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On a classification of integrable vectorial evolutionary equations

M Ju Balakhnev and A G Meshkov

Orel State Technical University, 29 Naugorskoe st., Orel, Russia, 302020
E-mail: a_meshkov@orel.ru, balakhnev@yandex.ru

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Abstract

A list of twenty five integrable vectorial evolutionary equations of the third order is presented. Each equation from the list possesses higher symmerties and higher conservation laws.

1 Introduction

Vector integrable equations appear first as some specializations of Jordan triple systems introduced in [1]. The most known of them are two vector modified Korteweg-de Vries equations [2], [3]:

$$\begin{aligned} \boldsymbol{u}_t &= \boldsymbol{u}_{xxx} + (\boldsymbol{u},\boldsymbol{u})\boldsymbol{u}_x \\ \boldsymbol{u}_t &= \boldsymbol{u}_{xxx} + (\boldsymbol{u},\boldsymbol{u})\boldsymbol{u}_x + (\boldsymbol{u},\boldsymbol{u}_x)\boldsymbol{u}. \end{aligned} \tag{1.1}$$

Here and below $\boldsymbol{u}(t,x)$ belongs to a N-dimensional vector space V with the scalar product $(\cdot\,,\cdot)$.

The further example is the N-component higher analog of the Landau-Lifshitz equation [4]:

$$\boldsymbol{u}_t = \left(\boldsymbol{u}_{xx} + \frac{3}{2}(\boldsymbol{u}_x,\boldsymbol{u}_x)\boldsymbol{u}\right)_x + \frac{3}{2}\langle\boldsymbol{u},\boldsymbol{u}\rangle\boldsymbol{u}_x, \quad \boldsymbol{u}^2 = 1. \tag{1.2}$$

Here $\langle\cdot,\cdot\rangle$ is a second scalar product. In applications a second scalar product are usually used for describing anisotropy of medium. There may be any realizations of scale products. These realizations are not important for the symmetry analysis, only bilinearity and differentiability are used therein.

The following two series of scalar variables

$$u_{[0,0]} = (\boldsymbol{u}_0, \boldsymbol{u}_0), \quad u_{[0,1]} = (\boldsymbol{u}_0, \boldsymbol{u}_1), \dots, \; u_{[i,k]} = (\boldsymbol{u}_i, \boldsymbol{u}_k), \quad i \leqslant k, \tag{1.3}$$

$$\tilde{u}_{[0,0]} = \langle \boldsymbol{u}_0, \boldsymbol{u}_0 \rangle, \quad \tilde{u}_{[0,1]} = \langle \boldsymbol{u}_0, \boldsymbol{u}_1 \rangle, \dots, \; \tilde{u}_{[i,k]} = \langle \boldsymbol{u}_i, \boldsymbol{u}_k \rangle, \quad i \leqslant k, \tag{1.4}$$

where $\boldsymbol{u}_k = \partial^k \boldsymbol{u}/\partial x^k$, are used. For arbitrary N these variables can be considered as independent.

A componentless version of the symmetry approach has been developed in [5] for vector evolutionary equations of the following type

$$\boldsymbol{u}_t = f_n \boldsymbol{u}_n + f_{n-1} \boldsymbol{u}_{n-1} + \cdots + f_0 \boldsymbol{u}_0, \tag{1.5}$$

where f_k are scalar smooth functions depending on variables (1.3) and (1.4).

Definition 1. If the coefficients f_i of equation (1.5) depend on variables (1.3) only, then equation (1.5) is said to be isotropic. If the coefficients f_i of equation (1.5) depend on both variables (1.3) and (1.4), then equation (1.5) is said to be anisotropic.

In recent years the following third order equation

$$\boldsymbol{u}_t = \boldsymbol{u}_3 + f_2 \boldsymbol{u}_2 + f_1 \boldsymbol{u}_1 + f_0 \boldsymbol{u}_0. \tag{1.6}$$

has been studied. A complete classification of the isotropic integrable equations (1.6) on a sphere has been obtained in [5]. A complete classification of the integrable divergent equations (1.6) has been presented in [6]. A complete list of the isotropic integrable equations (1.6) with $f_2 = 0$ has been presented in [7]. The anisotropic integrable equations (1.6) on a sphere have been completely classified in [8]. But at present there is no complete classification of integrable equations (1.6) in general form.

Definition 2. Denote as **ord** the order of variables: $\operatorname{ord} \boldsymbol{u}_n = n$, $\operatorname{ord} u_{[i,k]} = k$, $\operatorname{ord} \tilde{u}_{[i,k]} = k$ (because $i \leqslant k$). The order of a function F ($\operatorname{ord} F$) is the maximal order of its arguments.

This paper is devoted to a symmetry classification of the **isotropic** equations (1.6), where $\operatorname{ord} f_2 \leqslant 2, \operatorname{ord} f_1 \leqslant 2$ and $\operatorname{ord} f_0 \leqslant 1$. The latter condition has been set because the problem with $\operatorname{ord} f_0 \leqslant 2$ is extremely cumbersome (see Appendix).

If a point transformation preserves a form of an equation, then we call this transformation admissible for the equation under consideration.

It can be easily verified that equation (1.6) has three admissible point transformations:

$$x' = \alpha x + \beta t, \quad t' = \alpha^3 t, \quad \boldsymbol{u}' = \gamma \boldsymbol{u}, \tag{1.7}$$

$$\boldsymbol{u}' = \boldsymbol{u} \exp(ax + bt), \quad \text{if} \quad f_i(\lambda u_{[i,k]}, \lambda \tilde{u}_{[i,k]}) = f_i(u_{[i,k]}, \tilde{u}_{[i,k]}), \forall \lambda \neq 0, \tag{1.8}$$

$$\boldsymbol{u}' = \boldsymbol{u} f(u_{[0,0]}), \quad \left(f^2(u_{[0,0]}) u_{[0,0]}\right)' \neq 0. \tag{1.9}$$

Here α, β, γ, a and b are constant. The condition for f_i in (1.8) expresses homogeneity of the equation and the condition in (1.9) guarantees invertibility of the transformation.

2 Integrability conditions

It is well known that any equation integrable by the inverse spectral transform method possesses infinitely many conservation laws

$$D_t\,\rho_i = D_x\,\theta_i, \qquad i = 0, 1, 2, \ldots \tag{2.1}$$

Here conserved densities ρ_i and fluxes θ_i are functions of field variables and their spatial derivatives. D_x is the total differentiation operator with respect to x, D_t is the evolutionary differentiation. The operators D_x and D_t are defined by the following formulas:

$$\begin{aligned}
&D_x t = 0,\ D_x x = 1,\ D_x \boldsymbol{u}_i = \boldsymbol{u}_{i+1}, && D_x u_{[i,i]} = 2u_{[i,i+1]},\ D_x \tilde{u}_{[i,i]} = 2\tilde{u}_{[i,i+1]},\\
&D_x\, u_{[i,k]} = u_{[i+1,k]} + u_{[i,k+1]},\ i < k, && D_x\, \tilde{u}_{[i,k]} = \tilde{u}_{[i+1,k]} + \tilde{u}_{[i,k+1]},\ i < k,\\
&D_t t = 1,\ D_t x = 0,\ D_t \boldsymbol{u}_n = D_x^n \boldsymbol{F}, && \boldsymbol{F} = \boldsymbol{u}_3 + f_2\boldsymbol{u}_2 + f_1\boldsymbol{u}_1 + f_0\boldsymbol{u}_0,\\
&D_t\, u_{[i,k]} = \big(\boldsymbol{u}_i, D_x^k\boldsymbol{F}\big) + \big(\boldsymbol{u}_k, D_x^i\boldsymbol{F}\big), && D_t\, \tilde{u}_{[i,k]} = \big\langle \boldsymbol{u}_i, D_x^k\boldsymbol{F}\big\rangle + \big\langle \boldsymbol{u}_k, D_x^i\boldsymbol{F}\big\rangle.
\end{aligned}$$

Moreover, the usual rules for differentiations of sums, products and composite functions are implied.

The symmetry method deals with the so-called canonical conserved densities. These densities are usually obtained by using an asymptotic expansion of the logarithmic derivative of the Lax eigenfunction (see [9], Chapter 1, for example). Canonical conserved densities can be also obtained from a temporal Lax equation.

Equation (1.6) can be written in the form $(-\partial_t + D_x^3 + f_2 D_x^2 + f_1 D_x + f_0)\boldsymbol{u} = 0$. The main idea of the componentless version of the symmetry approach is to use the equation $(-\partial_t + D_x^3 + f_2 D_x^2 + f_1 D_x + f_0)\psi = 0$ as a temporal Lax equation. Then one ought to set

$$\psi = \exp\left(\int R\,dx\right),$$

to obtain a Riccaty-type equation:

$$(D_x + R)^2 R + f_2(D_x + R)R + f_1 R + f_0 = F, \quad F = \int R_t\,dx. \tag{2.2}$$

This equation has a formal solution in the following form

$$R = \lambda^{-1} + \sum_{n=0}^{\infty} \rho_n \lambda^n, \quad F = \lambda^{-3} + \sum_{n=0}^{\infty} \theta_n \lambda^n \tag{2.3}$$

Equations (2.2) and (2.3) imply the following recursion relation:

$$\rho_{n+2} = \frac{1}{3}\Big[\theta_n - f_0\,\delta_{n,0} - 2\,f_2\,\rho_{n+1} - f_2\,D_x\rho_n - f_1\,\rho_n\Big]$$
$$- \frac{1}{3}\Big[f_2\sum_{s=0}^{n}\rho_s\,\rho_{n-s} + \sum_{0\leqslant s+k\leqslant n}\rho_s\,\rho_k\,\rho_{n-s-k} + 3\sum_{s=0}^{n+1}\rho_s\,\rho_{n-s+1}\Big]$$
$$- D_x\Big[\rho_{n+1} + \frac{1}{2}\sum_{s=0}^{n}\rho_s\,\rho_{n-s} + \frac{1}{3}D_x\rho_n\Big], \quad n \geqslant 0.$$

Here $\delta_{i,j}$ is the Kronecker symbol, ρ_0 and ρ_1 are given by the formulas:

$$\rho_0 = -\frac{1}{3}\,f_2, \quad \rho_1 = \frac{1}{9}\,f_2^2 - \frac{1}{3}\,f_1 + \frac{1}{3}\,D_x\,f_2. \tag{2.4}$$

The second equation (2.2) may be rewritten as a conservation law $R_t = F_x$. Using the expansions (2.3) one has an infinite set of conservation laws (2.1). The evolutionary differentiation D_t appears in these equations because the functions ρ_n, $n = 0, 1, \ldots$ depend on a solution $\boldsymbol{u}$ of equation (1.6).

To use the recursion formula one ought to find functions θ_i from (2.1). Expressions for the further densities ρ_i contain the fluxes θ_j, $j \leqslant i-2$. For example,

$$\rho_2 = -\frac{1}{3}\,f_0 + \frac{1}{3}\,\theta_0 - \frac{2}{81}\,f_2^3 + \frac{1}{9}\,f_1\,f_2 - D_x\Big(\frac{1}{9}\,f_2^2 + \frac{2}{9}\,D_x\,f_2 - \frac{1}{3}\,f_1\Big),$$

and so on.

It is shown in [5] that even canonical densities are trivial, i. e. $\rho_{2n} = D_x\chi_n$, $n = 0, 1, \ldots$. Therefore, the necessary integrability conditions of equation (1.6) may be presented in the following form:

$$\frac{\delta}{\delta\boldsymbol{u}}D_t\rho_{2n+1} = 0, \quad \frac{\delta}{\delta\boldsymbol{u}}\rho_{2n} = 0, \quad n = 0, 1, \ldots, \tag{2.5}$$

where $\delta/\delta\boldsymbol{u}$ is the Euler operator.

3 List of integrable equations

Theorem. Isotropic equation (1.6) satisfying six integrability conditions (2.5) can be reduced by a suitable point transformation (1.7)–(1.9) to one of equations from the following list:

$$\boldsymbol{u}_t = \boldsymbol{u}_3 + \frac{3}{2}cu_{[0,0]}\,\boldsymbol{u}_1, \tag{3.1}$$

$$\boldsymbol{u}_t = \boldsymbol{u}_3 + 3c(u_{[0,0]}\,\boldsymbol{u}_1 + u_{[0,1]}\,\boldsymbol{u}), \tag{3.2}$$

$$\boldsymbol{u}_t=\boldsymbol{u}_3+\frac{3}{2}\Big(\frac{c^2\,{u_{[1,2]}}^2}{1+c\,u_{[1,1]}}-c\,u_{[2,2]}\Big)\boldsymbol{u}_1, \tag{3.3}$$

$$\boldsymbol{u}_t=\boldsymbol{u}_3-3\,\frac{u_{[1,2]}}{u_{[1,1]}}\,\boldsymbol{u}_2+\frac{3}{2}\,\frac{u_{[2,2]}}{u_{[1,1]}}\,\boldsymbol{u}_1, \tag{3.4}$$

$$\boldsymbol{u}_t=\boldsymbol{u}_3-3\,\frac{u_{[1,2]}}{u_{[1,1]}}\,\boldsymbol{u}_2+\frac{3}{2}\,\left(\frac{u_{[2,2]}}{u_{[1,1]}}+\frac{{u_{[1,2]}}^2}{{u_{[1,1]}}^2(1+au_{[1,1]})}\right)\boldsymbol{u}_1, \tag{3.5}$$

$$\boldsymbol{u}_t=\boldsymbol{u}_3-\frac{3}{2}\,(p+1)\,\frac{u_{[1,2]}}{p\,u_{[1,1]}}\,\boldsymbol{u}_2+\frac{3}{2}\,(p+1)\left(\frac{u_{[2,2]}}{u_{[1,1]}}-\frac{a\,{u_{[1,2]}}^2}{p^2\,u_{[1,1]}}\right)\boldsymbol{u}_1, \tag{3.6}$$
$$p=\sqrt{1+a\,u_{[1,1]}},$$

$$\boldsymbol{u}_t=\boldsymbol{u}_3-3\,\frac{u_{[1,2]}}{u_{[1,1]}}\,\boldsymbol{u}_2+3\,\frac{u_{[2,2]}}{u_{[1,1]}}\,\boldsymbol{u}_1, \tag{3.7}$$

$$\boldsymbol{u}_t=\boldsymbol{u}_3-3\,\frac{u_{[1,2]}}{u_{[1,1]}}\,\boldsymbol{u}_2+\frac{3}{2}\left(\frac{u_{[2,2]}}{u_{[1,1]}}+\frac{{u_{[1,2]}}^2}{{u_{[1,1]}}^2}+\frac{c\,{u_{[0,1]}}^2}{u_{[1,1]}}\right)\boldsymbol{u}_1, \tag{3.8}$$

$$\boldsymbol{u}_t=\boldsymbol{u}_3-3\,\frac{u_{[1,2]}}{u_{[1,1]}}\,\boldsymbol{u}_2+\frac{3}{2}\left(\frac{u_{[2,2]}}{u_{[1,1]}}+\frac{{u_{[1,2]}}^2}{{u_{[1,1]}}^2}+4a\frac{{u_{[0,1]}}^2}{u_{[1,1]}}-\frac{\big(u_{[1,2]}+q'\,u_{[0,1]}\big)^2}{u_{[1,1]}\,\big(u_{[1,1]}+q\big)}\right)\boldsymbol{u}_1, \tag{3.9}$$
$$q=a\,{u_{[0,0]}}^2+b\,u_{[0,0]}+c,\;\; q'=2a\,u_{[0,0]}+b,$$

$$\begin{aligned}\boldsymbol{u}_t&=\boldsymbol{u}_3-3\,\frac{u_{[1,2]}}{u_{[1,1]}}\,\boldsymbol{u}_2+\\&+\frac{3}{2}\left(\frac{u_{[2,2]}}{u_{[1,1]}}+\frac{{u_{[1,2]}}^2}{{u_{[1,1]}}^2}-\frac{4\,u_{[0,1]}u_{[1,2]}}{\big(u_{[0,0]}+a\big)\,u_{[1,1]}}+4\,\frac{{u_{[0,1]}}^2\,\big(u_{[1,1]}+b\big)}{\big(u_{[0,0]}+a\big)^2\,u_{[1,1]}}\right)\boldsymbol{u}_1\end{aligned} \tag{3.10}$$

$$\boldsymbol{u}_t=\boldsymbol{u}_3-3\,\frac{u_{[1,2]}}{u_{[1,1]}}\,\boldsymbol{u}_2+3\left(\frac{u_{[2,2]}}{u_{[1,1]}}+2\,\frac{u_{[0,2]}u_{[1,1]}-2\,u_{[0,1]}u_{[1,2]}+u^2_{[1,1]}}{(u_{[0,0]}+a)u_{[1,1]}}\right)\boldsymbol{u}_1 \tag{3.11}$$

$$\boldsymbol{u}_t = \boldsymbol{u}_3 - 3\,\frac{u_{[0,1]}}{u_{[0,0]}+a}\,\boldsymbol{u}_2 + 3\,\left(\frac{2\,{u_{[0,1]}}^2}{(u_{[0,0]}+a)^2} - \frac{u_{[0,2]}}{u_{[0,0]}+a}\right)\boldsymbol{u}_1, \tag{3.12}$$

$$\boldsymbol{u}_t = \boldsymbol{u}_3 - 3\,\frac{u_{[0,1]}}{u_{[0,0]}+a}\,\boldsymbol{u}_2 + \frac{3}{2}\left(\frac{5\,{u_{[0,1]}}^2}{(u_{[0,0]}+a)^2} + \frac{b - u_{[1,1]} - 2\,u_{[0,2]}}{u_{[0,0]}+a}\right)\boldsymbol{u}_1, \tag{3.13}$$

$$\begin{aligned}\boldsymbol{u}_t = \boldsymbol{u}_3 - \frac{3}{2}&\left(au_{[2,2]} - \frac{\big(au_{[1,2]}(1+bu_{[0,0]}) + bu_{[0,1]}(1-au_{[0,2]})\big)^2}{\zeta(1+bu_{[0,0]})}\right.\\ &\left. - \frac{b(1-au_{[0,2]})^2}{a(1+bu_{[0,0]})}\right)\boldsymbol{u}_1,\\ \zeta &= (1+au_{[1,1]})(1+bu_{[0,0]}) - abu_{[0,1]}^2,\end{aligned} \tag{3.14}$$

$$\begin{aligned}\boldsymbol{u}_t = \boldsymbol{u}_3 - \frac{3}{2}&\left(au_{[2,2]} - \frac{\big(au_{[0,0]}u_{[1,2]} + u_{[0,1]}(1-au_{[0,2]})\big)^2}{\xi\,u_{[0,0]}}\right.\\ &\left. - \frac{(1-au_{[0,2]})^2}{au_{[0,0]}}\right)\boldsymbol{u}_1,\\ \xi &= u_{[0,0]}(1+au_{[1,1]}) - au_{[0,1]}^2,\end{aligned} \tag{3.15}$$

$$\begin{aligned}\boldsymbol{u}_t = \boldsymbol{u}_3 &- 3\,\frac{u_{[0,1]}}{u_{[0,0]}+a}\boldsymbol{u}_2 - 3\left(\frac{u_{[1,1]}}{u_{[0,0]}+a} - \frac{{u_{[0,1]}}^2}{\big(u_{[0,0]}+a\big)^2}\right)\boldsymbol{u}_1 +\\ &+ 3b\,\big(u_{[0,0]}+a\big)\,\big(u_{[0,1]}\boldsymbol{u} + u_{[0,0]}\boldsymbol{u}_1\big),\end{aligned} \tag{3.16}$$

$$\begin{aligned}\boldsymbol{u}_t = \boldsymbol{u}_3 &- 3\,\frac{u_{[0,1]}}{u_{[0,0]}+a}\,\boldsymbol{u}_2\\ &- \frac{3}{2(u_{[0,0]}+a)}\left(b\,u_{[2,2]} - \frac{{u_{[0,1]}}^2}{u_{[0,0]}+a} - \frac{\big(b\,u_{[1,2]} + u_{[0,1]}\big)^2}{b\,u_{[1,1]} + u_{[0,0]} + a} + u_{[1,1]}\right)\boldsymbol{u}_1,\end{aligned} \tag{3.17}$$

$$\boldsymbol{u}_t = \boldsymbol{u}_3 - 3\,\frac{u_{[0,1]}}{u_{[0,0]}+a}\,\boldsymbol{u}_2 - \frac{3}{2}\left(\frac{u_{[1,1]}}{u_{[0,0]}+a} - \frac{2\,{u_{[0,1]}}^2}{\big(u_{[0,0]}+a\big)^2} + b\,u_{[0,0]}\right)\boldsymbol{u}_1, \tag{3.18}$$

$$\boldsymbol{u}_t = \boldsymbol{u}_3 - 3\,\frac{u_{[0,1]}}{u_{[0,0]}+a}\,\boldsymbol{u}_2 + \frac{3}{2}\left(\frac{u_{[2,2]}}{b\left(u_{[0,0]}+a\right)} - \frac{(u_{[0,2]}+b+b\,u_{[0,0]})^2}{b(u_{[0,0]}+c)(u_{[0,0]}+a)}\right.$$
$$+\frac{(u_{[0,2]}\,u_{[0,1]} - u_{[1,2]}(u_{[0,0]}+c) + (2\,u_{[0,0]}+c+a)b\,u_{[0,1]})^2}{b\,\eta\,(u_{[0,0]}+c)(u_{[0,0]}+a)} +$$
$$\left. + \frac{\left(\eta\left(u_{[0,0]}+a\right) + {u_{[0,1]}}^2(c-a)\right)}{\left(u_{[0,0]}+c\right)\left(u_{[0,0]}+a\right)^2}\right)\boldsymbol{u}_1, \tag{3.19}$$
$$\eta = {u_{[0,1]}}^2 + \left(b\left(u_{[0,0]}+a\right) - u_{[1,1]}\right)\left(u_{[0,0]}+c\right),$$

$$\boldsymbol{u}_t = \boldsymbol{u}_3 + 3\left(\frac{u_{[0,1]}u_{[0,2]} - u_{[0,0]}u_{[1,2]}}{\xi} + \frac{u_{[0,1]}}{u_{[0,0]}}\right)\boldsymbol{u}_2 + c_0\,\frac{\sqrt{\xi+q}}{\sqrt{u_{[0,0]}}}\boldsymbol{u}$$
$$+\frac{3}{2}\left(\frac{\left(q(u_{[1,2]}u_{[0,0]} - u_{[0,2]}u_{[0,1]}) - u_{[0,1]}\xi\,(c + 2\,a\,u_{[0,0]} + 3\,b\,\sqrt{u_{[0,0]}}\,)\right)^2}{q\,\xi^2(\xi+q)}\right.$$
$$\left.+\frac{u_{[2,2]}u_{[0,0]}}{\xi} - \frac{\left(u_{[0,2]}u_{[0,0]}+\xi\right)^2}{\xi u_{[0,0]}^2} + \frac{{u_{[0,1]}}^2 u_{[0,0]}\,(a\,c-b^2)}{q\,\xi}\right)\boldsymbol{u}_1, \tag{3.20}$$
$$\xi = u_{[1,1]}u_{[0,0]} - {u_{[0,1]}}^2, \quad q = u_{[0,0]}\left(a\,u_{[0,0]} + 2b\sqrt{u_{[0,0]}} + c\right),$$

$$\boldsymbol{u}_t = \boldsymbol{u}_3 + 3\left(\frac{u_{[0,1]}u_{[0,2]} - u_{[0,0]}u_{[1,2]}}{\xi} + \frac{u_{[0,1]}}{u_{[0,0]}}\right)\boldsymbol{u}_2 + c_0\left(1 + a\sqrt{u_{[0,0]}}\,\right)\boldsymbol{u}$$
$$+\frac{3}{2}\left(\frac{u_{[2,2]}u_{[0,0]}}{\xi} - \frac{\left(u_{[0,2]}u_{[0,0]}+\xi\right)^2}{\xi\,{u_{[0,0]}}^2}\right.$$
$$\left.+\frac{\left(u_{[0,0]}u_{[1,2]} - u_{[0,1]}u_{[0,2]} + q u_{[0,1]}\xi\right)^2}{\xi^2} + \frac{c_1 {u_{[0,1]}}^2}{\xi\,\left(1 + a\sqrt{u_{[0,0]}}\right)^2}\right)\boldsymbol{u}_1, \tag{3.21}$$
$$q = -\frac{2\,a\sqrt{u_{[0,0]}}+1}{\left(a\sqrt{u_{[0,0]}}+1\right)u_{[0,0]}}, \quad \xi = u_{[1,1]}u_{[0,0]} - {u_{[0,1]}}^2,$$

$$
\begin{aligned}
\boldsymbol{u}_t &= \boldsymbol{u}_3 + 3\left(\frac{u_{[0,1]}u_{[0,2]} - u_{[0,0]}u_{[1,2]}}{\xi} + \frac{u_{[0,1]}}{u_{[0,0]}}\right)\boldsymbol{u}_2 + c_0\sqrt{u_{[0,0]}}\,\boldsymbol{u} + \\
&+ \frac{3}{2}\left(\frac{u_{[0,0]}u_{[2,2]}}{\xi} - \frac{(u_{[0,0]}u_{[0,2]} + \xi)^2}{\xi\, {u_{[0,0]}}^2}\right. \\
&\quad \left. + \frac{({u_{[0,0]}}^2 u_{[1,2]} - u_{[0,0]}u_{[0,1]}u_{[0,2]} - 2\,\xi\, u_{[0,1]})^2}{\xi^2\, {u_{[0,0]}}^2} + \frac{c_1\, {u_{[0,1]}}^2}{\xi\, u_{[0,0]}}\right)\boldsymbol{u}_1, \\
&\xi = u_{[1,1]}u_{[0,0]} - {u_{[0,1]}}^2,
\end{aligned}
\tag{3.22}
$$

$$
\begin{aligned}
\boldsymbol{u}_t &= \boldsymbol{u}_3 + 3\left(\frac{u_{[0,1]}\,u_{[0,2]} - u_{[0,0]}\,u_{[1,2]}}{\xi} + \frac{u_{[0,1]}}{u_{[0,0]}}\right)\boldsymbol{u}_2 + c_0\left(1 + a\sqrt{u_{[0,0]}}\,\right)\boldsymbol{u} \\
&+ 3\left(\frac{u_{[2,2]}u_{[0,0]}}{\xi} - \frac{\left(2\,u_{[0,2]}u_{[0,0]} + \xi\, q u_{[0,0]} + 2\,\xi\right)^2}{4\,\xi\, {u_{[0,0]}}^2} + \frac{1}{4}\,\xi\, q^2\right. \\
&\quad \left. + 2\,\frac{q u_{[0,1]}\left(u_{[1,2]}u_{[0,0]} - u_{[0,1]}u_{[0,2]}\right)}{\xi} - 2\,\frac{{u_{[0,1]}}^2 q}{u_{[0,0]}}\right)\boldsymbol{u}_1, \\
&\xi = u_{[1,1]}u_{[0,0]} - {u_{[0,1]}}^2, \quad q = -\frac{a}{\left(1 + a\sqrt{u_{[0,0]}}\,\right)\sqrt{u_{[0,0]}}}\,,
\end{aligned}
\tag{3.23}
$$

$$
\begin{aligned}
\boldsymbol{u}_t &= \boldsymbol{u}_3 + 3\left(\frac{u_{[0,2]}u_{[0,1]}}{\xi} - \frac{u_{[1,2]}u_{[0,0]}}{\xi} + \frac{u_{[0,1]}}{u_{[0,0]}}\right)\boldsymbol{u}_2 + c_0\sqrt{u_{[0,0]}}\,\boldsymbol{u} \\
&+ 3\left(\frac{u_{[0,0]}u_{[2,2]}}{\xi} - \frac{{u_{[0,2]}}^2}{\xi} - 2\,\frac{u_{[0,1]}\left(u_{[1,2]}u_{[0,0]} - u_{[0,2]}u_{[0,1]}\right)}{\xi\, u_{[0,0]}} - \right. \\
&\quad \left. - \frac{u_{[0,2]}}{u_{[0,0]}} + 2\,\frac{{u_{[0,1]}}^2}{{u_{[0,0]}}^2}\right)\boldsymbol{u}_1, \quad \xi = u_{[1,1]}u_{[0,0]} - {u_{[0,1]}}^2.
\end{aligned}
\tag{3.24}
$$

$$
\begin{aligned}
\boldsymbol{u}_t = & \boldsymbol{u}_3 + \frac{3}{2}\left(\frac{b\,u_{[0,1]}u_{[0,2]} - b\,u_{[1,2]}\left(u_{[0,0]}+a\right)}{p\,(p-1)\left(u_{[0,0]}+a\right)^2} + \frac{2\,u_{[0,1]}}{p\left(u_{[0,0]}+a\right)}\right)\boldsymbol{u}_2 \\
& + \frac{3}{2}\left(\frac{b\,u_{[2,2]}}{(p-1)\left(u_{[0,0]}+a\right)} - \frac{b^2\left(u_{[1,2]}\left(u_{[0,0]}+a\right) - u_{[0,2]}\,u_{[0,1]}\right)^2}{p^2\,(p-1)\left(u_{[0,0]}+a\right)^4}\right. \\
& \qquad - \frac{b\,{u_{[0,2]}}^2}{\left(u_{[0,0]}+a\right)^2(p-1)} - \frac{2\,u_{[0,2]}}{u_{[0,0]}+a} + \frac{4\,{u_{[0,1]}}^2}{p^2\left(u_{[0,0]}+a\right)^2} \\
& \qquad \left. + \frac{2\,b\,u_{[0,1]}\,(p-2)\left(u_{[1,2]}\left(u_{[0,0]}+a\right) - u_{[0,2]}\,u_{[0,1]}\right)}{\left(u_{[0,0]}+a\right)^3(p-1)\,p^2}\right)\boldsymbol{u}_1,
\end{aligned} \tag{3.25}
$$

$$
p = \sqrt{1 + b\,\frac{u_{[1,1]}\left(u_{[0,0]}+a\right) - {u_{[0,1]}}^2}{\left(u_{[0,0]}+a\right)^2}}.
$$

Everywhere in formulas (3.1)–(3.25) a, b, c, c_i are arbitrary constants.

Comment. It follows from the zeroth condition of integrability (2.5) that f_2 is the total derivative of another function:

$$
f_2 = \frac{3}{2}D_x \ln h, \tag{3.26}
$$

where $\operatorname{ord} h \leqslant 1$ because $\operatorname{ord} f_2 \leqslant 2$. This implies

$$
\rho_0 = -\frac{1}{2}D_x \ln h, \quad \theta_0 = -\frac{1}{2}D_t \ln h. \tag{3.27}
$$

The direct computation provides that any second order conserved density ρ is the second power polynomial of $\boldsymbol{u}_2$:

$$
\rho = c_1 h\, u_{[2,2]} + h_1 u_{[1,2]}^2 + h_2 u_{[0,2]}^2 + h_3 u_{[1,2]} u_{[0,2]} + p\, u_{[1,2]} + q\, u_{[0,2]} + r, \tag{3.28}
$$

where c_1 is a constant, h_i, p, q and r are some fuctions of the first order. Using (3.26) and (2.4) one can find that

$$
\rho_1 = \frac{1}{4}(D_x \ln h)^2 - \frac{1}{3}f_1 + \frac{1}{3}D_x f_2.
$$

Comparing ρ_1 and (3.28) one concludes that f_1 takes the following form:

$$
f_1 = c_1\, h\, u_{[2,2]} + f_4\, u_{[1,2]}^2 + f_5\, u_{[1,2]}\, u_{[0,2]} + f_6\, u_{[0,2]}^2 + f_7\, u_{[1,2]} + f_8\, u_{[0,2]} + f_9. \tag{3.29}
$$

Further integrability conditions have proved to be very cumbersome and investigation of them have been performed by a computer. That is why we can not present this analysis in the paper. Note that the case $\operatorname{ord} f_0 = 2$ is extremely difficult for analysis and we have not been able to perform it. (Example of a system with $\operatorname{ord} f_0 = 2$ is given in Appendix).

Remark 1. Equations (3.1) and (3.2) differ from (1.1) by a dilatation of $\boldsymbol{u}$; equations (3.3) – (3.7) have been presented in [6]. Equations (3.14) and (3.15) have been presented in [7] in a different form.

Remark 2. Equation (3.8) with $c = 0$ coincides with (3.5) if $a = 0$ in it.

Remark 3. Equation (3.9) is reduced to (3.5) when $a = b = 0$ and it is reduced to (3.4) when $a = b = c = 0$.

Remark 4. Some equations can be obtained as limit cases of other equations:
(a) equation (3.5) is reduced to (3.4) when $a \to \infty$;
(b) equation (3.11) is reduced to (3.7) when $a \to \infty$;
(c) equation (3.10) with $b = ca^2/4$ is reduced to (3.8) when $a \to \infty$;
(d) equation (3.14) is reduced to (3.15) when $b \to \infty$;
(e) equation (3.17) with $b = ca$ is reduced to (3.3) when $a \to \infty$;
(f) equation (3.21) with $c_1 = c_1' a^2, c_0 = c_0'/a$ is reduced to (3.22) when $a \to \infty$;
(g) equation (3.23) is reduced to (3.24) when $a \to \infty$;
(h) equation (3.25) is reduced to (3.12) when $b \to 0$, $p \to -1$;
(i) equation (3.25) with $a = 0$ can be reduced to (3.24) when $b \to 0, p \to 1$;
(j) equation (3.25) with $a \neq 0$ and $b \to 0, p \to 1$ can be reduced to (3.11) by the following point transformation $\boldsymbol{u} = 2\boldsymbol{v}\sqrt{-a}/(1 + v_{[0,0]})$.

Remark 5. If one sets $c_1 = 0$ in (3.22), then, for new functions $\boldsymbol{v} = \boldsymbol{u}/|\boldsymbol{u}|,\ r = |\boldsymbol{u}|$, a triangular system is obtained where $\boldsymbol{v}$ satisfies the independent equation (11) from [5] with $a = 0$.

Remark 6. If one sets $a = 0$ in (3.25), then, for new functions $\boldsymbol{v} = \boldsymbol{u}/|\boldsymbol{u}|,\ r = |\boldsymbol{u}|$, a triangular system is obtained where $\boldsymbol{v}$ satisfies equation (14) from [5].

4 Differential substitutions and Bäcklund transformations

To prove the integrability of a new equation one ought to find a Lax representation or a zero curvature representation. Also, there are two other ways of proving it, namely (i) to find a Miura-type transformation between the new equation and an equation known to be integrable and (ii) to find an auto-Backlund transformation for the new equation.

A first order differential substitution

$$\boldsymbol{u} = h_1 \boldsymbol{v}_1 + h_0 \boldsymbol{v}. \tag{4.1}$$

generalizes to the Miura transformation. Here smooth functions h_0 and h_1 depend on $v_{[0,0]}, v_{[0,1]}$ and $v_{[1,1]}$. The function $\boldsymbol{u}$ is supposed to be a solution of some equation from the list, and $\boldsymbol{v}$ is supposed to be a solution of an equation of the type (1.6)

$$\boldsymbol{v}_t = \boldsymbol{v}_3 + D_x(g_2)\boldsymbol{v}_2 + g_1 \boldsymbol{v}_1 + g_0 \boldsymbol{v} \tag{4.2}$$

with unknown smooth functions $g_i(v_{[j,k]})$, where $\operatorname{ord} g_1 \leqslant 2$ and $\operatorname{ord} g_0 \leqslant 1$. The form of the coefficient $D_x(g_2)$ with $\operatorname{ord} g_2 \leqslant 1$ is necessary for integrability.

It is possible that higher order differential substitutions also exist. But computations of the higher order differential substitutions are extremely cumbersome, therefore only the first order differential substitutions have been computed.

Consider a general case of equation (1.6). By differentiating equation (4.1) with respect to t one obtains the equation

$$\boldsymbol{u}_3 + f_2\boldsymbol{u}_2 + f_1\boldsymbol{u}_1 + f_0\boldsymbol{u}_0 = D_t(h_1\boldsymbol{v}_1 + h_0\boldsymbol{v}), \tag{4.3}$$

where the right-hand side is differentiated according to (4.2). Hence, the right-hand side of (4.3) depends on $\boldsymbol{v}_k$ and $v_{[i,j]}$ only. The left-hand side of (4.3) depends on $\boldsymbol{u}_k$ and $u_{[i,j]}$. Using the differential consequences of (4.1)

$$\boldsymbol{u} = h_1\boldsymbol{v}_1 + h_0\boldsymbol{v}, \qquad \boldsymbol{u}_1 = D_x(h_1\boldsymbol{v}_1 + h_0\boldsymbol{v}), \qquad \boldsymbol{u}_2 = D_x^2(h_1\boldsymbol{v}_1 + h_0\boldsymbol{v}), \ldots$$

one can evaluate their scalar products by pairs and express all variables $u_{[i,j]}$ in terms of $v_{[k,l]}$. Thus, one can exclude the variables $\boldsymbol{u}_k$ and $u_{[i,j]}$ from (4.3). Then, it ought to split the obtained equation with respect to $\boldsymbol{v}_k, k = 0, 1, \ldots$ and then additional splitting is possible with respect to $v_{[i,j]}$. In this way one obtains an overdetermined partial differential system for h_i and g_k. If a solution of this system exists it provides a diffrential substitution and a corresponding equation (4.2).

The first order auto-Bäcklund transformation takes the following form

$$\boldsymbol{u}_1 = h_1\boldsymbol{v}_1 + h_2\boldsymbol{v} + h_3\boldsymbol{u}, \tag{4.4}$$

where both $\boldsymbol{u}$ and $\boldsymbol{v}$ satisfy (1.6), smooth functions h_i depend on $u_{[0,0]}, v_{[i,j]}$, $0 \leqslant i \leqslant j \leqslant 1$ and $w_i = (\boldsymbol{u}, \boldsymbol{v}_i), i \geqslant 0$. An algorithm for computation the auto-Bäcklund transformations is the same as for the diffrential substitutions. There is only one important difference between Bäcklund transformations and diffrential substitutions. A Bäcklund transformation must depend on a "spectral" parameter.

Differential substitutions. If $\boldsymbol{u}$ satisfies (3.2) with $c = -1$ and $\boldsymbol{v}$ satisfies (3.16), then

$$\boldsymbol{u} = \frac{\boldsymbol{v}_1}{\sqrt{v_{[0,0]} + a}} + \sqrt{-b}\,\sqrt{v_{[0,0]} + a}\,\boldsymbol{v}. \tag{4.5}$$

If $\boldsymbol{u}$ satisfies (3.1) with $c = -1$ and $\boldsymbol{v}$ satisfies (3.18) then

$$\boldsymbol{u} = \frac{\boldsymbol{v}_1}{\sqrt{v_{[0,0]} + a}} + \sqrt{b}\,\boldsymbol{v}. \tag{4.6}$$

If $\boldsymbol{u}$ satisfies (3.13) with $a = -k$, $b = -c_1\,k$ and $\boldsymbol{v}$ satisfies (3.21), then

$$\boldsymbol{u} = \sqrt{k}\left(\frac{a\,v_{[0,0]} + \sqrt{v_{[0,0]}}}{v_{[0,1]}}\boldsymbol{v}_1 - a\,\boldsymbol{v}\right). \tag{4.7}$$

where a is the parameter from (3.21).

If $\boldsymbol{u}$ satisfies (3.12) with $a=-k$ and $\boldsymbol{v}$ satisfies (3.23),then

$$\boldsymbol{u}=\sqrt{k}\left(\frac{a\,v_{[0,0]}+\sqrt{v_{[0,0]}}}{v_{[0,1]}}\boldsymbol{v}_1-a\,\boldsymbol{v}\right), \tag{4.8}$$

where a is the parameter from (3.23).

Bäcklund transformations. In all formulas below, μ is an arbitrary parameter. Equation (3.1) has the following auto-Bäcklund transformation:

$$\boldsymbol{u}_x+\boldsymbol{v}_x=\frac{1}{2}(\boldsymbol{u}-\boldsymbol{v})\sqrt{\mu-c(\boldsymbol{u}+\boldsymbol{v})^2}. \tag{4.9}$$

The auto-Bäcklund transformation for equation (3.2) is:

$$\boldsymbol{u}_x+\boldsymbol{v}_x=\mu(\boldsymbol{u}-\boldsymbol{v})-\frac{\mu+\sqrt{\mu^2-(c/2)(\boldsymbol{u}+\boldsymbol{v})^2}}{(\boldsymbol{u}+\boldsymbol{v})^2}\big((u_{[0,0]}+w_0)\boldsymbol{u}-(v_{[0,0]}+w_0)\boldsymbol{v}\big). \tag{4.10}$$

Equation (3.8) has the following auto-Bäcklund transformation:

$$\boldsymbol{u}_x=\left(\frac{g}{\sqrt{v_{[1,1]}}}-1\right)\left(\boldsymbol{v}_x-2\frac{(\boldsymbol{u}+\boldsymbol{v},\boldsymbol{v}_x)}{(\boldsymbol{u}+\boldsymbol{v})^2}(\boldsymbol{u}+\boldsymbol{v})\right), \tag{4.11}$$

where

$$g=\frac{1}{2}\sqrt{\mu\,(\boldsymbol{u}+\boldsymbol{v})^2-c\,(u_{[0,0]}-v_{[0,0]})^2}.$$

The auto-Bäcklund transformation for equation (3.9) takes the following form:

$$\boldsymbol{u}_1=-\frac{p\,(f^2+2f\eta+1)}{q\,(f^2-1)}\left(\boldsymbol{v}_1-\frac{2(\boldsymbol{u}+\boldsymbol{v},\boldsymbol{v}_1)}{(\boldsymbol{u}+\boldsymbol{v})^2}\,(\boldsymbol{u}+\boldsymbol{v})\right), \tag{4.12}$$

where

$$\eta^2=1+q^2/v_{[1,1]},\ p^2=au_{[0,0]}^2+bu_{[0,0]}+c,\ q^2=av_{[0,0]}^2+bv_{[0,0]}+c,$$
$$f^2=\frac{a(u_{[0,0]}-v_{[0,0]})^2-(p+q)^2-\mu\,(\boldsymbol{u}+\boldsymbol{v})^2}{a(u_{[0,0]}-v_{[0,0]})^2-(p-q)^2-\mu\,(\boldsymbol{u}+\boldsymbol{v})^2}.$$

The auto-Bäcklund transformation for equation (3.10) may be written in the following form:

$$\boldsymbol{u}_1=-\Big(\frac{p^2}{q^2}-\frac{p\,\eta}{q\sqrt{v_{[1,1]}}}\Big)\left(\boldsymbol{v}_1-\frac{2(\boldsymbol{u}+\boldsymbol{v},\boldsymbol{v}_1)}{(\boldsymbol{u}+\boldsymbol{v})^2}(\boldsymbol{u}+\boldsymbol{v})\right), \tag{4.13}$$

where $p^2 = u_{[0,0]} + a$, $q^2 = v_{[0,0]} + a$, $\eta^2 = \mu(\boldsymbol{u}+\boldsymbol{v})^2 - b(u_{[0,0]} - v_{[0,0]})^2$.

Equation (3.11) has the auto-Bäcklund transformation of the following form:

$$\boldsymbol{u}_1 = \left(\frac{p}{q} + \frac{\mu\,\eta}{q v_{[1,1]}}\right)\left(2\frac{(\boldsymbol{u}+\boldsymbol{v},\boldsymbol{v}_1)}{(\boldsymbol{u}+\boldsymbol{v})^2}(\boldsymbol{u}+\boldsymbol{v}) - \boldsymbol{v}_1\right), \tag{4.14}$$

where $p = u_{[0,0]} + a$, $q = v_{[0,0]} + a$, $\eta = 2v_{[0,1]}(w_0 - a) + v_{[0,1]}p - w_1 q$.

The auto-Bäcklund transformation for equation (3.12) is written:

$$\boldsymbol{u}_1 = \frac{p}{q}\boldsymbol{v}_1 + p\frac{\mu(w_1 q - v_{[0,1]}p) - aq(a - pq + w_0)}{\mu\, q^2(a - p\,q + w_0)}\,(\boldsymbol{u}-\boldsymbol{v})\,, \tag{4.15}$$

where $p^2 = u_{[0,0]} + a$, $q^2 = v_{[0,0]} + a$.

The auto-Bäcklund transformation for equation (3.13) takes the following form:

$$\boldsymbol{u}_1 = \frac{p}{q}\boldsymbol{v}_1 + p\frac{q\,w_1 - p\,v_{[0,1]} + q\,\eta}{(w_0 - p\,q + a)q^2}\,(\boldsymbol{u}-\boldsymbol{v})\,, \tag{4.16}$$

where $p^2 = u_{[0,0]} + a$, $q^2 = v_{[0,0]} + a$, $\eta^2 = \mu\,p\,q(w_0 - p\,q + a) - b\,(q-p)^2$.

Equation (3.17) has the auto-Bäcklund transformation of the following form

$$\boldsymbol{u}_1 = \frac{p}{q}\boldsymbol{v}_1 + p\frac{b\,\mu(w_1 - v_{[0,1]}) + \eta}{q\,(w_0 - p\,q + a)}\;(\boldsymbol{v}-\boldsymbol{u})\,, \tag{4.17}$$

where $\eta^2 = \mu\big(b\,v_{[1,1]} + q^2\big)\big(b\,\mu(u_{[0,0]} + v_{[0,0]} - 2w_0) - 2(w_0 - p\,q + a)\big)$, $p^2 = u_{[0,0]} + a$, $q^2 = v_{[0,0]} + a$.

The auto-Bäcklund transformation for equation (3.14) takes the following form

$$\boldsymbol{u}_1 - \boldsymbol{v}_1 = \left(\frac{f\eta}{1 + bv_{[0,0]}} - v_{[0,1]}\frac{b + \mu(bw_0 - 1)}{1 + bv_{[0,0]}} + \mu w_1\right)(\boldsymbol{u}+\boldsymbol{v}), \tag{4.18}$$

where

$$f^2 = b\mu^2(u_{[0,0]}v_{[0,0]} - w_0^2) - b(1 + 2\mu w_0) + 2\mu + \mu^2(\boldsymbol{u}+\boldsymbol{v})^2,$$
$$a\,\eta^2 = (1 + av_{[1,1]})(1 + bv_{[0,0]}) - abv_{[0,1]}^2.$$

The auto-Bäcklund transformation for equation (3.15) can be written as

$$\boldsymbol{u}_1 - \boldsymbol{v}_1 = \left(\frac{f\eta}{v_{[0,0]}} - v_{[0,1]}\frac{1 + \mu w_0}{v_{[0,0]}} + \mu w_1\right)(\boldsymbol{u}+\boldsymbol{v}), \tag{4.19}$$

where

$$f^2 = \mu^2(u_{[0,0]}v_{[0,0]} - w_0^2) - 1 - 2\mu w_0, \quad a\,\eta^2 = (1 + av_{[1,1]})v_{[0,0]} - av_{[0,1]}^2.$$

Remark 7. The auto-Bäcklund transformations (4.9), (4.10), (4.18) and (4.19) have been found in [7]. The auto-Bäcklund transformations for equations (3.3)–(3.7) have been found in [6].

Remark 8. It can be verified that auto-Bäcklund transformation (4.14) is reduced to auto-Bäcklund transformation (54) from [6] in accordance with Remark 4b.

Remark 9. It is obvious that the auto-Bäcklund transformation (4.18) is reduced to (4.19) when $b \to \infty$ in accordance with Remark 4d.

Corollaries.

1. It is clear from Remark 1 that integrability of equations (3.1)–(3.7) has been proved in the papers referred to above.
2. It follows from BTs (4.11)–(4.19) that equations (3.8), (3.9), (3.10), (3.11), (3.12), (3.13), (3.14), (3.15), and (3.17) are integrable.
3. It follows from differential substitutions (4.5)–(4.8) that equations (3.16), (3.18), (3.21) and (3.23) are connected with the integrable equations listed in the previous points 1,2, hence equations (3.16), (3.18), (3.21) and (3.23) are integrable too.

Thus, integrability of the five equations (3.19), (3.20), (3.22), (3.24) and (3.25) is not proved. These equations are cumbersome and their auto-Bäcklund transformations are even more cumbersome. Moreover, the benefit of these equations for applications is unclear, that is why we have not computed these five Bäcklund transformations.

Conclusion

For each equation from the list, eight integrability conditions have been verified. It was found that each equation possesses at least five higher order conserved densities. This is a weighty argument for exact integrability of all equations.

The obtained equations may be useful in future investigations of vectorial integrable equations, such as, for example, the hyperbolic equations.

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Appendix. An example of an equation with ord $f_0 = 2$

The following equation satisfies six conditions of integrability:

$$\boldsymbol{u}_t = \boldsymbol{u}_3 + \frac{3}{2}(\ln f)_x \boldsymbol{u}_2 + 3\left(\frac{u_{[0,1]} f_x^2}{f g^2} - \frac{\varphi f_x}{f g} - \frac{4(b+1)u_{[0,1]}^3}{3bg^2 u_{[0,0]}} + \frac{4u_{[0,1]}^3 h^2}{3b^2 g^4 u_{[0,0]}^3} - \right.$$
$$\left. - \frac{u_{[0,1]} h}{b u_{[0,0]}^2 f g^2}\left[\varphi - \frac{u_{[0,1]}(3g+4)f_x}{2g} - \frac{(bg^2+1) f u_{[0,1]}^2}{3bg^2 u_{[0,0]}}\right] + c\right)\boldsymbol{u} +$$
$$+ \frac{3}{2a}\left(u_{[2,2]} f - \frac{\varphi^2}{u_{[0,0]} f} - \frac{(b u_{[0,0]} g (f f_x u_{[0,0]} u_{[0,1]} - \varphi g) + u_{[0,1]}^2 f h)^2}{b^3 u_{[0,0]}^3 f g^6}\right)\boldsymbol{u}_1.$$

Here $g = u_{[0,0]}f - 1$, $h = b(u_{[0,0]}^2 f^2 - 1) - 1$, $\varphi = u_{[0,2]}f + a$, a, b and c are arbitrary constants. Besides, f is a root of the following cubic equation:

$$\big((u_{[0,1]}^2 - u_{[0,0]}u_{[1,1]})f + a\,u_{[0,0]}\big)(u_{[0,0]}f - 1)^2 + u_{[0,1]}^2 b^{-1} f = 0.$$

This equation contains the second order function f_0 (because $\operatorname{ord}\varphi = 2$). There are other such terribly cumbersome examples. That is why the complete classification of integrable equations (1.6) has not been obtained until now.

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