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Decomposition of symmetric tensor fields in the presence of a flat contact projective structure

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Abstract

Let M be an odd-dimensional Euclidean space endowed with a contact 1-form α. We investigate the space of symmetric contravariant tensor fields over M as a module over the Lie algebra of contact vector fields, i.e. over the Lie subalgebra made up of those vector fields that preserve the contact structure defined by α. If we consider symmetric tensor fields with coefficients in tensor densities (also called symbols), the vertical cotangent lift of the contact form α defines a contact invariant operator. We also extend the classical contact Hamiltonian to the space of symbols. This generalized Hamiltonian operator on the space of symbols is invariant with respect to the action of the projective contact algebra $sp(2n+2)$ the algebra of vector fields which preserve both the contact structure and the projective structure of the Euclidean space. These two operators lead to a decomposition of the space of symbols, except for some critical density weights, which generalizes a splitting proposed by V. Ovsienko in [18].

1 Introduction

In a paper of 1997, C. Duval and V. Ovsienko [7] considered the space $\mathcal{D}_\lambda(M)$ of differential operators acting on λ-densities ($\lambda \in \mathbb{R}$) on a manifold M as modules over the Lie algebra of vector fields $\mathrm{Vect}(M)$. The density weight λ allows to define a one parameter family of modules. The space of differential operators acting on half densities, which is very popular in mathematical physics in the context of geometric quantization, lies inside this family of representations. C. Duval and V. Ovsienko provided a first classification result for differential operators of order less or equal to two. Differential operators that are allowed to modify the weight of their arguments also appear in the mathematical literature, for instance in projective differential geometry (see [19, 22]). Hence the spaces $\mathcal{D}_{\lambda\mu}(M)$ (made of differential operators that map λ-densities into μ-densities) also deserve interest. The first classification result of [7] was then followed by a series of papers [10, 9, 8, 11, 14] where the classification of the spaces $\mathcal{D}_{\lambda\mu}(M)$ was finally settled. More recently, in [3], similar classification results were obtained for spaces of differential operators acting on differential forms.

The search for $\mathrm{Vect}(M)$-isomorphisms from a space of differential operators to another one, or more generally the search of $\mathrm{Vect}(M)$-invariant maps between such spaces, can be made easier by using the so-called projectively equivariant symbol calculus introduced and developed by P. Lecomte and V. Ovsienko in [12] : On the one hand the space $\mathcal{D}_{\lambda\mu}(\mathbb{R}^m)$ is naturally filtered by the order of differential operators. The action of $\mathrm{Vect}(\mathbb{R}^m)$ preserves the filtration and the associated graded space $\mathcal{S}_\delta(\mathbb{R}^m)$ (the space of symbols) is identified to contravariant symmetric tensor fields with coefficients in δ-densities, with $\delta = \mu - \lambda$.

On the other hand the projective group $PGL(m+1, \mathbb{R})$ acts locally on $\mathbb{R}^m$ by linear fractional transformations. The fundamental vector fields associated to this action generate a subalgebra of $\mathrm{Vect}(\mathbb{R}^m)$, the projective algebra $sl(m+1)$, which is obviously isomorphic to $sl(m+1, \mathbb{R})$.

P. Lecomte and V. Ovsienko proved in [12] that the spaces $\mathcal{D}_\lambda(\mathbb{R}^m)$ and $\mathcal{S}_0(\mathbb{R}^m)$ are canonically isomorphic as $sl(m+1)$-modules. The canonical bijections are called respectively projectively equivariant quantization and symbol map.

This result was extended in [13] and [6] : the spaces $\mathcal{D}_{\lambda\mu}(\mathbb{R}^m)$ and $\mathcal{S}_\delta(\mathbb{R}^m)$ are canonically isomorphic, provided δ does not belong to a set of critical values.

Once the projectively equivariant symbol map exists, the analysis of the filtered $sl(m+1)$-module $\mathcal{D}_{\lambda\mu}(\mathbb{R}^m)$ can be reduced to the study of the graded module $\mathcal{S}_\delta(\mathbb{R}^m)$.

Now, if we consider a contact manifold M of dimension $m = 2n+1$, it is sensible to consider the spaces $\mathcal{D}_{\lambda\mu}(M)$ as modules over the algebra of contact vector fields $\mathcal{C}(M)$, i.e. of those vector fields that preserve the contact structure. Using the projectively equivariant symbol map in order to perform local computations, we are led to consider the space of symbols over $\mathbb{R}^{2n+1}$ as a representation of the contact projective algebra $\mathcal{C}(\mathbb{R}^{2n+1}) \cap sl(2n+2) = sp(2n+2)$. Note that a similar problem was considered in [20] where differential forms where studied in the presence of a contact structure.

A first question in the analysis of a representation is to know whether it decomposes as a direct sum of invariant subspaces or not.

The answer to this question is known for the space $\mathcal{S}^1_0$, that is, the space of vector fields : in [18], the author shows that the space of vector fields over a contact manifold, viewed as a module over the Lie algebra of contact vector fields, splits as the direct sum of contact vector fields and of vector fields which are tangent to the contact distribution :

$$\mathcal{S}^1_0(M) = \mathrm{Vect}(M) = \mathcal{C}(M) \oplus TVect(M). \tag{1.1}$$

The construction is based on the well-known Hamiltonian operator X, which associates a contact vector field to every $-\frac{1}{n+1}$-density and is $\mathcal{C}(M)$-invariant.

In the present paper, we will define an extension of the operator X to the whole space of symbols over the euclidean space $\mathbb{R}^{2n+1}$ endowed with its standard contact structure. This extended operator is not invariant with respect to the action of the algebra $\mathcal{C}(\mathbb{R}^{2n+1})$ but only with respect to the action of $sp(2n+2)$.

However, this operator, together with the contact form viewed as an operator acting on symbols, allows to define a decomposition of $\mathcal{S}_\delta(\mathbb{R}^{2n+1})$ as a sum of $sp(2n+2)$-submodules, unless δ belongs to a set of singular values.

The paper is organised as follows. In section 2 we recall the definition of the basic material concerning densities, symbols and contact structures. We also recall the definition of the Lagrange bracket and of the Hamiltonian operator X acting on densities.

In section 3 we show how to view the contact form α as an invariant operator $i(\alpha)$. We give the definition of the extended operator X on the space of symbols and show its invariance with respect to the action of the contact projective algebra $sp(2n+2)$.

We show in the next section that these operators allow to define a representation of the Lie algebra $sl(2,\mathbb{R})$ on the space of symbols.

In section 5, we show that the contact hamiltonian operator X allows to define an $sp(2n+2)$-invariant right inverse of the operator $i(\alpha)$, except for some singular values of the density weight. This right inverse allows to show the existence of a direct sum decomposition of the space of symbols into $sp(2n+2)$-invariant subspaces.

In the final sections, we take another point of view and show how to obtain the decomposition by considering the natural filtration of the space of symbols of a given degree associated to the operator $i(\alpha)$. We finally use this filtration in order to analyse the decomposition in the case of singular values.

2 Basic objects

In this section, we recall the definitions of the basic objects that we will use throughout the paper, and we set our notations. As we continue, we denote by M a smooth, connected, Hausdorff and second countable manifold of dimension m.

2.1 Tensor densities and symbols

Let us denote by $\Delta^\lambda(M) \to M$ the line bundle of tensor densities of weight λ over M and by $\mathcal{F}_\lambda(M)$ of smooth sections of this bundle, i.e. the space $\Gamma(\Delta^\lambda(M))$. The Lie algebra of vector fields $\mathrm{Vect}(M)$ acts on $\mathcal{F}_\lambda(M)$ in a natural way. In local coordinates, any element F of $\mathcal{F}_\lambda(M)$ as a local expression

$$F(x) = f(x)|dx^1 \wedge \cdots \wedge dx^m|^\lambda$$

and the Lie derivative of F in the direction of a vector field $X = \sum_i X^i \frac{\partial}{\partial x^i}$ is given by

$$(L_X F)(x) = (\sum_i X^i \frac{\partial}{\partial x^i} f + \lambda(\sum_i \frac{\partial}{\partial x^i} X^i) f)|dx^1 \wedge \cdots \wedge dx^m|^\lambda. \tag{2.1}$$

Note that, as a vector space, $\mathcal{F}_\lambda(M)$ can be identified with the space of smooth functions on M, and thus formula (2.1) defines a one parameter family of deformations of the natural representation of $\mathrm{Vect}(M)$ on $C^\infty(M)$.

2.2 Symbols

We call the *symbol space of degree* k and denote by $\mathcal{S}^k_\delta(M)$ (or simply $\mathcal{S}^k_\delta$) the space of contravariant symmetric tensor fields of degree k, with coefficients in δ-densities ($\delta \in \mathbb{R}$) :

$$\mathcal{S}^k_\delta(M) = \Gamma(S^k TM \otimes \Delta^\delta(M)).$$

We also consider the whole symbol space

$$\mathcal{S}_\delta(M) = \bigoplus_{k\geq 0} \mathcal{S}^k_\delta(M).$$

As we continue, we will freely identify symbols with functions on T^*M that are polynomial along the fibre and we will denote by ξ their generic argument in the fibre of T^*M.

The action of the algebra $\mathrm{Vect}(M)$ on symbols is the natural extension of its action on densities (2.1) and on symmetric tensor fields. Let us write it down in order to illustrate the identification of symbols and functions on T^*M :

$$L_X u(x,\xi) = \sum_i X^i \partial_{x^i} u(x,\xi) + \delta(\sum_i \partial_{x^i} X^i) u(x,\xi) - \sum_{i,k} (\partial_{x^k} X^i)\xi_i \partial_{\xi_k} u(x,\xi). \quad (2.2)$$

The spaces of symbols appear in a series of recent papers concerning equivariant quantizations [12, 5, 4, 16]. Therefore we will not discuss them in full detail and refer the reader to these works for more information.

2.3 Contact manifolds

Here we will recall some basic facts about contact manifolds (see for instance [1, 2, 17]) and we set our notation. For the main part, we focus our attention to the Euclidean space endowed with its standard contact structure since we are only concerned with local phenomena on contact manifolds.

First recall that in general, a contact manifold is a manifold M of odd dimension $m = 2n+1$ endowed with a distribution of hyperplanes (the contact distribution) in the tangent space that is maximally non integrable. Locally, the distribution can be defined as the kernel of a 1-form α and the non-integrability condition means that $\alpha\wedge(d\alpha)^n \neq 0$. A contact manifold M is called *coorientable* if the contact distribution is the kernel of a globally defined 1-form α, such that $\alpha \wedge d\alpha^n \neq 0$. In this case, the form α is called a contact form on M. As we continue, we will always suppose that (M,α) is a coorientable contact manifold.

Recall also that all contact manifolds of dimension $m = 2n+1$ are locally isomorphic to $\mathbb{R}^{2n+1}$: there exist local coordinates (Darboux coordinates) such that the contact form writes

$$\alpha = \frac{1}{2}(\sum_{k=1}^n (p^k dq^k - q^k dp^k) - dt). \quad (2.3)$$

Since we are concerned with local computations on coorientable contact manifold, we will only consider, unless otherwise stated, the Euclidean space $M = \mathbb{R}^{2n+1}$ with coordinates $(q^1,\ldots,q^n,p^1,\ldots,p^n,t)$ endowed with the contact form α defined by (2.3).

2.3.1 Contact vector fields

A contact vector field over M is a vector field which preserves the contact structure. The set of such vector fields forms a subalgebra of $\mathrm{Vect}(M)$, denoted by $\mathcal{C}(M)$. In other words, we have

$$\mathcal{C}(M) = \{Z \in \mathit{Vect}(M) : \exists f_Z \in \Gamma(M\times\mathbb{R}^*) : L_Z\alpha = f_Z\alpha\}. \quad (2.4)$$

2.3.2 The Lagrange bracket and the operator X on densities

For every contact manifold M there exists a $\mathcal{C}(M)$-invariant bidifferential operator acting on tensor densities. This is the so-called Lagrange bracket

$$\{,\}_{\mathcal{L}} : \mathcal{F}_\lambda(M) \times \mathcal{F}_\mu(M) \to \mathcal{F}_{\lambda+\mu+\frac{1}{n+1}}(M),$$

which is given in Darboux coordinates by the following expression :

$$\begin{array}{rcl}\{f,g\}_{\mathcal{L}} & = & \sum_{k=1}^n(\partial_{p^k}f\partial_{q^k}g - \partial_{q^k}f\partial_{p^k}g) - \partial_t f E_s.g + \partial_t g E_s.f \\ & & +2(n+1)(\lambda f\partial_t g - \mu g \partial_t f),\end{array}$$

(where E_s stands for the operator $\sum_{k=1}^n(p^k\partial_{p^k} + q^k\partial_{q^k})$) for every $f \in \mathcal{F}_\lambda(M)$ and $g \in \mathcal{F}_\mu(M)$.

Now, the bilinear operator $\{,\}_{\mathcal{L}}$ can be viewed as a linear operator from $\mathcal{F}_\lambda(M)$ to the space $\mathcal{D}^1_{\mu,\lambda+\mu+\frac{1}{n+1}}(M)$ made of differential operators of order less or equal to one that map μ-densities into $\lambda+\mu+\frac{1}{n+1}$-densities. Namely,

$$\{,\}_{\mathcal{L}} : f \mapsto \{f,\cdot\}_{\mathcal{L}} : g \mapsto \{f,g\}_{\mathcal{L}}.$$

Since the Lagrange bracket is $\mathcal{C}(M)$-invariant, this correspondence is also a $\mathcal{C}(M)$-invariant operator from $\mathcal{F}_\lambda(M)$ to $\mathcal{D}_{\mu,\lambda+\mu+\frac{1}{n+1}}(M)$ (the later space is endowed with the Lie derivative given by the commutator).

Finally, we consider the principal operator, which associates to every differential operator its term of highest order. It is well known that it is a $\mathrm{Vect}(M)$-invariant operator

$$\sigma : \mathcal{D}^1_{\mu,\lambda+\mu+\frac{1}{n+1}}(M) \to \mathcal{S}^1_{\lambda+\frac{1}{n+1}}(M).$$

If we compose these operators, we obtain

$$X : \mathcal{F}_\lambda(M) \to \mathcal{S}^1_{\lambda+\frac{1}{n+1}}(M) : f \mapsto \sigma(\{f,\cdot\}_{\mathcal{L}}).$$

We then have the following immediate result.

Proposition 1. *The operator*

$$X : \mathcal{F}_\lambda(M) \to \mathcal{S}^1_{\lambda+\frac{1}{n+1}}(M)$$

is $\mathcal{C}(M)$-invariant.

Using the identification of symbols and polynomials, we can give the expression of X :

$$X(f)(\xi) = \sum_{k=1}^n(\xi_{q^k}\partial_{p^k}f - \xi_{p^k}\partial_{q^k}f) - \partial_t f\langle E_s,\xi\rangle + \xi_t(E_s.f + 2(n+1)\lambda f).$$

Let us close this section by the following result about contact vector fields.

Proposition 2. *The algebra $\mathcal{C}(\mathbb{R}^{2n+1})$ is exactly $X(\mathcal{F}_{\frac{-1}{n+1}}(\mathbb{R}^{2n+1}))$.*

2.3.3 The projective and symplectic algebras

We also consider the projective Lie algebra $sl(2n+2)$. It is the algebra of fundamental vector fields associated to the (local) action of the projective group $PGL(2n+2,\mathbb{R})$ on $\mathbb{R}^{2n+1}$. This algebra is generated by constant and linear vector fields and quadratic vector fields of the form $\eta\mathcal{E}$ for $\eta \in \mathbb{R}^{2n+1^*}$, where $\mathcal{E}$ is the Euler vector field. Finally, the contact projective algebra $sp(2n+2)$ is the intersection $\mathcal{C}(\mathbb{R}^{2n+1}) \cap sl(2n+2)$. Note that the algebra $sp(2n+2)$ can be obtained using proposition 2 by applying the operator X to polynomial functions of degree less or equal to 2. For more details on the structure of this algebra, we refer the reader to [15].

3 Invariant operators

Here we will define some operators related to the form α. We will then prove in the next sections that these operators commute with the actions of $\mathcal{C}(M)$ or of $sp(2n+2)$.

3.1 The contact form as an invariant operator

We denote by Ω the volume form defined by α, namely

$$\Omega = \alpha \wedge (d\alpha)^n$$

and by div the divergence associated Ω. We then have for every $Z \in \mathcal{C}(M)$

$$L_Z\Omega = div(Z)\Omega = (n+1)f_Z\Omega,$$

(where f_Z is defined in (2.4)) and therefore

$$f_Z = \frac{1}{n+1}div(Z).$$

We then introduce a density weight in order to turn the form α into a $\mathcal{C}(M)$- invariant tensor field : we consider

$$\alpha \otimes |\Omega|^{-\frac{1}{n+1}} \in \Gamma(T^*M \otimes \Delta^{-\frac{1}{n+1}}(M)).$$

and we can compute that the Lie derivative of this tensor field in the direction of any field of $\mathcal{C}(M)$ is vanishing.

As we continue, we will omit the factor $|\Omega|^{-\frac{1}{n+1}}$ unless this leads to confusion, and consider α as an invariant tensor field.

Now, since α is a 1-form, it defines a linear functional on vector fields. This map has a natural extension to symmetric contravariant tensor fields, which we denote by $i(\alpha)$. But since we want this map to be $\mathcal{C}(M)$-invariant, we have to take the density weight of α into account and consider symmetric tensor fields with coefficients in tensor densities. We then have this first elementary result :

Proposition 3. *The map*

$$i(\alpha) : \mathcal{S}^k_\delta \to \mathcal{S}^{k-1}_{\delta-\frac{1}{n+1}}$$

commutes with the action of $\mathcal{C}(M)$.

We can give the expression of the operator $i(\alpha)$ in terms of polynomials : for every $S \in \mathcal{S}^k_\delta$, there holds

$$i(\alpha)(S)(\xi) = \frac{1}{2}(\sum_i (p^i \partial_{\xi_{q^i}} - q^i \partial_{\xi_{p^i}}) - \partial_{\xi_t})(S(\xi)).$$

Remarks : The expression of the operator $i(\alpha)$ is independent of the density weight. This weight appears only in order to turn the map $i(\alpha)$ into an invariant map.

3.2 The Hamiltonian operator X

It turns out that the operator X given in section 2.3.2 extends to an operator on the space of symbols on $\mathbb{R}^{2n+1}$. We will prove that this operator is $sp(2n+2)$-invariant but not $\mathcal{C}(\mathbb{R}^{2n+1})$ equivariant.

Definition 1. We define the Hamiltonian operator X in Darboux coordinates by

$$X : \mathcal{S}^k_\delta \to \mathcal{S}^{k+1}_{\delta+\frac{1}{n+1}} : S \mapsto D(S) + a(k,\delta)\xi_t\, S,$$

where

$$D(S)(\xi) = \sum_i (\xi_{q^i}\partial_{p^i} - \xi_{p^i}\partial_{q^i})S(\xi) + \xi_t E_s(S)(\xi) - \langle E_s, \xi\rangle \partial_t S(\xi),$$

$$E_s = \sum_i (p^i \partial_{p^i} + q^i \partial_{q^i}),$$

and

$$a(k,\delta) = 2(n+1)\delta - k.$$

The main result is the following :

Proposition 4. *The operator $X : \mathcal{S}^k_\delta \to \mathcal{S}^{k+1}_{\delta+\frac{1}{n+1}}$ commutes with the action of the algebra $sp(2n+2)$. It does not commute with the action of $\mathcal{C}(\mathbb{R}^{2n+1})$ unless $k=0$.*

Proof. The result could be checked by hands, and the computations are made easier by using the structure of the algebra $sp(2n+2)$, such as the grading of this algebra (see [15]). However, we will present a few arguments based on the affine symbol map σ_{Aff} already used in [11, 4, 3]. Recall that this map transforms a differential operator into a symmetric contravariant tensor field, just as the principal symbol map does. The map σ_{Aff} is equivariant with respect to the actions on differential operators and on symmetric tensor fields of the affine algebra Aff, spanned by linear and constant vector fields. Hence the differential operator X is $sp(2n+2)\cap\mathrm{Aff}$-invariant iff its affine symbol $\sigma_{\mathrm{Aff}}(X)$ is. We now compute

$$(\sigma_{\mathrm{Aff}}(X)(\eta, S))(\xi) = (L_1(\xi,\eta) + a(k,\delta)\xi_t))S,$$

where

$$L_1(\xi,\eta) = \sum_i (\xi_{q^i}\eta_{p^i} - \xi_{p^i}\eta_{q^i}) + \xi_t\langle E_s, \eta\rangle - \langle E_s, \xi\rangle \eta_t.$$

Moreover, the polynomial $L_1(\xi, \eta)$ is the principal symbol of the Lagrange bracket and is therefore an invariant polynomial (note that this can also be checked using classical results of H. Weyl on invariant polynomials, see [21]). It was already mentioned in [15, section 6] that the polynomial $P(\xi) = \xi_t$ is $sp(2n+2) \cap \mathrm{Aff}$-invariant. Therefore the operator X commutes with the action of the subalgebra $sp(2n+2) \cap \mathrm{Aff}$.

Now, it is easy to verify that the algebra $sp(2n+2)$ is spanned by $sp(2n+2) \cap \mathrm{Aff}$ and the field

$$-\frac{1}{2} X_{t^2} = tE_s + t^2 \partial_t = tE,$$

where E is the Euler field defined by $E = E_s + t\partial_t$. The action of this field on $\mathcal{S}^k_\delta$ is given by the operator

$$tE - \langle E, \xi \rangle \partial_t + a(k, \delta) t,$$

and it is easy to check that it commutes with X.

For the second part of the result, we check in the same way that the commutator of X with the Lie derivative in the direction of the vector field $X_{q_1^3}$ does not vanish unless $k = 0$. ■

4 Representation of the algebra $sl(2, \mathbb{R})$

In general, the space of symbols is defined as the graded space

$$\mathcal{S}_\delta = \oplus_{k \in \mathbb{N}} \mathcal{S}^k_\delta.$$

The grading is natural if we see the space of symbols as the graded space associated to the filtered space of differential operators mapping λ-densities into $\lambda + \delta$-densities.

However, this grading is not suitable to deal with the operators $i(\alpha)$ and X since they modify the density weights of their arguments. It is then natural to define the graded space

$$R_\delta = \oplus_{k \in \mathbb{N}} R^k_\delta$$

where $R^k_\delta = \mathcal{S}^k_{\delta + \frac{k}{n+1}}$ (and thus $\mathcal{S}^k_\delta = R^k_{\delta - \frac{k}{n+1}}$).

As we continue we will omit the reference to δ and denote R^k instead of R^k_δ. It follows that from this definition the operators X and $i(\alpha)$ act on R_δ.

4.1 The operator H and the representation of $sl(2, \mathbb{R})$

We now investigate the relationships between X and $i(\alpha)$. We first compute the restriction of X to R^k and obtain the following elementary result

Proposition 5. *The restriction of the operator X to R^k is given by*

$$X(S)(\xi) = D(S)(\xi) + [2(n+1)\delta + k]\xi_t S(\xi)$$

for all $S \in R^k$.

In order to make the results of this section easier to state, we introduce a new operator H on R_δ :

Definition 2. The operator H is defined by its restrictions to R^k given by

$$H|_{R^k} = h_k \mathrm{Id} = -((n+1)\delta + k)\mathrm{Id}.$$

The main result of this section deals with the commutators of the operators $X, i(\alpha)$ and H on R_δ.

Proposition 6. *The operators H, $i(\alpha)$ and X define a representation of the algebra $sl(2,\mathbb{R})$ on the space R_δ. Specifically, the relations*

$$\begin{cases} [i(\alpha), X] &= H \\ [H, i(\alpha)] &= i(\alpha) \\ [H, X] &= -X \end{cases}$$

hold on R_δ.

Proof. The first relation is a simple computation : on R^k_δ, in view of Proposition 5, the commutator under consideration equals

$$\begin{aligned} & i(\alpha) \circ (D + (2(n+1)\delta + k)\xi_t) - (D + (2(n+1)\delta + k - 1)\xi_t) \circ i(\alpha) \\ & = [i(\alpha), D] + (2(n+1)\delta + k)[i(\alpha), \xi_t] + \xi_t \circ i(\alpha) \\ & = -\tfrac{1}{2}E_\xi - \tfrac{1}{2}(2(n+1)\delta + k)\mathrm{Id} = h_k \mathrm{Id}, \end{aligned}$$

(where E_ξ stands for the operator $\sum_{i=1}^{2n+1} \xi^i \partial_{\xi^i}$) since the commutators in the second line are given by

$$[i(\alpha), D] = -\frac{1}{2}E_\xi - \xi_t \circ i(\alpha)$$

and

$$[i(\alpha), \xi_t] = -\frac{1}{2}\mathrm{Id}.$$

For the second relation we have on R^k :

$$\begin{aligned} [H, i(\alpha)] &= H \circ i(\alpha) - i(\alpha) \circ H \\ &= (h_{k-1} - h_k) i(\alpha) \\ &= i(\alpha). \end{aligned}$$

The third one is proved in the same way. ■

Remark : In this representation of $sl(2,\mathbb{R})$, H corresponds to the action of an element in a Cartan subalgebra. The operator $i(\alpha)$ can be thought as the action of a generator of positive root space. In this setting, the elements of $ker\, i(\alpha)$ could be seen as highest weight vectors.

We end this section with a technical result.

Definition 3. We define

$$r(l,k) = -\frac{l}{2}(2(n+1)\delta + 2k + l - 1).$$

We can now state the result.

Proposition 7. *On R^k, we have*

$$\begin{cases} i(\alpha)\circ X^l - X^l \circ i(\alpha) &= r(l,k)X^{l-1}, \\ X\circ i(\alpha)^l - i(\alpha)^l \circ X &= -r(l,k-l+1)i(\alpha)^{l-1} \end{cases}$$

Proof. There holds

$$i(\alpha)\circ X^l - X^l\circ i(\alpha) = \sum_{r=0}^{l-1} X^r[i(\alpha),X]X^{l-r-1} = \sum_{r=0}^{l-1} X^r H X^{l-r-1}.$$

On R^k, this operator is

$$\sum_{r=0}^{l-1} h_{k+l-r-1}X^{l-1}.$$

We then compute

$$\begin{aligned} \textstyle\sum_{r=0}^{l-1} h_{k+l-r-1} &= -\textstyle\sum_{r=0}^{l-1}((n+1)\delta + k + l - r - 1) \\ &= -l((n+1)\delta + k + l - 1) + \tfrac{l(l-1)}{2} \\ &= -\tfrac{l}{2}(2(n+1)\delta + 2k + 2l - 2 - l + 1), \end{aligned}$$

hence the first result.

We proceed in the same way for the second part : We simply write

$$\begin{aligned} i(\alpha)^l\circ X - X\circ i(\alpha)^l &= \textstyle\sum_{r=0}^{l-1} i(\alpha)^r[i(\alpha),X]i(\alpha)^{l-r-1} \\ &= (\textstyle\sum_{r=0}^{l-1} h_{k-l+r+1})i(\alpha)^{l-1}. \end{aligned}$$

The result follows easily by the definition of H :

$$\begin{aligned} \textstyle\sum_{r=0}^{l-1} h_{k-l+r+1} &= -\textstyle\sum_{r=0}^{l-1}((n+1)\delta + k - l + r + 1) \\ &= -\textstyle\sum_{r=0}^{l-1}((n+1)\delta + k - r) \\ &= -\tfrac{l}{2}(2(n+1)\delta + 2k - l + 1). \end{aligned}$$

■

5 Decomposition of the space of symmetric tensors

In this section, we will obtain a decomposition of the space R^k by induction on k. The idea is that $i(\alpha)$ maps R^k to R^{k-1}. We will prove that this map is onto, and that there exist $sp(2n+2)$-invariant projectors $p_k : R^k \to R^k \cap ker\, i(\alpha)$, if δ does not belong to a set of singular values. These two facts will allow us to obtain the decomposition result.

5.1 Projectors from R^k to $R^k \cap ker\, i(\alpha)$

The Ansatz for the expression of this projector is given by the interpretation of R_δ as an $sl(2,\mathbb{R})$-module. More precisely it comes from the following conjecture.

Conjecture 1. *The algebra of all the invariant operators for the $sp(2n+2)$ action on R_δ is generated by the operators X, $i(\alpha)$ and the projectors onto the spaces R^k_δ.*

In particular if we want to build a projector Π_k from R^k onto a submodule of itself, it should be a linear combination of compositions of X and $i(\alpha)$ and all the monomials in this expression should have the same degree in X and $i(\alpha)$. Moreover we can order these compositions, using Proposition 7 and the projector should be of the form $\Pi_k = \sum_{l=0}^{\infty} b_{k,l} X^l \circ i(\alpha)^l$. But since $i(\alpha)^l_{|R^k} = 0$ for $l > k$ one gets the following Ansatz :

Ansatz 1. *The projector $p_k : R^k \to R^k \cap ker\, i(\alpha)$ should be of the form*

$$p_k = Id + \sum_{l=1}^{k} b_{k,l} X^l \circ i(\alpha)^l. \tag{5.1}$$

In the following, we will determine the expression of the constants $b_{k,l}$ such that p_k has values in $ker\, i(\alpha)$. This will allow us to prove that it is actually a projector. It turns out that there exists a set I_k of values of δ such that no operator of the form (5.1) can be a projector on $R^k \cap ker\, i(\alpha)$. In this situation, if the conjecture is true, the space R^k is not the direct sum of $R^k \cap ker\, i(\alpha)$ and of an $sp(2n+2)$ invariant subspace.

Definition 4. For every $k \geq 1$, we set

$$\begin{array}{rcl} I_k & = & \{\delta \in \mathbb{R} : \exists j \in \{1,\ldots,k\} : r(j,k-j) = 0\} \\ & = & \{-\frac{p}{2(n+1)} : p \in \{k-1,\ldots,2k-2\}\} \end{array}$$

We then have the following result.

Proposition 8. *If $\delta \notin I_k$, the operator $p_k : R^k \to R^k$ defined by*

$$p_k : R^k \to R^k : p_k = Id + \sum_{l=1}^{k} b_{k,l} X^l \circ i(\alpha)^l \tag{5.2}$$

where

$$b_{k,l} = \frac{(-1)^l}{\Pi_{j=1}^{l} r(j,k-j)},$$

is a projector onto $R^k \cap ker\, i(\alpha)$.

Proof. The restriction of p_k to $ker\, i(\alpha)$ is the identity mapping, in view of (5.2). It is then sufficient to prove that $Im\,(p_k) \subset ker\, i(\alpha)$. Then we deduce $Im\,(p_k) = ker\, i(\alpha)$ and $p_k^2 = p_k$.

We actually have the relation

$$i(\alpha) \circ p_k = 0.$$

Indeed, we have

$$\begin{array}{rcl} i(\alpha)\circ p_k & = & i(\alpha)+\sum_{l=1}^k b_{k,l} i(\alpha) X^l i(\alpha)^l \\ & = & i(\alpha)+\sum_{l=1}^k b_{k,l}(X^l i(\alpha)^{l+1}+r(l,k-l)X^{l-1}i(\alpha)^l). \end{array}$$

The result follows since the constants $b_{k,l}$ fulfill the equations

$$\left\{\begin{array}{lcl} 1+b_{k,1}r(1,k-1) & = & 0 \\ b_{k,l}+b_{k,l+1}r(l+1,k-l-1) & = & 0 \quad \forall l=1,\ldots,k-1, \end{array}\right.$$

and since $i(\alpha)^{k+1}\equiv 0$ on R^k. ■

This proposition suggests to define a new operator from R^{k-1} to R^k.

Definition 5. If $\delta \notin I_k$, we define

$$s_{k-1} : R^{k-1} \to R^k : S \mapsto (-\sum_{l=1}^k b_{k,l} X^l i(\alpha)^{l-1})(S). \tag{5.3}$$

We then have the following result that links s_{k-1} and $i(\alpha)$:

Lemma 1. *If $\delta \notin I_k$, then one has $i(\alpha)\circ s_{k-1} = Id$ on R^{k-1}.*

Proof. We just compute

$$\begin{array}{rcl} i(\alpha)\circ s_{k-1} & = & -\sum_{l=1}^k b_{k,l} i(\alpha) X^l i(\alpha)^{l-1} \\ & = & -\sum_{l=1}^k b_{k,l}(X^l i(\alpha)^l + r(l,k-l)X^{l-1}i(\alpha)^{l-1}) \\ & = & \mathrm{Id}, \end{array}$$

by using Proposition 7, and the definition of $b_{k,l}$. ■

From this Lemma, we obtain an important information about $i(\alpha)$.

Corollary 1. *For every k and δ, the map $i(\alpha) : R^k_\delta \to R^{k-1}_\delta$ is surjective.*

Proof. If $\delta \notin I_k$, the result follows from the existence of a right inverse. Since the expression of $i(\alpha)$ is independent of δ, the result holds true for every δ. ■

We can now state a first decomposition result.

Proposition 9. *If $\delta \notin I_k$, one has $R^k = ker\, i(\alpha) \oplus s_{k-1}(R^{k-1})$.*

Proof. Since the projector p_k is well defined, there is a decomposition

$$R^k = (ker\, i(\alpha)\cap R^k)\oplus V^k,$$

where $V^k = Im(\mathrm{Id} - p_k)$. It follows that the restriction of $i(\alpha)$ to V^k is injective. It is also surjective by corollary 1. By Lemma 1, s_{k-1} is the inverse of $i(\alpha)$ and thus $V^k = s_{k-1}(R^{k-1})$. ■

By applying successively the previous proposition we obtain a second result. We denote by s the operator on R_δ whose restriction to R^{k-1} is s_{k-1}.

Proposition 10. *If $\delta \notin \cup_{j=1}^{k} I_j$, then there holds*

$$R^k = \oplus_{l=0}^{k} s^l (R^{k-l} \cap ker\, i(\alpha)).$$

We will now compute the restriction of s^l to $(R^{k-l} \cap ker\, i(\alpha))$ and show that it is a scalar multiple of X^l.

Proposition 11. *Suppose that $\delta \notin \cup_{j=1}^{k} I_j$. Then the restriction of s^l to $R^{k-l} \cap ker\, i(\alpha)$ equals $c(l, k-l)X^l$, where*

$$c(l, k-l) = (\Pi_{i=1}^{l} r(i, k-l))^{-1}.$$

Proof. We first prove the existence of the constant $c(l, k-l)$ by showing that the restriction of s^j to $R^{k-l} \cap ker\, i(\alpha)$ equals $c(j)X^j$ $(c(j) \in \mathbb{R})$, for all $j = 1, \ldots, l$. For $j = 1$, the result follows from the very definition of s_{k-l}, and we obtain $c(1) = -b_{k-l+1,1}$. Suppose that it holds true for s^j. We then have on $R^{k-l} \cap ker\, i(\alpha)$

$$s^{j+1} = s \circ s^j = - \sum_{a=1}^{k-l+j+1} b_{k-l+j+1,a} X^a i(\alpha)^{a-1} s^j.$$

The last term is a multiple of X^{j+1}, by induction, since the composition $i(\alpha)^{a-1} s^j$ vanishes if $a - 1 > j$ and is equal to s^{j-a+1} if $a - 1 \leq j$.

Now, we compute the value of $c(j+1)$ by analysing the restriction of the operator $i(\alpha) \circ s^{j+1}$ to $R^{k-l} \cap ker\, i(\alpha)$. On the one hand, we have on $R^{k-l} \cap ker\, i(\alpha)$

$$i(\alpha) \circ s^{j+1} = s^j = c(j)X^j,$$

by induction. On the other hand, we also have

$$i(\alpha) \circ s^{j+1} = c(j+1) i(\alpha) X^{j+1} = c(j+1)(X^{j+1} i(\alpha) + r(j+1, k-l)X^j),$$

by Proposition 7.

Finally, we obtain the relation

$$(c(j) - r(j+1, k-l)c(j+1))X^j \equiv 0,$$

on $R^{k-l} \cap ker\, i(\alpha)$. Since X^j does not vanish identically on this space, this leads to

$$c(j) = r(j+1, k-l)c(j+1),$$

and the result follows. ■

6 Another point of view

In this section we investigate the relations of the maps $i(\alpha)$ and X with the filtration induced by $i(\alpha)$. We denote by $\mathcal{F}^{k,l}$ the space $R^k \cap ker\, i(\alpha)^l$. Since $i(\alpha)$ is a $\mathcal{C}(M)$-invariant operator, the spaces $\mathcal{F}^{k,l}$ are stable under the action of $\mathcal{C}(M)$. Moreover there is an obvious filtration of R^k defined by

$$0 = \mathcal{F}^{k,0} \subset \mathcal{F}^{k,1} \subset \cdots \subset \mathcal{F}^{k,k+1} = R^k. \tag{6.1}$$

We first prove that the maps $i(\alpha)$ and X respect this filtration and therefore induce mappings that deserve special interest on the associated graded spaces.

Proposition 12. *We have*

$$\begin{cases} i(\alpha)(\mathcal{F}^{k,l}) & \subset & \mathcal{F}^{k-1,l-1} \\ X(\mathcal{F}^{k-1,l-1}) & \subset & \mathcal{F}^{k,l} \end{cases}$$

for every $k \geq 1$ *and every* $l \in \{1, \ldots, k+1\}$.

Proof. The first result is a direct consequence of the definition of the filtration. For the second one, suppose that v is an element of $\mathcal{F}^{k-1,l-1}$ and compute, using Proposition 7,

$$i(\alpha)^l \circ X(v) = X \circ i(\alpha)^l(v) + r(l, k-l) i(\alpha)^{l-1}(v) = 0.$$

It follows that $X(v)$ is in $\mathcal{F}^{k,l}$. ∎

Let us introduce some more notation.

Definition 6. For every $l \in \{1, \ldots, k+1\}$ we denote by $\mathcal{G}^{k,l}$ the quotient space $\mathcal{F}^{k,l}/\mathcal{F}^{k,l-1}$. This space is naturally endowed with a representation of $\mathcal{C}(M)$. In particular we have $\mathcal{G}^{k,1} \cong \mathcal{F}^{k,1}$. We also set $\mathcal{G}^{k,0} = \{0\}$.

By Proposition 12, the maps $i(\alpha)$ and X induce maps on the graded space $\oplus_{k,l}\mathcal{G}^{k,l}$. Namely, for every $l \in \{1, \ldots, k+1\}$, we set :

$$\begin{cases} \widetilde{i(\alpha)} : \mathcal{G}^{k,l} \to \mathcal{G}^{k-1,l-1} : [u] \mapsto [i(\alpha)(u)] \\ \widetilde{X} : \mathcal{G}^{k-1,l-1} \to \mathcal{G}^{k,l} : [u] \mapsto [X(u)] \end{cases}$$

The main property of these maps is the following.

Proposition 13. *For every* $k \geq 1$ *and* $l \in \{1, \ldots, k+1\}$*, we have*

$$\widetilde{X} \circ \widetilde{i(\alpha)}|_{\mathcal{G}^{k,l}} = r(l-1, k-l+1) Id,$$

and

$$\widetilde{i(\alpha)} \circ \widetilde{X}|_{\mathcal{G}^{k-1,l-1}} = r(l-1, k-l+1) Id.$$

In particular, if $r(l-1, k-l+1)$ *does not vanish, the restricted map* $\widetilde{i(\alpha)} : \mathcal{G}^{k,l} \to \mathcal{G}^{k-1,l-1}$ *is invertible and the inverse map is given by* $\frac{1}{r(l-1,k-l+1)}\widetilde{X}$.

Proof. We only prove the first identity. The second one can be proved using the same arguments.

Let u be in $\mathcal{F}^{k,l}$. By definition, we have

$$\widetilde{X} \circ \widetilde{i(\alpha)}([u]) = [X \circ i(\alpha)(u)].$$

By Proposition 7, we have

$$\begin{aligned} i(\alpha)^{l-1} \circ X \circ i(\alpha)(u) &= X \circ i(\alpha)^l(u) + r(l-1, k-l+1) i(\alpha)^{l-1}(u) \\ &= r(l-1, k-l+1) i(\alpha)^{l-1}(u), \end{aligned}$$

so that $[X \circ i(\alpha)(u)] = r(l-1, k-l+1)[u]$, by the definition of $\mathcal{G}^{k,l}$. ∎

We get an immediate corollary about the decomposition of the filter $\mathcal{F}^{k,l}$ into stable subspaces.

Corollary 2. *Consider $k \geq 1$ and $l \in \{2, \ldots, k+1\}$ and suppose that $\prod_{j=1}^{l-1} r(j, k-l+1) \neq 0$. There holds*

$$\mathcal{F}^{k,l} = \mathcal{F}^{k,l-1} \oplus X^{l-1}(\mathcal{F}^{k-l+1,1}).$$

Proof. Consider the following short exact sequence of $sp(2n+2)$-modules

$$0 \longrightarrow \mathcal{F}^{k,l-1} \longrightarrow \mathcal{F}^{k,l} \longrightarrow \mathcal{G}^{k,l} \longrightarrow 0.$$

By the previous proposition the $sp(2n+2)$-modules $\mathcal{G}^{k,l}$ and $\mathcal{G}^{k-l+1,1}$ are isomorphic through $\widetilde{i(\alpha)}^{l-1}$ and $\widetilde{\phi} = \frac{1}{\prod_{j=1}^{l-1} r(j,k-l+1)} \widetilde{X}^{l-1}$. Denote by t the trivial isomorphism $\mathcal{G}^{k-l+1,1} \to \mathcal{F}^{k-l+1,1}$. It is then easy to check that the map

$$\phi = \frac{1}{\prod_{j=1}^{l-1} r(j, k-l+1)} X^{l-1} \circ t \circ \widetilde{i(\alpha)}^{l-1}$$

provides a section of the exact sequence above. ■

Note that applying successively this corollary, we recover the results of Proposition 10.

7 The singular situations

As we have seen in section 5, there are some values of the parameter δ such that the space of symbols does not decompose directly into submodules, by using the maps $i(\alpha)$ and X. This is due to the fact that the projector p_k (see expression (5.2)) is not well defined for some k. In this section we will briefly discuss the structure of the space R_δ in this situation. We begin with an obvious description of the set of singular values of the weight δ (recall Definition 4).

Proposition 14. *The set of singular values is given by*

$$I = \{-\frac{p}{2(n+1)} : p \in \mathbb{Z}_+\}.$$

Moreover for $p \in \mathbb{Z}_+$, $\delta = -\frac{p}{2(n+1)}$ belongs to I_k iff $k \in \{\lceil \frac{p+2}{2} \rceil, \ldots, p+1\}$.

In view of the previous proposition, for every singular value of δ, there exists a set of consecutive degrees k such that the projector p_k (see (5.2)) can not be defined. We will now comment on the structure of the corresponding modules R^k_δ. Of course, these modules still carry the filtration (6.1) induced by $i(\alpha)$. We will show that in some situations there exists a richer structure. Indeed, some of the filters (6.1) can be decomposed into the sum of submodules. Our analysis is based on the following trivial observation.

Observation 1. *For $2 \leq k' \leq k$ the restriction of the projector p_k to the filter $\mathcal{F}^{k,k'}$ is given by*

$$p_{k,k'} = Id + \sum_{l=1}^{k'-1} b_{k,l} X^l \circ i(\alpha)^l. \tag{7.1}$$

In the same way, the restriction of the operator s_{k-1} (see (5.3)) to $\mathcal{F}^{k-1,k'-1}$ is given by

$$s_{k-1,k'-1} = -\sum_{l=1}^{k'-1} b_{k,l} X^l i(\alpha)^{l-1}.$$

The following result is a straightforward adaptation of Propositions 8 and 9 and of Lemma 1.

Lemma 2. *Suppose that for some $k' \in \{2,\dots,k\}$ the operator $p_{k,k'}$ (see (7.1)) can be defined. Then we have the decomposition*

$$\mathcal{F}^{k,k'} = \mathcal{F}^{k,1} \oplus s_{k-1,k'-1}(\mathcal{F}^{k-1,k'-1}).$$

Now we can come to the final results :

Proposition 15 (Even Case). *Suppose that $\delta = -\frac{p}{2(n+1)}$ with p even, then*

- *For $k < \frac{p+2}{2}$, the decomposition given in Proposition 10 holds for R^k.*
- *The module $R^{\frac{p+2}{2}}$ only carries the filtration induced by $i(\alpha)$ (no filter of this filtration can be decomposed into the direct sum of $ker\, i(\alpha)$ and another submodule, using projectors of the type (7.1)).*
- *For every $u \in \{1,\dots,\frac{p}{2}\}$, and $k' \leq 2u+1$ the filter $\mathcal{F}^{\frac{p+2}{2}+u,k'}$ in $R^{\frac{p+2}{2}+u}$ can be decomposed in*

$$\mathcal{F}^{\frac{p+2}{2}+u,k'} = \mathcal{F}^{\frac{p+2}{2}+u,1} \oplus s_{\frac{p}{2}+u,k'-1}(\mathcal{F}^{\frac{p}{2}+u,k'-1}).$$

- *For $k > p+1$ the decomposition given in Proposition 9 holds for R^k.*

Proof. Using Proposition 14, we already know that the projector p_k from R^k onto $R^k \cap ker\, i(\alpha)$ is well-defined if $k < \frac{p+2}{2}$ or $k > p+1$, so that the decomposition given in Proposition 9 holds for R^k. Moreover, this implies the existence of the decomposition given in Proposition 10 for $k < \frac{p+2}{2}$, still by induction.

We now analyse the possible decomposition of the filters $\mathcal{F}^{k,k'}$ for $k' \in \{2,\dots,k+1\}$ and $k \in \{\frac{p+2}{2},\dots,p+1\}$. We first observe that the map $p_{k,k'}$ is well-defined iff the constants $b_{k,l}$ are well-defined for $1 \leq l \leq k'-1$, and that $b_{k,l}$ is well-defined iff $r(j,k-j) \neq 0$ for $1 \leq j \leq l$. It is easy to see that we have

$$r(j,k-j) = 0 \Leftrightarrow k = \frac{p+j+1}{2}.$$

Hence, for $k = \frac{p+2}{2}$, neither the constant $b_{k,1}$ nor the projector $p_{k,k'}$ for $k' \geq 2$ can be defined.

Finally, if $k = \frac{p+2}{2} + u$, $u \in \{1,\dots,\frac{p}{2}\}$, we have $r(j,k-j) = 0$ iff $j = 2u+1$. Hence $b_{k,l}$ is well-defined iff $2u+1 > l$ and $p_{k,k'}$ can be defined iff $2u+1 > k'-1$, i.e. $k' \leq 2u+1$ and the result follows by Lemma 2. ∎

In the same fashion, we can analyse the odd case.

Proposition 16 (Odd case). *Suppose that $\delta = -\frac{p}{2(n+1)}$ with p odd, then*

- *For $k < \frac{p+3}{2}$, the decomposition given in Proposition 10 holds for R^k.*
- *For every $u \in \{0, \ldots, \frac{p-1}{2}\}$, and $k' \in \{2, \ldots, 2u+2\}$ the filter $\mathcal{F}^{\frac{p+3}{2}+u,k'}$ in $R^{\frac{p+3}{2}+u}$ can be decomposed in*

$$\mathcal{F}^{\frac{p+3}{2}+u,k'} = \mathcal{F}^{\frac{p+3}{2}+u,1} \oplus s_{\frac{p+1}{2}+u,k'-1}(\mathcal{F}^{\frac{p+1}{2}+u,k'-1}).$$

- *For $k > p+1$ the decomposition given in Proposition 9 holds for R^k.*

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