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A Note on the Integrability of a Class of Nonlinear Ordinary Differential Equations

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Abstract

We study the integrability properties of the hierarchy of a class of nonlinear ordinary differential equations and point out some of the properties of these equations and their connection to the Ermakov-Pinney equation.

1 Introduction

It is well-known that the Ermakov-Pinney equation [4, 15] in its simplest form is given by

$$\omega'' = \frac{K}{\omega^3}, \tag{1.1}$$

where K is a constant. In most theoretical problems the sign of the constant K is immaterial and is usually rescaled to unity. In practical applications K is interpreted as the square of the angular momentum [18, 2] and hence it would be taken as positive to avoid 'collapse into the origin'. In the study of the time-dependent linear oscillator (in both the classical and quantal problem) the equation

$$\ddot{\rho} + \omega^2(t)\rho = \frac{1}{\rho^3} \tag{1.2}$$

occurs as the differential equation which determines the time-dependent rescaling of the space variable and the definition of 'new time' [10, 11, 12, 13, 6, 7, 8]. Equation (1.2) is the general form of (1.1).

Here we further discuss the integrability properties of a hierarchy of 'Euler' equations and state some of the properties of these equations [5] and their subsequent connection to the Ermakov-Pinney equation.

2 The 'Euler' hierarchy

We recall that Euler & Leach [5] considered the class of equations of the form

$$y^{(n+1)} = h(y, y^{(n)})y' \tag{2.1}$$

and in the case that (2.1) took the specific form

$$yy^{(4)} + \tfrac{5}{2}y'y''' = 0 \tag{2.2}$$

showed by means of a sequence of nonlocal transformations that it could be reduced to $\mathrm{d}^4Y/\mathrm{d}X^4 = 0$. If we assume that equation (2.1) admits the algebra

$$\Gamma_1 = \partial_x, \quad \Gamma_2 = x\partial_x + my\partial_y, \quad \Gamma_3 = x^2\partial_x + 2mxy\partial_y, \tag{2.3}$$

we obtain the equation

$$yy^{(n+1)} + \alpha y^{(n)}y' = 0,\ \alpha = \frac{n+1}{n-1}, n \neq 1 \tag{2.4}$$

provided $m = \frac{1}{2}(n-1)$ which is a particular case of a class of equations considered for their interesting integrability properties in [14].

The Ermakov-Pinney connection to (2.4) is found in the existence of the integrating factor, $y^{\frac{n+1}{n-1}-1}$. The corresponding generalised Ermakov-Pinney equation is

$$y^{(n)} = \frac{K}{y^{(n+1)/(n-1)}}, \tag{2.5}$$

where K is the constant of integration. Equation (2.5) has the algebra $sl(2,R)$ with representation

$$\begin{aligned} \Gamma_1 &= \partial_x, \\ \Gamma_2 &= x\partial_x + \frac{n+1}{2(n-1)}y\partial_y, \\ \Gamma_3 &= x^2\partial_x + \frac{n+1}{(n-1)}xy\partial_y. \end{aligned} \tag{2.6}$$

We note that (2.4) has a four-dimensional algebra, *videlicet* $sl(2,R) \oplus A_1$, since it is homogenous in both x and y[1].

3 Singularity analysis of equations (2.4)

We consider the singularity analysis for the class of equations (2.4) for $n = 2, 3, 4, 5, 6, 7$. A detailed presentation of the singularity analysis to ordinary differential equations can be found in [3, 17, 16]. The corresponding set of equations is given by

$$E_2:\ yy^{(3)} + 3y'y'' = 0 \tag{3.1}$$

[1]This is in contrast to (2.2) which has just the three Lie point symmetries, ∂_x, $x\partial_x$ and $y\partial_y$, that is, the algebra $A_2 \oplus A_1$. Interestingly, when (2.2) is reduced, the resulting third-order equation has three Lie point symmetries due to the presence of a hidden symmetry in (2.2)[9].

$$E_3: \; yy^{(4)} + 2y'y^{(3)} = 0 \tag{3.2}$$

$$E_4: \; yy^{(5)} + \frac{5}{3}y'y^{(4)} = 0 \tag{3.3}$$

$$E_5: \; yy^{(6)} + \frac{6}{4}y'y^{(5)} = 0 \tag{3.4}$$

$$E_6: \; yy^{(7)} + \frac{7}{5}y'y^{(6)} = 0 \tag{3.5}$$

$$E_7: \; yy^{(8)} + \frac{8}{6}y'y^{(7)} = 0. \tag{3.6}$$

Table 1: The exponents of the leading-order term for the above equations are given below:

Equation	Exponents
E_2	0, 1; $\frac{1}{2}$
E_3	0, 1, 2; 1
E_4	0, 1, 2, 3; $\frac{3}{2}$
E_5	0, 1, 2, 3, 4; 2
E_6	0, 1, 2, 3, 4, 5; $\frac{5}{2}$
E_7	0, 1, 2, 3, 4, 5, 6; 3

In general the exponents of the leading-order term are the integers 0, $n(1); n/2$, where n is the number of the equation. We note that the exponents are non-negative integers apart from the fractions found in the equations of odd number.

4 Observations

We have found that in the singularity analysis of the class of equations (2.4) the exponent of the leading-order term is either a non-negative integer or $\frac{1}{2}n$. This means that only those members of the class of equations given by (2.4) with n an even integer are candidates for the standard singularity analysis and then can only possess the Painlevé Property in the weak form. In this context we also note that the equations corresponding to even values of n are those which can be integrated without needing a nontrivial integrating factor. The resonance analysis for the $n = 3, 5$ and 7 cases also gives the following for the half integer exponent.

Equation	Resonances
E_2	$-1,\ 0,\ 1$
E_4	$-1,\ 0,\ 1,\ \pm\sqrt{\frac{3}{2}}$
E_6	$-1,\ 0,\ 1$, others

The resonances listed as 'others' are messily irrational and are not worth recording. The additional resonances for higher values of even n are much worse. Consequently the equations cannot possess the weak Painlevé Property apart from E_2. In the case of a zero exponent for the leading-order term one can substitute a MacLaurin expansion. The arbitrary constants of integration are the first $n+2$ coefficients in the expansion. The standard approach for the positive exponents is to make the transformation $y \longrightarrow 1/w$. In the case of E_2 the resonances are 0, -1 and -2. This indicates a Left Painlevé Series. There are no incompatibilities at the resonances. The situation becomes more complex with increasing order of the equation. In the case of E_3 the double exponent of 1 leads to a double zero and so a logarithmic term must be introduced. On the other hand the exponent 2 leads to a Left Painlevé Series. A new feature arises when one examines E_3. For the exponent 1 there is a standard Right Painlevé Series in fractional powers and for 3 a Left Painlevé Series. For the exponent 2 one has resonances of mixed sign and so the series is a complete Laurent expansion [1].

In terms of the trivial, that is, constant, integrating factor we may write

$$E_2: \quad \left(y^2\right)''' = 0, \tag{4.1}$$

that is, E_1 is really a linear equation of the third order. In a similar way we can write

$$E_4: \quad \left(yy'' - \tfrac{1}{3}y'^2\right)''' = 0, \tag{4.2}$$

which loses the linearity of (4.1), but can be written as

$$w'' = \frac{A_0 + A_1x + A_2x^2}{w^5}, \quad w = y^{1/3}, \tag{4.3}$$

which is of generalised Emden-Fowler structure and a generalised Ermakov-Pinney equation. For the next case we have

$$E_6: \quad \left(yy^{(4)} - \tfrac{8}{5}y'y''' + \tfrac{9}{10}y''^2\right)''' = 0, \tag{4.4}$$

which does not appear to admit the type of simplification mentioned above. A symmetry analysis of

$$yy'' - \tfrac{1}{3}y'^2 = 0 \tag{4.5}$$

gives eight Lie-point symmetries whilst

$$yy^{(4)} - \tfrac{8}{5}y'y''' + \tfrac{9}{10}y''^2 = 0 \tag{4.6}$$

gives ∂_x, $y\partial_y$, $x\partial_x$ and $x^2\partial_x + 5xy\partial_y$.

Although the members of the hierarchy with odd values of n may be integrated once to obtain the equivalent Ermakov-Pinney equation, they do not admit the ease of partial integrability shown by the members of the hierarchy for even n.

5 Conclusion

The equations (2.4) have the same basic singularity properties for even values of n. For n odd the members of the hierarchy do not allow for an easy integration. In terms of the reduced equation which is written in terms of the third derivative we find that E_4 reduces to a second-order ordinary differential equation with eight Lie point symmetries and thereafter we get an equation which has the same symmetries as the original equation. For example the next reduced equation in the hierarchy,

$$yy^{(6)} - \tfrac{12}{7}y'y^{(5)} + \tfrac{15}{7}y''y''' - \tfrac{8}{7}y'''^2 = 0,$$

gives the symmetries $\partial_x, y\partial_y, x\partial_x$ and $x^2\partial_x + 7xy\partial_y$. For general n the symmetries are the same for the first three plus $x^2\partial_x + (n-2)xy\partial_y$ or, as one may prefer,

$$\Gamma_1 = \partial_x,\ \Gamma_2 = x\partial_x + \tfrac{1}{2}(n-2)y\partial_y,\ \Gamma_3 = x^2\partial_x + (n-2)xy\partial_y \text{ and } \Gamma_4 = y\partial_y$$

with the algebra $sl(2,R) \oplus A_1$.

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References

[1] Andriopoulos K & Leach PGL, An interpretation of the presence of both positive and negative nongeneric resonances in the singularity analysis *Phys. Lett. A* **359** (2006), 199-203.

[2] Camiz P, Geradi A, Marchioro C, Presutti E & Scacciatelli E, Exact solution of a time-dependent quantal harmonic oscillator with a singular perturbation, *J. Math. Phys.* **12** (1971), 2040-2043.

[3] Conte R, Singularities of differential equations and integrability, Introduction to Methods of Complex Analysis and Geometry for Classical Mechanics and Nonlinear Waves, Editors: Benest D & Froeschlé C, Éditions Frontières, Gif-sur-Yvette, (1994), 49-143.

[4] Ermakov V, Second-order differential equations. Conditions of complete integrability, *Universita Izvestia Kiev Series III* (trans Harin AO), **9** (1880), 1-25.

[5] Euler N & Leach PGL, First integrals and reduction of a class of nonlinear higher order ordinary differential equations, *J. Math. Anal. Appl.*, **287** (2003), 337-347.

[6] Leach PGL, On a direct method for the determination of an exact invariant for the time-dependent harmonic oscillator, *J. Aust. Math. Soc., Series B*, **20** (1977), 97-105.

[7] Leach PGL, Invariants and wave-functions for some time-dependent harmonic oscillator-type Hamiltonians, *J. Math. Phys.*, **18** (1977), 1902-1907.

[8] Leach PGL, Berry's phase and wave functions for time-dependent Hamiltonian systems *J.Phys.A: Math.Gen.*, **23** (1990), 2695-2699.

[9] Leach PGL & Govinder KS, Hidden symmetries and integration of the generalized Emden-Fowler equation of index two, *Proc. 14th IMACS World Congress on Comput. App. Math.*, Ames WF ed (Georgia Institute of Technology, Atlanta), (1994), 300-303.

[10] Lewis HR Jr, Classical and quantum systems with time-dependent harmonic oscillator-type Hamiltonians, *Phys. Rev. Lett.*, **18** (1967), 510-512.

[11] Lewis HR Jr, Motion of a time-dependent harmonic oscillator and of a charged particle in a time-dependent, axially symmetric, electromagnetic field *Phys. Rev.* **172** (1968), 1313-1315.

[12] Lewis HR Jr, Class of exact invariants for classical and quantum time-dependent harmonic oscillators, *J. Math. Phys.*, **9** (1968), 1976-1986

[13] Lewis H Ralph Jr & Riesenfeld WB, An exact quantum theory of the time-dependent harmonic oscillator and of a charged particle in a time-dependent electromagnetic field, *J. Math. Phys.*, **10** (1969), 1458-1473.

[14] Moyo S and Leach PGL, Reduction properties of ordinary differential equations of maximal symmetry, *Proc. Global Integrability of Field Theories (GIFT 2006)*, (Cockcroft Institute, Daresbury, UK), ISBN: 3-86644-035-9 (2006), 253 - 266.

[15] Pinney E, The nonlinear differential equation $y''(x) + p(x)y + cy^{-3} = 0$, *Proc. Amer. Math. Soc.*, **1** (1980), 681.

[16] Ramani A, Grammaticos B & Bountis T, The Painlevé property and singularity analysis of integrable and nonintegrable systems, *Phys. Rep.*, **180** (1989), 159-245.

[17] Tabor M, Chaos and Integrability in Nonlinear Dynamics, (John Wiley, New York), (1989).

[18] Whittaker ET, A Treatise on the Analytical Dynamics of Particles and Rigid Bodies, (Dover, New York), (1944).