z + C(x)y + F(x). \tag{2.9}$$

The substitution of these expressions into (2.4) and (2.5) results in

$$B = \frac{1}{4}\alpha'' y^2 - C'(x)yz + yS'(x) + \alpha(x)v + T(x,z), \tag{2.10}$$

where

$$T = \frac{1}{4}\alpha'' z^2 + F'(x)z + U(x),\ C(x) = A. \tag{2.11}$$

The replacement of η^1, η^2 and B in (2.6) leads to the following equation

$$(\frac{1}{2}\alpha' y - Az + S(x))v_y + (\frac{1}{2}\alpha' z + Ay + F(x))v_z$$
$$+\frac{1}{4}\alpha'''(y^2+z^2) + yS''(x) + \alpha'(x)v + F''(x)z + U'(x) = 0. \tag{2.12}$$

First of all if v is arbitrary, then we get the translation in x partial Noether operator with $B = v$ and obvious integral which is the Hamiltonian itself.

The differentiation of (2.12) with respect to x yields

$$(\frac{1}{2}\alpha'' y + S'(x))v_y + (\frac{1}{2}\alpha'' z + F'(x))v_z + \frac{1}{4}\alpha''''(y^2+z^2)$$
$$+yS'''(x) + \alpha''(x)v + F'''(x)z + U''(x) = 0. \tag{2.13}$$

In order to solve (2.13), two cases arise, viz. $\alpha'' \neq 0$ and $\alpha'' = 0$.

Case 1: $\alpha'' \neq 0$.

The division of equation (2.13) with α'' and then differentiation with respect to x gives

$$(\frac{S'}{\alpha''})' v_y + (\frac{F'}{\alpha''})' v_z + \frac{1}{4}(\frac{\alpha'''}{\alpha''})'(y^2+z^2) + (\frac{S'''}{\alpha''})' y + (\frac{F'''}{\alpha''})' z + (\frac{U''}{\alpha''})' = 0. \tag{2.14}$$

In equation (2.14) v satisfies the first order partial differential equation of the following form

$$\lambda_1 v_y + \lambda_2 v_z + \lambda_3(y^2+z^2) + \lambda_4 y + \lambda_5 z + \lambda_6 = 0. \tag{2.15}$$

For case 1 (when $\alpha'' \neq 0$), there are three subcases

Case 1.1: $\lambda_1 \neq 0,\ \lambda_2 \neq 0$.
Case 1.2: $\lambda_1 \neq 0,\ \lambda_2 = 0$.
Case 1.3: $\lambda_1 = 0,\ \lambda_2 \neq 0$.

For case 2 (when $\alpha'' = 0$), we obtain

$$S'(x)v_y + F'(x)v_z + yS'''(x) + F'''(x)z + U''(x) = 0. \tag{2.16}$$

or

$$\lambda_1 v_y + \lambda_2 v_z + \lambda_3 y + \lambda_4 z + \lambda_5 = 0. \tag{2.17}$$

In order to solve (2.17) the following three subcases are considered.

Case 2.1: $\lambda_1 \neq 0,\ \lambda_2 \neq 0.$
Case 2.2: $\lambda_1 \neq 0,\ \lambda_2 = 0.$
Case 2.3: $\lambda_1 = 0,\ \lambda_2 \neq 0.$

We provide details of the calculations for the subcase 1.1.1 of case 1.1. All the possibilities are summarized in a table (see below).

Case 1.1: $\lambda_1 \neq 0,\ \lambda_2 \neq 0.$
The following subcass arise.

Case 1.1.1: $A \neq 0,\ S(x) \neq 0,\ F(x) \neq 0,\ U(x) \neq 0.$
The equation (2.15) gives

$$v = -\frac{\lambda_1^2\lambda_3 z^3}{3\lambda_2^3} - \frac{\lambda_3 y^2 z}{\lambda_2} + \frac{\lambda_1\lambda_3 y z^2}{\lambda_2^2} - \frac{\lambda_3 z^3}{3\lambda_2} + \frac{\lambda_1\lambda_4 z^2}{2\lambda_2^2} - \frac{\lambda_4 y z}{\lambda_2} - \frac{\lambda_5 z^2}{2\lambda_2}$$

$$-\frac{\lambda_6}{\lambda_2} z + f(\lambda_2 y - \lambda_1 z), \tag{2.18}$$

where

$$\lambda_2 y - \lambda_1 z = \phi \text{ or } y = \frac{\lambda_1 z}{\lambda_2} + \frac{\phi}{\lambda_2}. \tag{2.19}$$

The substitution of v (in terms of y from (2.19)) in equation (2.13) and then separation with respect to powers of z gives

$$\lambda_3 = 0,\ -\frac{\alpha'\lambda_1\lambda_4}{\lambda_2^2} + \frac{A\lambda_4}{\lambda_2} - \frac{\alpha'\lambda_5}{\lambda_2} - \frac{A\lambda_1\lambda_5}{\lambda_2^2} + \frac{1}{4}\alpha''' + \frac{\alpha'''\lambda_1^2}{4\lambda_2^2} = 0, \tag{2.20}$$

$$\left(\frac{\alpha'\lambda_1}{2\lambda_2} - A\right)(\lambda_2 f'(\phi)) - \frac{\lambda_4}{\lambda_2}\left(\frac{\alpha'\phi}{2\lambda_2} + S(x)\right) + \left(\frac{\alpha'}{2} + \frac{A\lambda_1}{\lambda_2}\right)\left(-\frac{\lambda_4\phi}{\lambda_2^2} - \frac{\lambda_6}{\lambda_2} - \lambda_1 f'(\phi)\right)$$

$$-\frac{\lambda_5}{\lambda_2}\left(\frac{A\phi}{\lambda_2} + F(x)\right) + \frac{\alpha'''\lambda_1\phi}{2\lambda_2^2} + \frac{\lambda_1}{\lambda_2}S''(x) - \frac{\alpha'\lambda_4\phi}{\lambda_2^2} - \frac{\alpha'\lambda_6}{\lambda_2} + F''(x) = 0, \tag{2.21}$$

$$\left(\frac{\alpha'\phi}{2\lambda_2} + S(x)\right)(\lambda_2 f'(\phi)) + \left(\frac{A\phi}{\lambda_2} + F(x)\right)\left(-\frac{\lambda_4\phi}{\lambda_2^2} - \frac{\lambda_6}{\lambda_2} - \lambda_1 f'(\phi)\right)$$

$$+\frac{\alpha'''\phi^2}{4\lambda_2^2} + \frac{\phi}{\lambda_2}S''(x) + \alpha' f(\phi) + U'(x) = 0. \tag{2.22}$$

Routine but lengthy calculations lead to

$$v = -\frac{\lambda_6}{\lambda_2} z + f(\phi), \tag{2.23}$$

where

$$f(\phi) = d_4 + d_5\phi \text{ and } \phi = \lambda_2 y - \lambda_1 z. \tag{2.24}$$

In this case we find that $\lambda_4 = 0,\ \lambda_5 = 0$ and

$$\alpha = d_1 + d_2 x + d_3 x^2, \tag{2.25}$$

$$S(x) = -\frac{3d_5\lambda_2}{2}(\frac{d_2x^2}{2} + \frac{d_3x^3}{3}) + \frac{A\lambda_6 x^2}{2\lambda_2} + \frac{A\lambda_1 d_5 x^2}{2} + d_6 x + d_7, \tag{2.26}$$

$$F(x) = \frac{d_5}{2}(A\lambda_2 + \frac{A\lambda_1^2}{\lambda_2})x^2 + \frac{3\lambda_6}{2\lambda_2}(\frac{d_2x^2}{2} + \frac{d_3x^3}{3}) + \frac{A\lambda_1\lambda_6 x^2}{2\lambda_2^2}$$
$$-\frac{\lambda_1}{\lambda_2}[-\frac{3}{2}d_5\lambda_2(\frac{d_2x^2}{2} + \frac{d_3x^3}{3}) + \frac{A\lambda_6x^2}{2\lambda_2} + \frac{A\lambda_1 d_5x^2}{2}] + d_8x + d_9, \tag{2.27}$$

$$U(x) = -d_5\lambda_2[-\frac{3}{2}d_5\lambda_2(\frac{d_2x^3}{6} + \frac{d_3x^4}{12}) + \frac{A\lambda_6x^3}{6\lambda_2} + \frac{A\lambda_1d_5x^3}{6} + \frac{d_6x^2}{2}$$
$$+d_7x] + (\frac{\lambda_6}{\lambda_2} + \lambda_1 d_5)[\frac{d_5}{6}(A\lambda_2 + \frac{A\lambda_1^2}{\lambda_2})x^3 + \frac{3\lambda_6}{2\lambda_2}(\frac{d_2x^3}{6} + \frac{d_3x^4}{12})$$
$$+\frac{A\lambda_1\lambda_6x^3}{6\lambda_2^2} - \frac{\lambda_1}{\lambda_2}(-\frac{3}{2}d_5\lambda_2(\frac{d_2x^3}{6} + \frac{d_3x^4}{12}) + \frac{A\lambda_6x^3}{6\lambda_2}$$
$$+\frac{A\lambda_1d_5x^3}{6}) + \frac{d_8x^2}{2} + d_9x] - d_4(d_2x + d_3x^2) + d_{10}. \tag{2.28}$$

The potential function for cases (1.1.2-2.3), as listed in the table in section 3 are

Case 1.1.2: $A = 0,\ S(x) = 0,\ F(x) = 0.$

$$v = -\frac{\lambda_4}{2\lambda_1}(y^2 + z^2) + \frac{1}{y^2}f(\frac{z}{y}). \tag{2.29}$$

Case 1.1.3: $A = 0,\ F(x) = \frac{\lambda_2}{\lambda_1}S(x).$

$$v = -\frac{\lambda_4}{2\lambda_1}(y^2 + z^2) + \frac{\nu}{(z - \frac{\lambda_2}{\lambda_1}y)^2}. \tag{2.30}$$

Case 1.1.4: $A \neq 0,\ S(x) \neq 0,\ F(x) \neq 0,\ U(x) = 0.$

$$v = -\frac{\lambda_4}{2\lambda_1}(y^2 + z^2). \tag{2.31}$$

Case 1.2: $\lambda_1 \neq 0,\ \lambda_2 = 0.$

At this point, the following cases should be considered.

Case 1.2.1: $A \neq 0,\ S(x) \neq 0,\ F(x) \neq 0,\ U(x) \neq 0.$

$$v = -\frac{\lambda_4}{2\lambda_1}(y^2+z^2) - \frac{\lambda_6}{\lambda_1}y + e_1 z + e_2. \tag{2.32}$$

Case 1.2.2: $A \neq 0,\ S(x) = 0,\ F(x) = 0.$

$$v = -\frac{\lambda_4}{2\lambda_1}(y^2+z^2) + \frac{\mu}{y^2+z^2}. \tag{2.33}$$

Case 1.3: $\lambda_1 = 0,\ \lambda_2 \neq 0.$

$$v = -\frac{\lambda_5}{2\lambda_2}(y^2+z^2) - \frac{\lambda_6}{\lambda_2}z + f_1 y + f_2. \tag{2.34}$$

Case 2.1: $\lambda_1 \neq 0,\ \lambda_2 \neq 0.$

The subcases of case 2.1:

Case 2.1.1: $A \neq 0,\ S(x) \neq 0,\ F(x) \neq 0,\ U(x) \neq 0.$

$$v = -\frac{\lambda_5}{\lambda_2}z + g_1(\lambda_2 y - \lambda_1 z) + g_2. \tag{2.35}$$

Case 2.1.2: $A \neq 0,\ d_2 = 0,\ S(x) = 0,\ F(x) = 0.$

$$v = \frac{U'(x)}{A}\arcsin\frac{y}{\sqrt{y^2+z^2}} + f(y^2+z^2). \tag{2.36}$$

Case 2.1.3: $A \neq 0,\ d_2 \neq 0,\ S(x) = 0,\ F(x) = 0.$

$$v = \frac{1}{y^2+z^2} f((y^2+z^2)^A \exp(-d_2 \arctan\frac{z}{y})). \tag{2.37}$$

Case 2.1.4: $A = 0,\ d_2 = 0,\ F(x) = \frac{\lambda_2}{\lambda_1}S(x).$

$$v = -\frac{\lambda_3}{2\lambda_1}(y^2+z^2) + f(\lambda_2 y - \lambda_1 z). \tag{2.38}$$

Case 2.2: $\lambda_1 \neq 0,\ \lambda_2 = 0.$

The following cases need to be considered.

Case 2.2.1: $A \neq 0,\ d_2 \neq 0,\ \lambda_3 = 0.$

$$v = -\frac{\lambda_5}{\lambda_1}y + h_1 z + h_2. \tag{2.39}$$

Case 2.2.2: $A \neq 0,\ d_2 \neq 0,\ \lambda_3 \neq 0.$

$$v = -\frac{\lambda_3}{2\lambda_1}(y^2+z^2) - \frac{\lambda_5}{\lambda_1}y + h_1 z + h_2. \tag{2.40}$$

Case 2.3: $\lambda_1 = 0,\ \lambda_2 \neq 0.$

$$v = -\frac{\lambda_5}{\lambda_2}z + k_1 y + k_2.$$

where $\lambda_i s,\ e_i s,\ f_i s,\ g_i s,\ h_i s$ and $k_i s$ are constants.

Comparison of Lagrangian and partial Lagrangian approaches.

In [6], Damianou and Sophocleous have obtained the Noether point symmetries for a two degrees of freedom Lagrangian system and the results for one degree of freedom system were also reviewed in their paper. They did not provide the complete classification for the Noether symmetries as all the higher dimensional symmetry cases are missing for the two-dimensional Lagrangian system. Moreover, the first integrals corresponding to Noether symmetries for the system under study were also not given in their paper.

In this paper we have provided the complete classification for the partial Noether operators and first integrals are constructed by means of a partial Lagrangian approach with the help of partial Noether operators for a system with two degrees of freedom. For partial Noether operators obtained herein we have recovered all the cases as given in the case of Noether symmetries and we have got some new results summarized in the table that have not been obtained in the earlier work [6]. The system of determining equations obtained for partial Noether operators are similar to the case of Noether symmetries [6]. The reason being that $\delta L/\delta y$ and $\delta L/\delta z$ are independent of derivatives and the algebras for both cases are isomorphic. We give an alternative viewpoint to construct potential functions using the notion of a partial Lagrangian. In fact, a Lagrangian exists for the system under consideration but we wanted to see the effectiveness of a partial Lagrangian approach. We used a similar classification criteria for partial Noether operators as the authors in [6] have performed for the case of Noether symmetries. Then the first integrals are obtained by utilizing a partial Noether's theorem with the help of partial Noether operators via a partial Lagrangian.

The authors in [6] could also have provide the complete classification for a two degrees of freedom Lagrangian system and the first integrals could have been constructed by using a classical Noether's theorem but they did not do so.

The partial Noether operators and B in each case are given in the following table (section 3) by choosing each constant equal to one and the rest equal to zero.

3 First Integrals

If X in (1.4) is a partial Noether operator corresponding to the partial Lagrangian (2.1) for the system (1.2), then a first integral of (1.2) associated with X is constructed from the formula (1.9).

For each case, the first integrals are given in the following table. Precisely, the partial Noether operators and the first integrals of the two cases that arise are listed in the table. Note that there are three subcases in the first case and three subcases in the second case as given in the table.

Partial Operators X_i	Guage Terms B	First Integrals I_i
Case 1.1.1 $(A \neq 0,\ S(x) \neq 0,\ F(x) \neq 0,\ U(x) \neq 0)$		
$X_1 = (-z + \frac{\lambda_6}{2\lambda_2}x^2)\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}$,	$B = \frac{\lambda_6}{\lambda_2}xy$,	$I_1 = y'z - yz' + \frac{\lambda_6}{\lambda_2}xy - \frac{\lambda_6}{2\lambda_2}x^2y'$,
$X_2 = \frac{\partial}{\partial x}$,	$B = -\frac{\lambda_6}{\lambda_2}z$,	$I_2 = -\frac{\lambda_6}{\lambda_2}z + \frac{1}{2}(y'^2 + z'^2)$,
$X_3 = x\frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial y} + (\frac{z}{2} + \frac{3\lambda_6}{4\lambda_2}x^2)\frac{\partial}{\partial z}$,	$B = \frac{\lambda_6}{2\lambda_2}xz + \frac{\lambda_6^2}{4\lambda_2^2}x^3$,	$I_3 = \frac{\lambda_6}{2\lambda_2}xz + \frac{\lambda_6^2}{4\lambda_2^2}x^3 - \frac{3\lambda_6}{4\lambda_2}x^2z' + \frac{x}{2}(y'^2 + z^2) - \frac{1}{2}(yy' + zz')$,
$X_4 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} + (xz + \frac{\lambda_6}{2\lambda_2}x^3)\frac{\partial}{\partial z}$,	$B = \frac{1}{2}(y^2 + z^2) + \frac{\lambda_6}{2\lambda_2}x^2z + \frac{\lambda_6^2}{8\lambda_2^2}x^4$,	$I_4 = \frac{1}{2}(y^2 + z^2) + \frac{\lambda_6}{2\lambda_2}x^2z + \frac{\lambda_6^2}{8\lambda_2^2}x^4 - \frac{\lambda_6}{2\lambda_2}x^3z' + \frac{x^2}{2}(y'^2 + z^2) - x(yy' + zz')$,
$X_5 = x\frac{\partial}{\partial y}$,	$B = y$,	$I_5 = y - xy'$,
$X_6 = \frac{\partial}{\partial y}$,	$B = 0$,	$I_6 = -y'$,
$X_7 = x\frac{\partial}{\partial z}$,	$B = z + \frac{\lambda_6}{2\lambda_2}x^2$,	$I_7 = z - xz' + \frac{\lambda_6}{2\lambda_2}x^2$,
$X_8 = \frac{\partial}{\partial z}$.	$B = \frac{\lambda_6}{\lambda_2}x$.	$I_8 = -z' + \frac{\lambda_6}{\lambda_2}x$.
Case 1.1.2 $(A = 0,\ S(x) = 0,\ F(x) = 0)$		
$X_1 = -\frac{\lambda_1}{4\lambda_4}\frac{\partial}{\partial x}$,	$B = \frac{1}{8}(y^2 + z^2) - \frac{\lambda_1}{4\lambda_4 y^2}f(\frac{z}{y})$,	$I_1 = \frac{1}{8}(y^2 + z^2) - \frac{\lambda_1}{4\lambda_4 y^2}f(\frac{z}{y}) - \frac{\lambda_1}{8\lambda_4}(y'^2 + z'^2)$,
$X_2 = \exp(2\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times [\frac{\partial}{\partial x} + \sqrt{\frac{\lambda_4}{\lambda_1}}(y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z})]$,	$B = \exp(2\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times [\frac{\lambda_4}{2\lambda_1}(y^2 + z^2) + \frac{1}{y^2}f(\frac{z}{y})]$,	$I_2 = \exp(2\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times [\frac{\lambda_4}{2\lambda_1}(y^2 + z^2) + \frac{1}{y^2}f(\frac{z}{y}) + \frac{1}{2}(y'^2 + z'^2) - \sqrt{\frac{\lambda_4}{\lambda_1}}(yy' + zz')]$,
$X_3 = \exp(-2\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times [\frac{\partial}{\partial x} - \sqrt{\frac{\lambda_4}{\lambda_1}}(y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z})]$.	$B = \exp(-2\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times [\frac{\lambda_4}{2\lambda_1}(y^2 + z^2) + \frac{1}{y^2}f(\frac{z}{y})]$.	$I_3 = \exp(-2\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times [\frac{\lambda_4}{2\lambda_1}(y^2 + z^2) + \frac{1}{y^2}f(\frac{z}{y}) + \frac{1}{2}(y'^2 + z'^2) + \sqrt{\frac{\lambda_4}{\lambda_1}}(yy' + zz')]$.
Case 1.1.3 $(A = 0,\ F(x) = \frac{\lambda_2}{\lambda_1}S(x))$		
$X_1 = -\frac{\lambda_1}{4\lambda_4}\frac{\partial}{\partial x}$,	$B = \frac{1}{8}(y^2 + z^2) - \frac{\lambda_1\nu}{4\lambda_4(z - \frac{\lambda_2}{\lambda_1}y)^2}$,	$I_1 = \frac{1}{8}(y^2 + z^2) - \frac{\lambda_1\nu}{4\lambda_4(z - \frac{\lambda_2}{\lambda_1}y)^2} - \frac{\lambda_1}{8\lambda_4}(y'^2 + z'^2)$,
$X_2 = \exp(2\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times [\frac{\partial}{\partial x} + \sqrt{\frac{\lambda_4}{\lambda_1}}(y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z})]$,	$B = \exp(2\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times [\frac{\lambda_4}{2\lambda_1}(y^2 + z^2) + \frac{\nu}{(z - \frac{\lambda_2}{\lambda_1}y)^2}]$,	$I_2 = \exp(2\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times [\frac{\lambda_4}{2\lambda_1}(y^2 + z^2) + \frac{\nu}{(z - \frac{\lambda_2}{\lambda_1}y)^2} + \frac{1}{2}(y'^2 + z'^2) - \sqrt{\frac{\lambda_4}{\lambda_1}}(yy' + zz')]$,
$X_3 = \exp(-2\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times [\frac{\partial}{\partial x} - \sqrt{\frac{\lambda_4}{\lambda_1}}(y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z})]$,	$B = \exp(-2\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times [\frac{\lambda_4}{2\lambda_1}(y^2 + z^2) + \frac{\nu}{(z - \frac{\lambda_2}{\lambda_1}y)^2}]$,	$I_3 = \exp -(2\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times [\frac{\lambda_4}{2\lambda_1}(y^2 + z^2) + \frac{\nu}{(z - \frac{\lambda_2}{\lambda_1}y)^2} + \frac{1}{2}(y'^2 + z'^2) + \sqrt{\frac{\lambda_4}{\lambda_1}}(yy' + zz')]$,

$X_4=\exp(\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times$ $[\frac{\partial}{\partial y}+\frac{\lambda_2}{\lambda_1}\frac{\partial}{\partial z}]$, $X_5=\exp(-\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times$ $[\frac{\partial}{\partial y}+\frac{\lambda_2}{\lambda_1}\frac{\partial}{\partial z}]$.	$B=\sqrt{\frac{\lambda_4}{\lambda_1}}\exp(\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times$ $[y+\frac{\lambda_2}{\lambda_1}z]$, $B=-\sqrt{\frac{\lambda_4}{\lambda_1}}\exp(-\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times$ $[y+\frac{\lambda_2}{\lambda_1}z]$.	$I_4=\exp(\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times$ $[\sqrt{\frac{\lambda_4}{\lambda_1}}(y+\frac{\lambda_2}{\lambda_1}z)-y'-\frac{\lambda_2}{\lambda_1}z']$, $I_5=-\exp(-\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times$ $[\sqrt{\frac{\lambda_4}{\lambda_1}}(y+\frac{\lambda_2}{\lambda_1}z)+y'+\frac{\lambda_2}{\lambda_1}z']$.
Case 1.1.4 $(A\neq 0,\ S(x)\neq 0,\ F(x)\neq 0,\ U(x)=0)$		
$X_1=-\frac{\lambda_1}{4\lambda_4}\frac{\partial}{\partial x}$,	$B=\frac{1}{8}(y^2+z^2)$,	$I_1=\frac{1}{8}(y^2+z^2)-\frac{\lambda_1}{8\lambda_4}(y'^2+z'^2)$,
$X_2=\exp(2\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times$ $[\frac{\partial}{\partial x}+\sqrt{\frac{\lambda_4}{\lambda_1}}(y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z})]$,	$B=\exp(2\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times$ $[\frac{\lambda_4}{2\lambda_1}(y^2+z^2)]$,	$I_2=\exp(2\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times$ $[\frac{\lambda_4}{2\lambda_1}(y^2+z^2)+\frac{1}{2}(y'^2+z'^2)$ $-\sqrt{\frac{\lambda_4}{\lambda_1}}(yy'+zz')]$,
$X_3=\exp(-2\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times$ $[\frac{\partial}{\partial x}-\sqrt{\frac{\lambda_4}{\lambda_1}}(y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z})]$,	$B=\exp(-2\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times$ $[\frac{\lambda_4}{2\lambda_1}(y^2+z^2)]$,	$I_3=\exp(-2\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times$ $[\frac{\lambda_4}{2\lambda_1}(y^2+z^2)+\frac{1}{2}(y'^2+z'^2)$ $+\sqrt{\frac{\lambda_4}{\lambda_1}}(yy'+zz')]$,
$X_4=\exp(\sqrt{\frac{\lambda_4}{\lambda_1}}x)\frac{\partial}{\partial y}$,	$B=y\sqrt{\frac{\lambda_4}{\lambda_1}}\exp(\sqrt{\frac{\lambda_4}{\lambda_1}}x)$,	$I_4=\exp(\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times$ $[\sqrt{\frac{\lambda_4}{\lambda_1}}y-y']$,
$X_5=\exp(-\sqrt{\frac{\lambda_4}{\lambda_1}}x)\frac{\partial}{\partial y}$,	$B=-y\sqrt{\frac{\lambda_4}{\lambda_1}}\exp(-\sqrt{\frac{\lambda_4}{\lambda_1}}x)$,	$I_5=-\exp(-\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times$ $[\sqrt{\frac{\lambda_4}{\lambda_1}}y+y']$,
$X_6=\exp(\sqrt{\frac{\lambda_4}{\lambda_1}}x)\frac{\partial}{\partial z}$,	$B=z\sqrt{\frac{\lambda_4}{\lambda_1}}\exp(\sqrt{\frac{\lambda_4}{\lambda_1}}x)$,	$I_6=\exp(\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times$ $[\sqrt{\frac{\lambda_4}{\lambda_1}}z-z']$,
$X_7=\exp(-\sqrt{\frac{\lambda_4}{\lambda_1}}x)\frac{\partial}{\partial z}$,	$B=-z\sqrt{\frac{\lambda_4}{\lambda_1}}\exp(-\sqrt{\frac{\lambda_4}{\lambda_1}}x)$,	$I_7=-\exp(-\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times$ $[\sqrt{\frac{\lambda_4}{\lambda_1}}z+z']$,
$X_8=-z\frac{\partial}{\partial y}+y\frac{\partial}{\partial z}$.	$B=0$.	$I_8=zy'-yz'$.
Case 1.2.1 $(A\neq 0,\ S(x)\neq 0,\ F(x)\neq 0,\ U(x)\neq 0)$		
$X_1=-z\frac{\partial}{\partial y}+(y+\frac{\lambda_6}{\lambda_4})\frac{\partial}{\partial z}$,	$B=0$,	$I_1=y'z-yz'-\frac{\lambda_6}{\lambda_4}z'$,
$X_2=\frac{\partial}{\partial x}$,	$B=-\frac{\lambda_4}{2\lambda_1}(y^2+z^2)$ $-\frac{\lambda_6}{\lambda_1}y$,	$I_2=-\frac{\lambda_4}{2\lambda_1}(y^2+z^2)-\frac{\lambda_6}{\lambda_1}y$ $+\frac{1}{2}(y'^2+z'^2)$,
$X_3=\exp(2\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times$ $[\frac{\partial}{\partial x}+(\sqrt{\frac{\lambda_4}{\lambda_1}}y+\frac{\lambda_6}{\sqrt{\lambda_1\lambda_4}})\frac{\partial}{\partial y}$ $+\sqrt{\frac{\lambda_4}{\lambda_1}}z\frac{\partial}{\partial z}]$,	$B=\exp(2\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times$ $[\frac{\lambda_4}{2\lambda_1}(y^2+z^2)+\frac{\lambda_6}{\lambda_1}y$ $+\frac{\lambda_6^2}{2\lambda_1\lambda_4}]$,	$I_3=\exp(2\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times$ $[\frac{\lambda_4}{2\lambda_1}(y^2+z^2)+\frac{\lambda_6}{\lambda_1}y$ $+\frac{\lambda_6^2}{2\lambda_1\lambda_4}+\frac{1}{2}(y'^2+z'^2)$ $-\sqrt{\frac{\lambda_4}{\lambda_1}}(yy'+zz')-\frac{\lambda_6}{\sqrt{\lambda_1\lambda_4}}y']$,
$X_4=\exp(-2\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times$ $[\frac{\partial}{\partial x}-(\sqrt{\frac{\lambda_4}{\lambda_1}}y+\frac{\lambda_6}{\sqrt{\lambda_1\lambda_4}})\frac{\partial}{\partial y}$ $-\sqrt{\frac{\lambda_4}{\lambda_1}}z\frac{\partial}{\partial z}]$,	$B=\exp(-2\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times$ $[\frac{\lambda_4}{2\lambda_1}(y^2+z^2)+\frac{\lambda_6}{\lambda_1}y$ $+\frac{\lambda_6^2}{2\lambda_1\lambda_4}]$,	$I_4=\exp(-2\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times$ $[\frac{\lambda_4}{2\lambda_1}(y^2+z^2)+\frac{\lambda_6}{\lambda_1}y$ $+\frac{\lambda_6^2}{2\lambda_1\lambda_4}+\frac{1}{2}(y'^2+z'^2)$ $+\sqrt{\frac{\lambda_4}{\lambda_1}}(yy'+zz')+\frac{\lambda_6}{\sqrt{\lambda_1\lambda_4}}y']$,
$X_5=\exp(\sqrt{\frac{\lambda_4}{\lambda_1}}x)\frac{\partial}{\partial y}$,	$B=\exp(\sqrt{\frac{\lambda_4}{\lambda_1}}x)[\sqrt{\frac{\lambda_4}{\lambda_1}}y$ $+\frac{\lambda_6}{\sqrt{\lambda_1\lambda_4}}]$,	$I_5=\exp(\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times$ $[\sqrt{\frac{\lambda_4}{\lambda_1}}y+\frac{\lambda_6}{\sqrt{\lambda_1\lambda_4}}-y']$,

$X_6=\exp(-\sqrt{\frac{\lambda_4}{\lambda_1}}x)\frac{\partial}{\partial y}$,	$B=\exp(-\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times[-\sqrt{\frac{\lambda_4}{\lambda_1}}y-\frac{\lambda_6}{\sqrt{\lambda_1\lambda_4}}]$,	$I_6=-\exp(-\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times[\sqrt{\frac{\lambda_4}{\lambda_1}}y+\frac{\lambda_6}{\sqrt{\lambda_1\lambda_4}}+y']$,
$X_7=\exp(\sqrt{\frac{\lambda_4}{\lambda_1}}x)\frac{\partial}{\partial z}$,	$B=\sqrt{\frac{\lambda_4}{\lambda_1}}z\exp(\sqrt{\frac{\lambda_4}{\lambda_1}}x)$,	$I_7=\exp(\sqrt{\frac{\lambda_4}{\lambda_1}}x)[\sqrt{\frac{\lambda_4}{\lambda_1}}z-z']$,
$X_8=\exp(-\sqrt{\frac{\lambda_4}{\lambda_1}}x)\frac{\partial}{\partial z}$.	$B=-\sqrt{\frac{\lambda_4}{\lambda_1}}z\exp(-\sqrt{\frac{\lambda_4}{\lambda_1}}x)$.	$I_8=-\exp(-\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times[\sqrt{\frac{\lambda_4}{\lambda_1}}z+z']$.
Case 1.2.2 $(A\neq 0,\ S(x)=0,\ F(x)=0)$		
$X_1=-\frac{\lambda_1}{4\lambda_4}\frac{\partial}{\partial x}$,	$B=\frac{1}{8}(y^2+z^2)-\frac{\lambda_1\mu}{4\lambda_4}(y^2+z^2)$,	$I_1=\frac{1}{8}(y^2+z^2)-\frac{\lambda_1\mu}{4\lambda_4(y^2+z^2)}-\frac{\lambda_1}{8\lambda_4}(y'^2+z'^2)$,
$X_2=\exp(2\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times[\frac{\partial}{\partial x}+\sqrt{\frac{\lambda_4}{\lambda_1}}(y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z})]$,	$B=\exp(2\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times[\frac{\lambda_4}{2\lambda_1}(y^2+z^2)+\frac{\mu}{y^2+z^2}]$,	$I_2=\exp(2\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times[\frac{\lambda_4}{2\lambda_1}(y^2+z^2)+\frac{\mu}{y^2+z^2}+\frac{1}{2}(y'^2+z'^2)-\sqrt{\frac{\lambda_4}{\lambda_1}}(yy'+zz')]$,
$X_3=\exp(-2\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times[\frac{\partial}{\partial x}-\sqrt{\frac{\lambda_4}{\lambda_1}}(y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z})]$,	$B=\exp(-2\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times[\frac{\lambda_4}{2\lambda_1}(y^2+z^2)+\frac{\mu}{y^2+z^2}]$,	$I_3=\exp(-2\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times[\frac{\lambda_4}{2\lambda_1}(y^2+z^2)+\frac{\mu}{y^2+z^2}+\frac{1}{2}(y'^2+z'^2)+\sqrt{\frac{\lambda_4}{\lambda_1}}(yy'+zz')]$,
$X_4=-z\frac{\partial}{\partial y}+y\frac{\partial}{\partial z}$.	$B=0$.	$I_4=zy'-yz'$.
Case 1.3 $(\lambda_1=0,\ \lambda_2\neq 0)$		
$X_1=-(z+\frac{\lambda_6}{\lambda_5})\frac{\partial}{\partial y}+y\frac{\partial}{\partial z}$,	$B=0$,	$I_1=y'z-yz'+\frac{\lambda_6}{\lambda_5}y'$,
$X_2=\frac{\partial}{\partial x}$,	$B=-\frac{\lambda_5}{2\lambda_2}(y^2+z^2)-\frac{\lambda_6}{\lambda_2}z$,	$I_2=-\frac{\lambda_5}{2\lambda_2}(y^2+z^2)-\frac{\lambda_6}{\lambda_2}z+\frac{1}{2}(y'^2+z'^2)$,
$X_3=\exp(2\sqrt{\frac{\lambda_5}{\lambda_2}}x)\times[\frac{\partial}{\partial x}+\sqrt{\frac{\lambda_5}{\lambda_2}}y\frac{\partial}{\partial y}+(\sqrt{\frac{\lambda_5}{\lambda_2}}z+\frac{\lambda_6}{\sqrt{\lambda_2\lambda_5}})\frac{\partial}{\partial z}]$,	$B=\exp(2\sqrt{\frac{\lambda_5}{\lambda_2}}x)\times[\frac{\lambda_5}{2\lambda_2}(y^2+z^2)+\frac{\lambda_6}{\lambda_2}z+\frac{\lambda_6^2}{2\lambda_2\lambda_5}]$,	$I_3=\exp(2\sqrt{\frac{\lambda_5}{\lambda_2}}x)\times[\frac{\lambda_5}{2\lambda_2}(y^2+z^2)+\frac{\lambda_6}{\lambda_2}z+\frac{\lambda_6^2}{2\lambda_2\lambda_5}+\frac{1}{2}(y'^2+z'^2)-\sqrt{\frac{\lambda_5}{\lambda_2}}(yy'+zz')-\frac{\lambda_6}{\sqrt{\lambda_2\lambda_5}}z']$,
$X_4=\exp(-2\sqrt{\frac{\lambda_5}{\lambda_2}}x)\times[\frac{\partial}{\partial x}-\sqrt{\frac{\lambda_5}{\lambda_2}}y\frac{\partial}{\partial y}-(\sqrt{\frac{\lambda_5}{\lambda_2}}z+\frac{\lambda_6}{\sqrt{\lambda_2\lambda_5}})\frac{\partial}{\partial z}]$,	$B=\exp(-2\sqrt{\frac{\lambda_5}{\lambda_2}}x)\times[\frac{\lambda_5}{2\lambda_2}(y^2+z^2)+\frac{\lambda_6}{\lambda_2}z+\frac{\lambda_6^2}{2\lambda_2\lambda_5}]$,	$I_4=\exp(-2\sqrt{\frac{\lambda_5}{\lambda_2}}x)\times[\frac{\lambda_5}{2\lambda_2}(y^2+z^2)+\frac{\lambda_6}{\lambda_2}z+\frac{\lambda_6^2}{2\lambda_2\lambda_5}+\frac{1}{2}(y'^2+z'^2)+\sqrt{\frac{\lambda_5}{\lambda_2}}(yy'+zz')+\frac{\lambda_6}{\sqrt{\lambda_2\lambda_5}}z']$,
$X_5=\exp(\sqrt{\frac{\lambda_5}{\lambda_2}}x)\frac{\partial}{\partial z}$,	$B=\exp(\sqrt{\frac{\lambda_5}{\lambda_2}}x)\times[\sqrt{\frac{\lambda_5}{\lambda_2}}z+\frac{\lambda_6}{\sqrt{\lambda_2\lambda_5}}]$,	$I_5=\exp(\sqrt{\frac{\lambda_5}{\lambda_2}}x)\times[\sqrt{\frac{\lambda_5}{\lambda_2}}z+\frac{\lambda_6}{\sqrt{\lambda_2\lambda_5}}-z']$,
$X_6=\exp(-\sqrt{\frac{\lambda_5}{\lambda_2}}x)\frac{\partial}{\partial z}$,	$B=-\exp(-\sqrt{\frac{\lambda_5}{\lambda_2}}x)\times[\sqrt{\frac{\lambda_5}{\lambda_2}}z+\frac{\lambda_6}{\sqrt{\lambda_2\lambda_5}}]$,	$I_6=-\exp(-\sqrt{\frac{\lambda_5}{\lambda_2}}x)\times[\sqrt{\frac{\lambda_5}{\lambda_2}}z+\frac{\lambda_6}{\sqrt{\lambda_2\lambda_5}}+z']$,
$X_7=\exp(\sqrt{\frac{\lambda_5}{\lambda_2}}x)\frac{\partial}{\partial y}$,	$B=\sqrt{\frac{\lambda_5}{\lambda_2}}y\exp(\sqrt{\frac{\lambda_5}{\lambda_2}}x)$,	$I_7=\exp(\sqrt{\frac{\lambda_5}{\lambda_2}}x)[\sqrt{\frac{\lambda_5}{\lambda_2}}y-y']$,
$X_8=\exp(-\sqrt{\frac{\lambda_5}{\lambda_2}}x)\frac{\partial}{\partial y}$.	$B=-\sqrt{\frac{\lambda_5}{\lambda_2}}y\exp(-\sqrt{\frac{\lambda_5}{\lambda_2}}x)$.	$I_8=-\exp(-\sqrt{\frac{\lambda_5}{\lambda_2}}x)\times[\sqrt{\frac{\lambda_5}{\lambda_2}}y+y']$.

Case 2.1.1 $(A \neq 0,\ S(x) \neq 0,\ F(x) \neq 0,\ U(x) \neq 0)$		
$X_1 = (-z + \frac{\lambda_5}{2\lambda_2}x^2)\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}$,	$B = \frac{\lambda_5}{\lambda_2}xy$,	$I_1 = y'z - yz' + \frac{\lambda_5}{\lambda_2}xy - \frac{\lambda_5}{2\lambda_2}x^2y'$,
$X_2 = \frac{\partial}{\partial x}$,	$B = -\frac{\lambda_5}{\lambda_2}z$,	$I_2 = -\frac{\lambda_5}{\lambda_2}z + \frac{1}{2}(y'^2 + z'^2)$,
$X_3 = x\frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial y} + (\frac{z}{2} + \frac{3\lambda_5}{4\lambda_2}x^2)\frac{\partial}{\partial z}$,	$B = \frac{\lambda_5}{2\lambda_2}xz + \frac{\lambda_5^2}{4\lambda_2^2}x^3$,	$I_3 = \frac{\lambda_5}{2\lambda_2}xz + \frac{\lambda_5^2}{4\lambda_2^2}x^3 - \frac{3\lambda_5}{4\lambda_2}x^2z' + \frac{x}{2}(y'^2 + z^2) - \frac{1}{2}(yy' + zz')$,
$X_4 = x\frac{\partial}{\partial y}$,	$B = y$,	$I_4 = y - xy'$,
$X_5 = \frac{\partial}{\partial y}$,	$B = 0$,	$I_5 = -y'$,
$X_6 = x\frac{\partial}{\partial z}$,	$B = z + \frac{\lambda_5}{2\lambda_2}x^2$,	$I_6 = z - xz' + \frac{\lambda_5}{2\lambda_2}x^2$,
$X_7 = \frac{\partial}{\partial z}$.	$B = \frac{\lambda_5}{\lambda_2}x$.	$I_7 = -z' + \frac{\lambda_5}{\lambda_2}x$.
Case 2.1.2 $(A \neq 0,\ d_2 = 0,\ S(x) = 0,\ F(x) = 0)$		
$X_1 = -z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}$,	$B = U(x)$,	$I_1 = U(x) + zy' - yz'$,
$X_2 = \frac{\partial}{\partial x}$.	$B = v$.	$I_2 = v + \frac{1}{2}(y'^2 + z'^2)$.
Case 2.1.3 $(A \neq 0,\ d_2 \neq 0,\ S(x) = 0,\ F(x) = 0)$		
$X_1 = \frac{\partial}{\partial x}$,	$B = v$,	$I_1 = v + \frac{1}{2}(y'^2 + z'^2)$,
$X_2 = x\frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial y} + \frac{z}{2}\frac{\partial}{\partial z}$,	$B = xv$,	$I_2 = xv + \frac{x}{2}(y'^2 + z'^2) - \frac{1}{2}(yy' + zz')$,
$X_3 = -z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}$.	$B = 0$.	$I_3 = zy' - yz'$.
Case 2.1.4 $(A = 0,\ d_2 = 0,\ F(x) = \frac{\lambda_2}{\lambda_1}S(x))$		
$X_1 = \frac{\partial}{\partial x}$,	$B = v$,	$I_1 = v + \frac{1}{2}(y'^2 + z'^2)$,
$X_2 = \exp(\sqrt{\frac{\lambda_3}{\lambda_1}}x)(\frac{\partial}{\partial y} + \frac{\lambda_2}{\lambda_1}\frac{\partial}{\partial z})$,	$B = \sqrt{\frac{\lambda_3}{\lambda_1}}\exp(\sqrt{\frac{\lambda_3}{\lambda_1}}x) \times (y + \frac{\lambda_2}{\lambda_1}z)$,	$I_2 = \exp(\sqrt{\frac{\lambda_3}{\lambda_1}}x)[\sqrt{\frac{\lambda_3}{\lambda_1}}(y + \frac{\lambda_2}{\lambda_1}z) - y' - z']$,
$X_3 = \exp(-\sqrt{\frac{\lambda_3}{\lambda_1}}x)(\frac{\partial}{\partial y} + \frac{\lambda_2}{\lambda_1}\frac{\partial}{\partial z})$.	$B = -\sqrt{\frac{\lambda_3}{\lambda_1}}\exp(-\sqrt{\frac{\lambda_3}{\lambda_1}}x) \times (y + \frac{\lambda_2}{\lambda_1}z)$.	$I_3 = -\exp(-\sqrt{\frac{\lambda_3}{\lambda_1}}x)[\sqrt{\frac{\lambda_3}{\lambda_1}}(y + \frac{\lambda_2}{\lambda_1}z) + y' + z']$.
Case 2.2.1 $(\lambda_1 \neq 0,\ \lambda_2 = 0,\ \lambda_3 = 0)$		
$X_1 = -z\frac{\partial}{\partial y} + (y - \frac{\lambda_5}{2\lambda_1}x^2)\frac{\partial}{\partial z}$,	$B = -\frac{\lambda_5}{\lambda_1}xz$,	$I_1 = y'z - yz' - \frac{\lambda_5}{\lambda_1}xz + \frac{\lambda_5}{2\lambda_1}x^2z'$,
$X_2 = \frac{\partial}{\partial x}$,	$B = -\frac{\lambda_5}{\lambda_1}y$,	$I_2 = -\frac{\lambda_5}{\lambda_1}y + \frac{1}{2}(y'^2 + z'^2)$,
$X_3 = x\frac{\partial}{\partial x} + (\frac{y}{2} + \frac{3\lambda_5}{4\lambda_1}x^2)\frac{\partial}{\partial y} + \frac{z}{2}\frac{\partial}{\partial z}$,	$B = \frac{\lambda_5}{2\lambda_1}xy + \frac{\lambda_5^2}{4\lambda_1^2}x^3$,	$I_3 = \frac{\lambda_5}{2\lambda_1}xy + \frac{\lambda_5^2}{4\lambda_1^2}x^3 - \frac{3\lambda_5}{4\lambda_1}x^2y' + \frac{x}{2}(y'^2 + z^2) - \frac{1}{2}(yy' + zz')$,
$X_4 = x\frac{\partial}{\partial y}$,	$B = y + \frac{\lambda_5}{2\lambda_1}x^2$,	$I_4 = y - xy' + \frac{\lambda_5}{2\lambda_1}x^2$,
$X_5 = \frac{\partial}{\partial y}$,	$B = \frac{\lambda_5}{\lambda_1}x$,	$I_5 = \frac{\lambda_5}{\lambda_1}x - y'$,
$X_6 = x\frac{\partial}{\partial z}$,	$B = z$,	$I_6 = z - xz'$,
$X_7 = \frac{\partial}{\partial z}$.	$B = 0$.	$I_7 = -z'$.

Case 2.2.2 $(\lambda_1 \neq 0,\ \lambda_2 = 0,\ \lambda_3 \neq 0)$		
$X_1 = -z\frac{\partial}{\partial y} + (y + \frac{\lambda_5}{\lambda_3})\frac{\partial}{\partial z}$,	$B = 0$,	$I_1 = -y'z + yz' + \frac{\lambda_5}{\lambda_3}z'$,
$X_2 = \frac{\partial}{\partial x}$,	$B = -\frac{\lambda_3}{2\lambda_1}(y^2+z^2) - \frac{\lambda_5}{\lambda_1}y$,	$I_2 = -\frac{\lambda_3}{2\lambda_1}(y^2+z^2) - \frac{\lambda_5}{\lambda_1}y$ $+\frac{1}{2}(y'^2+z'^2)$,
$X_3 = \exp(\sqrt{\frac{\lambda_3}{\lambda_1}}x)\frac{\partial}{\partial y}$,	$B = \exp(\sqrt{\frac{\lambda_3}{\lambda_1}}x)(y\sqrt{\frac{\lambda_3}{\lambda_1}}$ $+\frac{\lambda_5}{\sqrt{\lambda_1\lambda_3}})$,	$I_3 = \exp(\sqrt{\frac{\lambda_3}{\lambda_1}}x)(y\sqrt{\frac{\lambda_3}{\lambda_1}}$ $+\frac{\lambda_5}{\sqrt{\lambda_1\lambda_3}} - y')$,
$X_4 = \exp(-\sqrt{\frac{\lambda_3}{\lambda_1}}x)\frac{\partial}{\partial y}$,	$B = -\exp(-\sqrt{\frac{\lambda_3}{\lambda_1}}x)(y\sqrt{\frac{\lambda_3}{\lambda_1}}$ $+\frac{\lambda_5}{\sqrt{\lambda_1\lambda_3}})$,	$I_4 = -\exp(-\sqrt{\frac{\lambda_3}{\lambda_1}}x)(y\sqrt{\frac{\lambda_3}{\lambda_1}}$ $+\frac{\lambda_5}{\sqrt{\lambda_1\lambda_3}} + y')$,
$X_5 = \exp(\sqrt{\frac{\lambda_3}{\lambda_1}}x)\frac{\partial}{\partial z}$,	$B = z\sqrt{\frac{\lambda_3}{\lambda_1}}\exp(\sqrt{\frac{\lambda_3}{\lambda_1}}x)$,	$I_5 = \exp(\sqrt{\frac{\lambda_3}{\lambda_1}}x)(z\sqrt{\frac{\lambda_3}{\lambda_1}} - z')$,
$X_6 = \exp(-\sqrt{\frac{\lambda_3}{\lambda_1}}x)\frac{\partial}{\partial z}$.	$B = -z\sqrt{\frac{\lambda_3}{\lambda_1}}\exp(-\sqrt{\frac{\lambda_3}{\lambda_1}}x)$.	$I_6 = -\exp(-\sqrt{\frac{\lambda_3}{\lambda_1}}x)(z\sqrt{\frac{\lambda_3}{\lambda_1}} + z')$.
Case 2.3 $(\lambda_1 = 0,\ \lambda_2 \neq 0)$		
$X_1 = (-z + \frac{\lambda_5}{2\lambda_2}x^2)\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}$,	$B = \frac{\lambda_5}{\lambda_2}xy$,	$I_1 = y'z - yz' + \frac{\lambda_5}{\lambda_2}xy$ $-\frac{\lambda_5}{2\lambda_2}x^2y'$,
$X_2 = \frac{\partial}{\partial x}$,	$B = -\frac{\lambda_5}{\lambda_2}z$,	$I_2 = -\frac{\lambda_5}{\lambda_2}z + \frac{1}{2}(y'^2+z'^2)$,
$X_3 = x\frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial y}$ $+(\frac{z}{2} + \frac{3\lambda_5}{4\lambda_2}x^2)\frac{\partial}{\partial z}$,	$B = \frac{\lambda_5}{2\lambda_2}xz + \frac{\lambda_5^2}{4\lambda_2^2}x^3$,	$I_3 = \frac{\lambda_5}{2\lambda_2}xz + \frac{\lambda_5^2}{4\lambda_2^2}x^3 - \frac{3\lambda_5}{4\lambda_2}x^2z'$ $+\frac{x}{2}(y'^2+z^2) - \frac{1}{2}(yy'+zz')$,
$X_4 = x\frac{\partial}{\partial z}$,	$B = z + \frac{\lambda_5}{2\lambda_2}x^2$,	$I_4 = z - xz' + \frac{\lambda_5}{2\lambda_2}x^2$,
$X_5 = \frac{\partial}{\partial z}$,	$B = \frac{\lambda_5}{\lambda_2}x$,	$I_5 = \frac{\lambda_5}{\lambda_2}x - z'$,
$X_6 = x\frac{\partial}{\partial y}$,	$B = y$,	$I_6 = y - xy'$,
$X_7 = \frac{\partial}{\partial y}$.	$B = 0$.	$I_7 = -y'$.

4 Conclusion

We have studied the partial Noether operators corresponding to a partial Lagrangian for a Hamiltonian system with two degrees of freedom. This problem was studied before via Noether symmetries in [6] wherein the authors did not provide the first integrals. In this work we have obtained both the partial Noether operators and the corresponding first integrals. We investigated the effectiveness of the partial Lagrangian approach which has provided all the first integrals. This study provides an alternative way to construct first integrals for equations for which we do not need a Lagrangian. The previous work [6], does not give the complete classification for the Hamiltonian system considered. In this paper we gave the complete classification for the underlying system via a partial Lagrangian approach and we have obtained more general results that were not discussed in [6]. This approach can give rise to further studies to classify systems which are not variational and to derive first integrals from a partial Lagrangian viewpoint.

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