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Lagrangian formulation of a generalized Lane-Emden equation and double reduction

C Masood KHALIQUE, [‡] *Fazal M MAHOMED* [†] *and Ben MUATJETJEJA* [‡]

[‡] *International Institute for Symmetry Analysis and Mathematical Modelling Department of Mathematical Sciences, North-West University, Mafikeng Campus Private Bag X2046, Mmabatho 2735, South Africa*
E-mail: Masood.Khalique@nwu.ac.za

[†] *Centre for Differential Equations, Continuum Mechanics and Applications School of Computational and Applied Mathematics, University of the Witwatersrand Wits 2050, South Africa*
E-mail: Fazal.Mahomed@wits.ac.za

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Abstract

We classify the Noether point symmetries of the generalized Lane-Emden equation $y''+ny'/x+f(y)=0$ with respect to the standard Lagrangian $L = x^n y'^2/2 - x^n \int f(y)dy$ for various functions $f(y)$. We obtain first integrals of the various cases which admit Noether point symmetry and find reduction to quadratures for these cases. Three new cases are found for the function $f(y)$. One of them is $f(y) = \alpha y^r$, where $r \neq 0, 1$. The case $r = 5$ was considered previously and only a one-parameter family of solutions was presented. Here we provide a complete integration not only for $r = 5$ but for other r values. We also give the Lie point symmetries for each case. In two of the new cases, the single Noether symmetry is also the only Lie point symmetry.

1 Introduction

The Lane-Emden-type equation

$$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + f(y) = 0, \tag{1.1}$$

for various forms of $f(y)$, has been used to model several phenomena in mathematical physics and astrophysics. The most popular form of $f(y)$ is when $f(y) = y^r$, where r is a constant. In this case eqn (1.1) takes the form

$$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + y^r = 0. \tag{1.2}$$

This equation was used to model the thermal behaviour of a spherical cloud of gas acting under the mutual attraction of its molecules and subject to the classical laws of thermodynamics. Eqn (1.2) was first proposed by Lane (see Thompson [39]). It was studied in more detail by Emden [13] and Fowler [16]. Numerical and perturbation approaches to solve eqn (1.2) have been considered in Horedt [20, 21], Bender [5] and Lema [29]. Other methods have also been invoked for the solution of eqn (1.2) (see Roxbough and Stocken [36], Adomian et al [2], Shawagfeh [38], Burt [8], Wazwaz [41] and Liao [27]).

It is known that for $r = 0, 1$ and 5, eqn (1.2) has exact solutions (Chandrasekhar [9], Davis [11], Datta [10] and Wrubel [43]). In fact for $r = 5$, a one-parameter solution of eqn (1.2) is normally given.

The Lane-Emden equation (1.2) appears not only in the study of stellar structures but in other applications as well. The interested reader is refereed to the works of Meerson et al [31], Gnutzmann and Ritschel [17], and Bahcall [3, 4].

A more general form of (1.2), in which the coefficient of y' is considered an arbitrary function of x, was investigated for first integrals by Leach [26]. Moreover, transformation properties of a more general Emden-Fowler equation were considered in the works [15, 32]. The reader is also referred to the review paper by Wong [42] which mentions more than 140 references on the topic.

Another form of $f(y)$ is given by

$$f(y) = (y^2 - C)^{3/2}. \tag{1.3}$$

Inserting (1.3) into eqn (1.1) gives us the "white-dwarf" equation introduced by Chandrasekhar [9] in his study of the gravitational potential of degenerate white-dwarf stars. In fact, when $C = 0$ this equation reduces to Lane-Emden equation with index $r = 3$.

Another nonlinear form of $f(y)$ is the exponential function

$$f(y) = e^y. \tag{1.4}$$

Substituting (1.4) into eqn (1.1) results in a model that describes isothermal gas spheres where the temperature remains constant.

Eqn (1.1) with

$$f(y) = e^{-y}$$

gives a model that appears in the theory of thermionic currents when one seeks to determine the density and electric force of an electron gas in the neighbourhood of a hot body in thermal equilibrium was thoroughly investigated by Richardson [35].

Furthermore, the eqn (1.1) appears in eight additional cases for the function $f(y)$. The interested reader is referred to Davis [11] for more detail.

The equation

$$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + e^{\beta y} = 0, \tag{1.5}$$

where β is a constant, has also been studied by Emden [13].

The so-called generalized Lane-Emden equation of the first kind

$$x\frac{d^2y}{dx^2} + \alpha\frac{dy}{dx} + \beta x^\nu y^n = 0, \tag{1.6}$$

where α, β, ν and n are real, has been recently looked at in Goenner and Havas [18] and Goenner [19]. In Goenner [19], the author uncovered symmetries of eqn (1.6) to explain integrability of (1.6) for certain values of the parameters considered in Goenner and Havas [18]. The reader is also referred to the works ([23], [24], [6], [7]) for symmetries and solutions of Emden-type equations.

In this paper we investigate the Noether point symmetries of the generalized Lane-Emden equation

$$\frac{d^2y}{dx^2} + \frac{n}{x}\frac{dy}{dx} + f(y) = 0, \tag{1.7}$$

where n is a real constant and $f(y)$ as yet arbitrary. It should be pointed out that equation (1.7) for a power function $f(y) = y^r$ is linked to the Emden-Fowler equation $y'' + p(X)y^r = 0$ by means of the change of independent variables $X = x^{1-n}$, $n \neq 1$ and $X = \ln x$ for $n = 1$. The transformation properties in these instances are known (see [15, 32, 40]).

For $n = 2$, Wazwaz [41] considered eqn (1.7) for some particular functions $f(y)$ using an algorithm based on the Adomian decomposition method. Our approach is completely different and provides a complete Noether point symmetry classification of eqn (1.7) for different forms of $f(y)$. We also give the Lie point symmetries for the relevant cases. Moreover, we give reductions of eqn (1.7) in terms of quadratures for seven forms of the function $f(y)$. Among these are three new cases given in Section 3.

The outline of the paper is as follows. In the next section we present the preliminaries of the Noether point symmetry approach and in Section 3 we provide the Noether symmetry classification of eqn (1.7) for various functions $f(y)$ which is done here for the first time. Then in the same Section 3 we determine the double reductions of eqn (1.7) for the functions for which eqn (1.7) has Noether point symmetries, including three new Cases 4.2, 5.1 and 5.2. Concluding remarks are mentioned in Section 4.

2 Preliminaries on Noether symmetry classification and reduction

We first collect some relevant definitions and theorems from the literature which we utilize in what follows. Two of the theorems are not well-known. They are given here due to their importance for the calculations of the next section.

Consider the point type vector field

$$X = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} \tag{2.1}$$

which has first prolongation

$$X^{[1]} = X + (\eta_x + \eta_y y' - \xi_x y' - \xi_y y'^2)\frac{\partial}{\partial y'}. \tag{2.2}$$

Now we focus attention on an arbitrary second-order ODE

$$y'' = E(x, y, y') \tag{2.3}$$

that has Lagrangian $L(x, y, y')$. That is, (2.3) is equivalent to the Euler-Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial L}{\partial y'}\right) - \frac{\partial L}{\partial y} = 0. \tag{2.4}$$

Definition 1
The operator X is called a Noether point symmetry generator corresponding to a Lagrangian $L(x, y, y')$ of eqn (2.3) if there exists a gauge function $B(x, y)$ such that

$$X^{[1]}(L) + D(\xi)L = D(B), \tag{2.5}$$

where

$$D = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'} + \cdots . \tag{2.6}$$

The utility of an available Noether point symmetry generator lies in the following three theorems.

Theorem 1 (Noether [33])
Suppose that X is a Noether point symmetry generator corresponding to a Lagrangian $L(x, y, y')$ of eqn (2.3). Then

$$I = \xi L + (\eta - y'\xi)\frac{\partial L}{\partial y'} - B \tag{2.7}$$

is a first integral of eqn (2.3) associated with the operator X.

Proof. See, e.g., [34, 22].

Theorem 2
The first integral I, associated with the Noether point symmetry X, satisfies

$$X^{[1]}I = 0, \tag{2.8}$$

i.e., X is a point symmetry generator of the first integral I of eqn (2.3).

Proof. See [37, 25]. References [22, 25] contain more general proofs in more general settings.

Theorem 3
Suppose that for a Lagrangian $L(x, y, y')$ of eqn (2.3) there corresponds a Noether point symmetry generator. Then eqn (2.3) has solution in terms of quadratures.

Proof. See [25]. One can deduce this by invoking Theorems 1 and 2.

The approach pursued here was also utilized in [40] to deduce new solutions.

3 Integration of eqn (1.7) for different fs

We consider eqn (1.7). Its standard Lagrangian is

$$L = \frac{1}{2}x^n y'^2 - x^n \int f(y)dy. \tag{3.1}$$

The substitution of the Lagrangian (3.1) into the determining eqn (2.5) and separation with respect to the powers of y' yields the linear overdetermined system of four PDEs

$$\begin{aligned}
&\xi_y = 0,\\
&\eta_y = \frac{1}{2}(\xi_x - nx^{-1}\xi),\\
&x^n\eta_x = B_y,\\
&-nx^{n-1}\xi\int f(y)dy - x^n\eta f(y) - x^n\xi_x\int f(y)dy = B_x.
\end{aligned} \tag{3.2}$$

After straightforward manipulations, the system (3.2) results in

$$\begin{aligned}
&\xi = a(x),\\
&\eta = \frac{1}{2}[a' - nx^{-1}a]y + b(x),\\
&B = \frac{1}{4}x^n[a'' - n\left(\frac{a}{x}\right)']y^2 + b'x^ny + c(x),
\end{aligned} \tag{3.3}$$

$$\begin{aligned}
&[-nx^{n-1}a - a'x^n]\int f(y)dy + [-\frac{1}{2}x^na'y\\
&+\frac{1}{2}nx^{n-1}ay - x^nb]f(y)\\
&= \frac{1}{4}a'''x^ny^2 + \frac{1}{2}nx^{n-2}a'y^2 - \frac{1}{2}nx^{n-3}ay^2\\
&-\frac{1}{4}n^2x^{n-1}\left(\frac{a}{x}\right)'y^2 + b''x^ny + b'nx^{n-1}y + c'
\end{aligned} \tag{3.4}$$

The analysis of eqn (3.4) prompts the following eight cases.

Case 1. $n \neq 0$, $f(y)$ arbitrary but not of the form contained in cases 3, 4, 5 and 6. We find that $\xi = 0$, $\eta = 0$, and $B = k$, constant. Hence, there is no Noether point symmetry for this case.

Noether point symmetries exist in the following cases.

Case 2. $n = 0$, $f(y)$ arbitrary but not linear as in case 3.
We obtain $\xi = 1$, $\eta = 0$ and $B = k$, k a constant.
Therefore we have a single Lie and Noether symmetry generator $X = \dfrac{\partial}{\partial x}$.

The integration for this case is trivial even without a Noether symmetry. The use of the Noether integral (2.7) results in

$$I = \frac{1}{2}y'^2 + \int f(y)dy$$

from which, setting $I = C$, one easily gets quadrature. Note that this case includes the autonomous Ermakov equation [14] ($f(y) = \alpha y^{-3} + \beta y$) which has $sl(2,\Re)$ symmetry algebra. It turns out that $sl(2,\Re)$ is also the Noether symmetry algebra.

Case 3. $f(y)$ is linear in y
This case is well-known and the corresponding eqn (1.7) has $sl(3,\Re)$ symmetry algebra and five Noether point symmetries associated with the standard Lagrangian of the differential equation (1.7) (see [28, 30]).

Case 4. $f(y) = \alpha y^2 + \beta y + \gamma$, where α, β and γ are constants, with $\alpha \neq 0$. There are three subcases. They are

4.1. $n = 5$, $\beta = 0$ and $\gamma = 0$. We obtain $\xi = x$, $\eta = -2y$ and $B = k$, constant. This is contained in Case 5.1 below.

4.2. $n = 5$, $\beta^2 = 4\alpha\gamma$. We get $\xi = x$, $\eta = -(2y + \beta/\alpha)$ and $B = \dfrac{\beta\gamma}{6\alpha}x^6$.

We have a single Noether and Lie point symmetry generator

$$X = x\frac{\partial}{\partial x} - \left(2y + \frac{\beta}{\alpha}\right)\frac{\partial}{\partial y}.$$

The application of Theorem 1, due to Noether, results in

$$I = -\frac{1}{2}x^6 y'^2 - \frac{1}{3}\alpha x^6 y^3 - \frac{1}{2}\beta x^6 y^2 - \gamma x^6 y - 2x^5 yy' - \frac{\beta}{\alpha}x^5 y' - \frac{1}{6}\frac{\beta\gamma}{\alpha}x^6.$$

Thus, the reduced equation is

$$\frac{1}{2}x^6 y'^2 + \frac{1}{3}\alpha x^6 y^3 + \frac{1}{2}\beta x^6 y^2 + \gamma x^6 y + 2x^5 yy' + \frac{\beta}{\alpha}x^5 y' + \frac{1}{6}\frac{\beta\gamma}{\alpha}x^6 = C, \tag{3.5}$$

where C is an arbitrary constant. By Theorem 2, X is also a symmetry generator of I as well as the reduced eqn (3.5). Invoking Theorem 3, we can solve eqn (3.5). In order to solve the first-order ODE (3.5), we use an invariant of X (see [1]) as the dependent variable. This invariant is obtained by solving the characteristic equation associated with X, viz.,

$$\frac{dx}{x} = \frac{dy}{-(2y + \beta/\alpha)}.$$

The solution of this ODE gives the invariant

$$u = x^2 y + \frac{\beta}{2\alpha}x^2.$$

Eqn (3.5) in terms of u, after some calculations, is

$$C = 2u^2 - \frac{1}{2}x^2 u'^2 - \frac{\alpha}{3}u^3.$$

This last first-order ODE is variables separable as

$$\frac{du}{\pm\sqrt{4y^2 - (2/3)\alpha u^3 - 2C}} = \frac{dx}{x}.$$

Hence we have quadrature or double reduction of our equation for the given f.

4.3. $n = 5/3$, $\beta = 0$ and $\gamma = 0$. We find $\xi = x^{1/3}$, $\eta = -\dfrac{2}{3}x^{-2/3}y$ and $B = \dfrac{2}{9}y^2 + k$, k is constant. This is subsumed in Case 5.2 below.

Case 5. $f(y) = \alpha y^r$, α and r are constants with $\alpha \neq 0$ and $r \neq -3, 0, 1$. Here we have two subcases.

5.1. $n = \dfrac{r+3}{r-1}$. We obtain $\xi = x$, $\eta = \dfrac{2}{1-r}y$ and $B = k$, constant. This is also the single Lie point generator.

By the use of Theorems 1, 2 and 3, we find that the solution of eqn (1.7) for the above n and f is

$$y = u^{\frac{2}{r-1}} x^{\frac{2}{1-r}}, \tag{3.6}$$

where u is given by

$$\int \frac{du}{\pm\sqrt{4(1-r)^{-2}u^2 - 2\alpha(1+r)^{-1}u^{1+r} - C_1}} = \ln xC_2, \tag{3.7}$$

in which C_1 and C_2 are arbitrary constants of integration. Note that for $r = 5$, one gets $n = 2$ and we have the general solution given in eqn (3.7). Only a one-parameter family of solutions is known in the literature (see, e.g., [12]). Here we have determined the two-parameter family of solutions.

5.2. $n = \dfrac{r+3}{r+1}$, with $r \neq -1$. We have $\xi = x^{\frac{r-1}{r+1}}$, $\eta = -\dfrac{2}{r+1}x^{-\frac{2}{r+1}}y$ and $B = \dfrac{2}{(r+1)^2}\,y^2 + k$, k is constant. Here we obtain a second Lie point generator corresponding to $\xi = x$, $\eta = \dfrac{2}{1-r}y$. One can utilize double reduction method of Lie. We choose to use the Noether invariant approach.

Again we invoke Theorems 1, 2 and 3. In this subcase we deduce that the solution of eqn (1.7) is

$$y = ux^{-\frac{2}{r+1}}, \tag{3.8}$$

where u is defined by

$$\int \frac{du}{\pm\sqrt{-2\alpha(r+1)^{-1}u^{r+1} + C_1}} = \frac{r+1}{2}x^{\frac{2}{r+1}} + C_2, \tag{3.9}$$

in which C_1 and C_2 are arbitrary constants.

Case 6. $f(y) = \alpha\exp(\beta y) + \gamma y + \delta$, α, β, γ and δ are constants with $\alpha, \beta \neq 0$.

If $n = 1$, $\gamma = 0$ and $\delta = 0$, we deduce $\xi = x$, $\eta = -2/\beta$ and $B = k$, k a constant. The second Lie point symmetry generator is $Y = x\ln x\dfrac{\partial}{\partial x} - \dfrac{2}{\beta}(1+\ln x)\dfrac{\partial}{\partial y}$. One can use double reduction method of Lie here. However, we use the Noether integral approach.

The invocation of Theorems 1, 2 and 3, upon using the Noether symmetry, gives rise to the solution of eqn (1.7) for this case to be

$$y = \frac{2}{\beta}\ln\left(\frac{u}{x}\right), \tag{3.10}$$

where u is given by

$$\int \frac{du}{\pm u\sqrt{1 - (1/2)\alpha\beta u^2 + C_1}} = \ln xC_2, \tag{3.11}$$

in which C_1 and C_2 are integration constants.

Case 7. $f(y) = \alpha\ln y + \gamma y + \delta$, where α, γ and δ are constants with $\alpha \neq 0$.
If $n = 0$ and $\delta = 0$, we obtain $\xi = 1$, $\eta = 0$ and $B = k$, k a constant. This reduces to Case 2.

Case 8. $f(y) = \alpha y\ln y + \gamma y + \delta$, where α, γ and δ are constants with $\alpha \neq 0$.
If $n = 0$, we obtain $\xi = 1$, $\eta = 0$ and $B = k$, k a constant. This reduces to Case 2.

4 Concluding remarks

We have completely classified the Noether point symmetries of the generalized Lane-Emden equation (1.7) with respect to the standard Lagrangian (3.1). This has been done for the first time here. Eight cases arose out of which seven cases resulted in Noether point symmetries. For each of these we presented the Lie point symmetries and obtained the first integral and also reduction to quadrature of the corresponding Lane-Emden equation (1.7). Three new cases were found. These correspond to Cases 4.2, 5.1 and 5.2 of Section 3. It is interesting to mention that we have obtained a two-parameter family of solutions in Case 5.1 for $n = 2$ and $r = 5$ for which only a one-parameter family of solutions is known in the literature.

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