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Algebraic Discretization of the Camassa-Holm and Hunter-Saxton Equations

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Abstract

The Camassa-Holm (CH) and Hunter-Saxton (HS) equations have an interpretation as geodesic flow equations on the group of diffeomorphisms, preserving the H^1 and $\dot{H}^1$ right-invariant metrics correspondingly. There is an analogy to the Euler equations in hydrodynamics, which describe geodesic flow for a right-invariant metric on the infinite-dimensional group of diffeomorphisms preserving the volume element of the domain of fluid flow and to the Euler equations of rigid body whith a fixed point, describing geodesics for a left-invariant metric on SO(3). The CH and HS equations are integrable bi-hamiltonian equations and one of their Hamiltonian structures is associated to the Virasoro algebra. The parallel with the integrable SO(3) top is made explicit by a discretization of both equation based on Fourier modes expansion. The obtained equations represent integrable tops with infinitely many momentum components. An emphasis is given on the structure of the phase space of these equations, the momentum map and the space of canonical variables.

1 Introduction

The geometric interpretation of the Camassa-Holm equation [6] as a geodesic flow equation on the group of diffeomorphisms, preserving the H^1 right-invariant metric was noticed firstly by Misiołek [42] and developed further in many recent publications, e.g. [37, 25, 7, 14, 15, 39, 13]. The CH equation has also an interpretation in the context of water waves propagation [6, 34, 35, 19, 20, 32, 29]. The spectral problem for the CH equation on the line is developed in [2, 8, 10, 11, 17, 36], the periodic spectral problem – in [16, 48]. The CH solutions are investigated in a variety of recent papers, e.g. in [4, 5, 9, 12, 21, 22, 23, 26, 30, 46]. Hierarchies of CH equations are studied in [11, 31, 33], different modifications are studied in [41, 47].

There are different forms of the CH equation, containing linear term with a first derivative u_x; with a third derivative u_{xxx} (called sometimes Dullin-Gottwald-Holm equation [19, 20, 43, 44, 49]), or without such terms. These terms can be put in or removed from the equation independently by Galilean transformations.

We will be interested in the CH equation of the form

$$m_t + au_{xxx} + 2mu_x + m_x u = 0, \qquad m = u - u_{xx}, \tag{1.1}$$

with a being an arbitrary constant. It can be written in Hamiltonian form

$$m_t = \{m, H_1\}, \tag{1.2}$$

where, assuming that m is 2π periodic in x, i.e. $m(x) = m(x + 2\pi)$, the Poisson bracket and the Hamiltonian are

$$\{F, G\} \equiv -\int_0^{2\pi} \frac{\delta F}{\delta m}\Big(a\partial^3 + m\partial + \partial \circ m\Big)\frac{\delta G}{\delta m}\mathrm{d}x, \tag{1.3}$$

$$H_1 = \frac{1}{2}\int_0^{2\pi} mu\mathrm{d}x. \tag{1.4}$$

The equation (1.1) is bi-Hamiltonian with a second Hamiltonian representation $m_t = \{m, H_2\}_2$, where

$$\{F, G\}_2 \equiv -\int_0^{2\pi} \frac{\delta F}{\delta m}(\partial - \partial^3)\frac{\delta G}{\delta m}\mathrm{d}x, \tag{1.5}$$

$$H_2 = \frac{1}{2}\int_0^{2\pi} (u^3 + uu_x^2 - \frac{a}{2}u_x^2)\mathrm{d}x. \tag{1.6}$$

One can notice that the integral

$$H_0 = \int_0^{2\pi} m\mathrm{d}x \tag{1.7}$$

is a Casimir for the second Poisson bracket (1.5).

The relation of the first Poisson bracket (1.3) to the Virasoro algebra can be seen as follows [18]. The 2π-periodic function allows a Fourier decomposition

$$m(x, t) = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} L_n(t)e^{inx} + \frac{a}{2}, \tag{1.8}$$

(the reality of m can be achieved by $L_{-n} = \bar{L}_n$). Then the Fourier coefficients L_n close a classical Virasoro algebra of central charge $c = -24\pi a$ with respect to the Poisson bracket (1.3):

$$i\{L_n, L_m\} = (n - m)L_{n+m} - 2\pi a(n^3 - n)\delta_{n+m,0}. \tag{1.9}$$

The CH equation in the form

$$m_t + 2\omega u_x + 2mu_x + m_x u = 0, \qquad m = u - u_{xx}, \tag{1.10}$$

can be obtained from (1.1) via $u \to u + a$, and apparently $\omega = 3a/2$.
Since

$$H_0 = \frac{L_0}{2\pi} + \pi a \tag{1.11}$$

is an integral of motion (Casimir), so is L_0.

The first Hamiltonian is

$$H_1 = \frac{1}{4\pi}\sum_{n\in\mathbb{Z}} \frac{L_n L_{-n}}{1+n^2} + \frac{a}{2}L_0 + \frac{2\pi a^2}{8}. \tag{1.12}$$

From (1.9) and (1.12) we obtain the 'Camassa-Holm top' equations on the Virasoro group, which are a discretization of the Camassa-Holm equation (1.1)

$$i\dot{L}_k = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} \frac{k+n}{1+n^2} L_n L_{k-n} + \frac{a}{2}\frac{3k-k^3}{1+k^2}L_k, \tag{1.13}$$

(the dot is a t-derivative). This equation is analogous to the Euler top (rigid body) equation on the Lie group SO(3)

$$\dot{M}_k = \sum_{p,l=1}^{3} \varepsilon_{kpl}\Omega_p M_l, \qquad M_k \equiv I_k \Omega_k$$

for the quadratic Hamiltonian

$$H_E = \frac{1}{2}\sum_{p=1}^{3} M_p \Omega_p,$$

where I_k ($k = 1, 2, 3$) are three constants – the principle inertia momenta. The phase space is embedded in the Lie coalgebra so(3)* as a coadjoint orbit. The Lie-Poisson bracket, related to the so(3)* coalgebra is

$$\{M_n, M_m\} = \varepsilon_{nmk} M_k. \tag{1.14}$$

The inertia operator **I**: so(3)→ so(3)* (see e.g. [1]) relates the parametrization on the so(3) algebra given by the functions Ω_k and the parametrization on the co-algebra so(3)* given by the functions $M_k = I_k\Omega_k$. Note that the Poisson bracket (1.14) has a Casimir

$$K = \Omega_1^2 + \Omega_2^2 + \Omega_3^2, \tag{1.15}$$

constraining the phase space on a sphere. Since the Lie-Poisson bracket is degenerate on so(3)*, the coadjoint orbits (which are spheres centered at the origin) are labelled by the value of the Casimir K.

For the CH top (1.13) the coadjoint orbits are embedded in the Virasoro algebra (parameterized by the functions L_k) due to the Lie-Poisson bracket (1.9).

2 Lax representation for the discrete Camassa-Holm equation and integrals of motion

The Lax pair for the discrete CH equation (1.13) can be obtained from the Lax pair for (1.10),

$$\Psi_{xx} = \Big(\frac{1}{4} + \lambda(m + \frac{a}{2})\Big)\Psi \tag{2.1}$$

$$\Psi_t = \Big(\frac{1}{2\lambda} - u + a\Big)\Psi_x + \frac{u_x}{2}\Psi, \tag{2.2}$$

as follows. We take the expansions

$$\Psi = \sum_{n\in\mathbb{Z}} \Psi_{\frac{n}{2}} e^{i\frac{n}{2}x}, \tag{2.3}$$

$$u = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} u_n e^{inx} + \frac{a}{2}, \qquad u_n = \frac{L_n}{1+n^2}. \tag{2.4}$$

Then (2.1) gives

$$\frac{1}{\lambda}\Psi_{\frac{n}{2}} = \sum_{p\in\mathbb{Z}} \mathcal{L}_{\frac{n}{2},\frac{n}{2}-p}\Psi_{\frac{n}{2}-p}, \tag{2.5}$$

where

$$\mathcal{L}_{\frac{n}{2},\frac{n}{2}-p} = -\frac{4}{n^2+1}\Big(\frac{L_p}{2\pi} + a\delta_{p,0}\Big),$$

or

$$\mathcal{L}_{\frac{n}{2}-q,\frac{n}{2}-p} = -\frac{4}{(n-2q)^2+1}\Big(\frac{L_{p-q}}{2\pi} + a\delta_{p,q}\Big) \tag{2.6}$$

Now from (2.2), (2.3), (2.4) and (2.5) it follows

$$\dot{\Psi}_{\frac{n}{2}} = \sum_{p\in\mathbb{Z}} \mathcal{A}_{\frac{n}{2},\frac{n}{2}-p}\Psi_{\frac{n}{2}-p}, \tag{2.7}$$

where

$$\mathcal{A}_{\frac{n}{2},\frac{n}{2}-p} = -\frac{i}{4\pi}\Big(2n\frac{p^2+1}{n^2+1} + n - 3p\Big)u_p + in\Big(\frac{1}{4} - \frac{1}{n^2+1}\Big)a\delta_{p,0},$$

or

$$\begin{aligned}\mathcal{A}_{\frac{n}{2}-q,\frac{n}{2}-p} &= -\frac{i}{4\pi}\Big(2(n-2q)\frac{(p-q)^2+1}{(n-2q)^2+1}+n-3p+q\Big)u_{p-q} \\ &+ i(n-2q)\Big(\frac{1}{4}-\frac{1}{(n-2q)^2+1}\Big)a\delta_{p,q}. \end{aligned} \tag{2.8}$$

Differentiating (2.5) with respect to t we obtain

$$\frac{1}{\lambda}\dot{\Psi}_{\frac{n}{2}} = \sum_{p\in\mathbb{Z}}\dot{\mathcal{L}}_{\frac{n}{2},\frac{n}{2}-p}\Psi_{\frac{n}{2}-p} + \sum_{p\in\mathbb{Z}}\mathcal{L}_{\frac{n}{2},\frac{n}{2}-p}\dot{\Psi}_{\frac{n}{2}-p},$$

and with the further substitution from (2.7),

$$\frac{1}{\lambda}\sum_{q\in\mathbb{Z}}\mathcal{A}_{\frac{n}{2},\frac{n}{2}-q}\Psi_{\frac{n}{2}-q} = \sum_{p\in\mathbb{Z}}\dot{\mathcal{L}}_{\frac{n}{2},\frac{n}{2}-p}\Psi_{\frac{n}{2}-p} + \sum_{p,q\in\mathbb{Z}}\mathcal{L}_{\frac{n}{2},\frac{n}{2}-q}\mathcal{A}_{\frac{n}{2}-q,\frac{n}{2}-p}\Psi_{\frac{n}{2}-p},$$

$$\sum_{q\in\mathbb{Z}}\mathcal{A}_{\frac{n}{2},\frac{n}{2}-q}\Big(\frac{1}{\lambda}\Psi_{\frac{n}{2}-q}\Big) = \sum_{p\in\mathbb{Z}}\dot{\mathcal{L}}_{\frac{n}{2},\frac{n}{2}-p}\Psi_{\frac{n}{2}-p} + \sum_{p,q\in\mathbb{Z}}\mathcal{L}_{\frac{n}{2},\frac{n}{2}-q}\mathcal{A}_{\frac{n}{2}-q,\frac{n}{2}-p}\Psi_{\frac{n}{2}-p},$$

and finally, the substitution of (2.5) gives

$$\begin{aligned}&\sum_{p,q\in\mathbb{Z}}\mathcal{A}_{\frac{n}{2},\frac{n}{2}-q}\mathcal{L}_{\frac{n}{2}-q,\frac{n}{2}-p}\Psi_{\frac{n}{2}-p} = \\ &\quad\sum_{p\in\mathbb{Z}}\dot{\mathcal{L}}_{\frac{n}{2},\frac{n}{2}-p}\Psi_{\frac{n}{2}-p} + \sum_{p,q\in\mathbb{Z}}\mathcal{L}_{\frac{n}{2},\frac{n}{2}-q}\mathcal{A}_{\frac{n}{2}-q,\frac{n}{2}-p}\Psi_{\frac{n}{2}-p},\end{aligned} \tag{2.9}$$

or in matrix form,

$$\dot{\mathcal{L}} = [\mathcal{A},\mathcal{L}]. \tag{2.10}$$

After some lengthy computations one can verify that (2.10) gives (1.13). The integrals of motion are given by $I_k = \mathrm{tr}(\mathcal{L}^k)$. For example,

$$I_1 = \mathrm{tr}(\mathcal{L}) = \sum_{p\in\mathbb{Z}}\mathcal{L}_{\frac{n}{2}-p,\frac{n}{2}-p} = -4\Big(\frac{L_0}{2\pi}+a\Big)\sum_{p\in\mathbb{Z}}\frac{1}{(n-2p)^2+1}$$

produces, up to an overall constant, the Casimir H_0, (1.11).

$$\begin{aligned} I_2 &= \mathrm{tr}(\mathcal{L}^2) = \sum_{p,q\in\mathbb{Z}}\mathcal{L}_{\frac{n}{2}-p,\frac{n}{2}-q}\mathcal{L}_{\frac{n}{2}-q,\frac{n}{2}-p} \\ &= \frac{4}{\pi^2}\sum_{p,q\in\mathbb{Z}}\frac{L_{p-q}L_{q-p}}{[(n-2p)^2+1][(n-2q)^2+1]} \\ &+ \frac{16a}{\pi}(L_0+\pi a)\sum_{p\in\mathbb{Z}}\frac{1}{[(n-2p)^2+1]^2}. \end{aligned} \tag{2.11}$$

With partial fractions decomposition with respect to n one can derive the identity

$$\frac{1}{[(n-2p)^2+1][(n-2q)^2+1]} = \frac{1/4}{(p-q)^2+1}\Big\{\frac{(n-2q)+(p-q)}{(p-q)[(n-2q)^2+1]} - \frac{(n-2p)+(q-p)}{(p-q)[(n-2p)^2+1]}\Big\}.$$

Further, using the fact that all expressions that change sign under $p-q \to -(p-q)$ are zero, due to the summation over all integer numbers, we have

$$\frac{4}{\pi^2}\sum_{p,q\in\mathbb{Z}}\frac{L_{p-q}L_{q-p}}{[(n-2p)^2+1][(n-2q)^2+1]} = \frac{1}{\pi^2}\sum_{p,q\in\mathbb{Z}}\frac{L_{p-q}L_{q-p}}{1+(p-q)^2}\Big\{\frac{1}{(n-2p)^2+1}+\frac{1}{(n-2q)^2+1}\Big\} = \frac{2}{\pi^2}\sum_{p\in\mathbb{Z}}\frac{L_pL_{-p}}{1+p^2}\sum_{q\in\mathbb{Z}}\frac{1}{(n-2q)^2+1}.$$

Thus, the new integral that appears is $\sum_{p\in\mathbb{Z}}\frac{L_pL_{-p}}{1+p^2}$, giving H_1, the first Hamiltonian (1.12).

3 Oscillator algebra, Miura transformation and momentum map

Let us introduce now the oscillator algebra

$$i\{a_n, a_m\} = \frac{2\pi a}{\kappa^2} n\delta_{n+m,0}, \tag{3.1}$$

where κ is an arbitrary constant. Clearly, a_0 is a Casimir due to (3.1). One can easily verify the following oscillator representation of the Virasoro algebra [38, 24]:

$$L_n = -\kappa(n-1)a_n + \frac{\kappa^2}{4\pi a}\sum_{k\in\mathbb{Z}} a_k a_{n-k}. \tag{3.2}$$

This representation is also known as Sugawara construction. Further, it is evident that

$$i\{a_n, L_m\} = na_{n+m} + \frac{2\pi a}{\kappa} n(n+1)\delta_{n+m,0}. \tag{3.3}$$

Since a_k satisfy the 'canonical' Poisson brackets they are natural candidates for the coordinates in the phase-space. Thus, L_n has an interpretation of a momentum and (3.2) gives the momentum map. The momentum map in terms of field variables is given by the Miura transformation. This can be seen as follows. Defining

$$v = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}} a_k e^{ikx} + \frac{a}{\kappa} \tag{3.4}$$

from (3.2) and (1.8) we have the analog of the Miura transformation:

$$m = i\kappa v_x + \frac{\kappa^2}{2a}v^2 + \frac{a}{2} \tag{3.5}$$

The reality can be achieved by taking κ purely imaginary, $a_k = \bar{a}_{-k}$ for $k \neq 0$ and $\kappa = 2\pi i a/\Im(a_0)$.
Here we notice that the Casimir (1.7) due to (3.5) leads to the restriction

$$\int_0^{2\pi} v^2(x,t)\mathrm{d}x = \mathrm{const}, \tag{3.6}$$

which reduces the evolution of $v(x,t)$ on the L^2-sphere. In terms of the canonical coordinates this condition is

$$\sum_{k>0} |a_k|^2 = \mathrm{const}, \tag{3.7}$$

since a_0 is a constant. It shows that the time evolution of the canonical variables, given by

$$\dot{a}_n = \{a_n, H_1\}$$

is constrained on the infinite-dimensional l_2-sphere, a condition, similar to the one that we see in the $so(3)$ example (1.15).

When $a = 0$, the Sugawara construction for the Virasoro modes in the case of zero central charge is

$$L_n = \frac{1}{2\tilde{\kappa}}\sum_{k\in\mathbb{Z}} a_k a_{n-k},$$

where $\tilde{\kappa}$ is an arbitrary constant and

$$i\{a_n, a_m\} = \tilde{\kappa} n \delta_{n+m,0}. \tag{3.8}$$

The Casimir with respect to the first Poisson bracket (1.3) with $a \equiv 0$ is

$$\int_0^{2\pi} \sqrt{m}\mathrm{d}x = \sqrt{\frac{\pi}{\tilde{\kappa}}}\int_0^{2\pi} v\mathrm{d}x = 2\pi\sqrt{\frac{\pi}{\tilde{\kappa}}}a_0,$$

i.e. this is the Casimir a_0 of (3.8).
With the expansions

$$m = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} L_n e^{inx}, \qquad v = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} a_n e^{inx}$$

the Sugawara construction takes the form $m = \frac{\pi}{\kappa}v^2$. Since the integral $\int_0^{2\pi} m\mathrm{d}x = \text{const}$ is a Casimir, we have again

$$\int_0^{2\pi} v^2 \mathrm{d}x = \text{const},$$

leading to (3.7).

4 The Hunter-Saxton equation

The Hunter-Saxton (HS) equation

$$u_{xxt} + 2u_x u_{xx} + uu_{xxx} = 0$$

describes the propagation of waves in a massive director field of a nematic liquid crystal [27], with the orientation of the molecules described by the field of unit 1 vectors $n(x,t) = (\cos u(x,t), \sin u(x,t))$, where x is the space variable in a reference frame moving with the linearized wave velocity, and t is a 'slow time variable'. A linear term au_{xxx} can be generated by a shift $u \to u + a$:

$$u_{xxt} + au_{xxx} + 2u_x u_{xx} + uu_{xxx} = 0. \tag{4.1}$$

The HS equation is a short-wave limit of the CH equation, and can be obtained if one takes $m = -u_{xx}$. The Hamiltonian representation (1.2) – (1.4) for this equation is also valid. The HS equation (4.1) is an integrable, bi-Hamiltonian equation with a second Hamiltonian representation $m_t = \{m, H_2\}_2$, where

$$\{F,G\}_2 \equiv \int_0^{2\pi} \frac{\delta F}{\delta m}\partial^3 \frac{\delta G}{\delta m}\mathrm{d}x, \tag{4.2}$$

$$H_2 = \frac{1}{2}\int_0^{2\pi}(u - \frac{a}{2})u_x^2 \mathrm{d}x. \tag{4.3}$$

The HS Lax pair is

$$\Psi_{xx} = \lambda m \Psi, \tag{4.4}$$

$$\Psi_t = \Big(\frac{1}{2\lambda} - u - a\Big)\Psi_x + \frac{u_x}{2}\Psi. \tag{4.5}$$

The analytic and geometric aspects of the HS equation are discussed in a variety of recent papers, e.g. [28, 3, 40] and the references therein.

Again, assuming periodicity and using the expansions

$$\Psi = \sum_{n\in\mathbb{Z}} \Psi_n e^{inx}, \qquad u = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} u_n e^{inx}$$

we obtain the discrete form of the HS equation:

$$in\dot{u}_n - an^2u_n - \frac{1}{2}\sum_{k\in\mathbb{Z}} k(n+k)u_k u_{n-k} = 0.$$

In a similar manner from the Lax pair we obtain the matrix Lax representation for the discrete HS equation

$$\dot{\mathcal{L}}^{HS} = [\mathcal{A}^{HS}, \mathcal{L}^{HS}]. \tag{4.6}$$

where

$$\mathcal{L}^{HS}_{n,n-p} = -\frac{p^2}{n^2}u_p, \qquad \mathcal{A}^{HS}_{n,n-p} = \frac{i}{2}\Big(-\frac{p^2}{n} - 2n + 3p\Big)u_p - ina\delta_{p,0}.$$

The momentum map (the Sugawara construction) for the HS equation remains the same as for the CH equation. However, it becomes degenerated in the case $a = 0$, since $m = -u_{xx}$ and the Casimir $\int_0^{2\pi} m\mathrm{d}x = 0$. Then $\int_0^{2\pi} v^2\mathrm{d}x = 0$, which, for real variables is only possible when $v \equiv 0$, i.e. $m \equiv 0$.

5 Conclusions

At the examples of the CH and HS equations we have shown that the integrable systems with quadratic Hamiltonians are equivalent to integrable tops (possibly with infinitely many components), associated to the algebra of their Poisson brackets. An example for the two dimensional Euler equations in fluid mechanics is presented in [50], another example for the KdV superequation in [38, 45].

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