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Cocycles and stream functions in quasigeostrophic motion

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Abstract

We present a geometric version of the Lie algebra 2-cocycle connected to quasigeostrophic motion in the β-plane approximation. We write down an Euler equation for the fluid velocity, corresponding to the evolution equation for the stream function in quasigeostrophic motion.

1 Introduction

The equation for quasigeostrophic motion in β-plane approximation written for the stream function $\psi(x_1, x_2)$ of the geostrophic fluid velocity is [1]

$$\partial_t \Delta\psi = -\{\Delta\psi, \psi\} - \beta\partial_{x_1}\psi, \tag{1.1}$$

with β the gradient of the Coriolis parameter. A treatment of (1.1) as an Euler-Poincaré equation can be found in [2] and [3].

In [4] is shown that the quasigeostrophic motion in β-plane approximation is Euler equation on a central extension of the Lie algebra of exact divergence free vector fields on the flat 2-torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ with volume form $dx_1 \wedge dx_2$. Considering the basis

$$l_n = e^{in\cdot x}(-n_2\partial_{x_1} + n_1\partial_{x_2}), \quad n \in \mathbb{Z}^2 \tag{1.2}$$

of hamiltonian vector fields on $\mathbb{T}^2$, the Lie bracket in the central extension is of the form

$$[l_n, l_m] = i(n_1m_2 - n_2m_1)l_{n+m} + i\beta m_1\delta(n+m)l_0, \tag{1.3}$$

with l_0 the central element and $\beta \in \mathbb{R}$.

In this letter we present a geometric version of the Lie algebra 2-cocycle describing the central extension (1.3). For a $2k$-dimensional compact symplectic manifold (M, ω), each closed 1-form θ on M provides a Lie algebra 2-cocycle, called Roger cocycle, on the Lie algebra of hamiltonian vector fields on M [5]:

$$\sigma_\theta(H_f, H_g) = \int_M f\theta(H_g)\omega^k, \tag{1.4}$$

where f and g are hamiltonian functions with zero integral for the hamiltonian vector fields H_f and H_g on M. On the 2-torus the volume form $\omega = dx_1 \wedge dx_2$ is a symplectic form and the Lie algebra of exact divergence free vector fields is the Lie algebra of hamiltonian vector fields. The central Lie algebra extension given by the Roger 2-cocycle associated to the differential 1-form $\theta = \beta dx_2$ coincides with (1.3).

The extendability of the cocycle (1.4) to the Lie algebra of symplectic vector fields on a compact symplectic manifold is studied in [6]. For the 2-torus, the cocycle σ_θ can always be extended to the cocycle $\bar{\sigma}_\theta$ on the Lie algebra of symplectic (i.e. divergence free) vector fields, uniquely determined by the conditions $\bar{\sigma}_\theta(\partial_{x_1}, \partial_{x_2}) = \bar{\sigma}_\theta(\partial_{x_1}, H_f) = \bar{\sigma}_\theta(\partial_{x_2}, H_f) = 0$, for all smooth functions f [7].

Euler equation on the central extension of the Lie algebra of divergence free vector fields on the flat 2-torus given by $\bar{\sigma}_\theta$ is

$$\partial_t u = -\nabla_u u - \psi_u \theta^\sharp - \operatorname{grad} p, \tag{1.5}$$

where the zero integral function ψ_u is uniquely determined by u through $d\psi_u = i_u\omega - \langle i_u\omega\rangle$. Here $\langle\rangle$ denotes the average of a 1-form on the torus: $\langle \alpha dx_1 + \beta dx_2\rangle = (\int_{\mathbb{T}^2} \alpha\omega)dx_1 + (\int_{\mathbb{T}^2} \beta\omega)dx_2$ and $\sharp$ is the Riemannian lift with respect to the flat metric on the torus. Equation (1.5) generalizes the equation of motion of a perfect fluid with velocity field u and pressure function p.

The equation of motion of a perfect fluid on a Riemannian manifold M of dimension at least two,

$$\partial_t u = -\nabla_u u - \operatorname{grad} p, \tag{1.6}$$

is a geodesic equation on the group of volume preserving diffeomorphisms of M with right invariant L^2 metric [8][9]. The only Riemannian manifolds M with the property that the group of exact volume preserving diffeomorphisms is a totally geodesic subgroup of the group of volume preserving diffeomorphisms with the right invariant L^2 metric are twisted products of a flat torus with a manifold with vanishing first Betti number [10]. It follows that on the flat 2-torus equation (1.6) preserves the property of the velocity field to possess stream functions [11] and the evolution equation for the stream function ψ is

$$\partial_t \Delta\psi = -\{\Delta\psi, \psi\}. \tag{1.7}$$

We show that also equation (1.5) preserves the property of u to possess stream functions, when the 1-form θ on the 2-torus has constant coefficients. Writing the evolution equation for the stream function in the special case $\theta = \beta dx_2$, we find again the quasigeostrophic motion in β-plane approximation (1.1).

2 Cocycles on Lie algebras of symplectic vector fields

A bilinear skew-symmetric map $\sigma : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ is a 2-cocycle on the Lie algebra $\mathfrak{g}$ if it satisfies the condition

$$\sum_{cycl} \sigma([X_1, X_2], X_3) = 0, \quad X_1, X_2, X_3 \in \mathfrak{g}.$$

It determines a central Lie algebra extension $\hat{\mathfrak{g}} := \mathfrak{g} \times_\sigma \mathbb{R}$ of $\mathfrak{g}$ by $\mathbb{R}$ with Lie bracket

$$[(X_1, a_1), (X_2, a_2)] = ([X_1, X_2], \sigma(X_1, X_2)), \quad X_i \in \mathfrak{g}, \quad a_i \in \mathbb{R}. \tag{2.1}$$

There is a 1-1 correspondence between the second Lie algebra cohomology group $H^2(\mathfrak{g})$ and equivalence classes of central Lie algebra extensions $0 \to \mathbb{R} \to \hat{\mathfrak{g}} \to \mathfrak{g} \to 0$. When G is infinite dimensional, there are two obstructions for the integrability of the central Lie algebra extension $\mathfrak{g} \times_\sigma \mathbb{R}$ to a Lie group extension of the connected Lie group G [12]: the period group $\Pi_\sigma \subset \mathbb{R}$ (the group of spherical periods of the left invariant 2-form σ^l on G defined by σ) has to be discrete and the flux homomorphism $F_\sigma : \pi_1(G) \to H^1(\mathfrak{g})$ has to vanish ($F_\sigma([\gamma]) = [I_\gamma]$ and $I_\gamma(X) = -\int_\gamma i_{X^r}\sigma^l$ for $X \in \mathfrak{g}$).

For a $2k$-dimensional compact symplectic manifold (M, ω), each closed 1-form θ on M provides a Roger Lie algebra 2-cocycle (1.4) on the Lie algebra of hamiltonian vector fields on M, where f and g are hamiltonian functions with zero integral for the hamiltonian vector fields H_f and H_g on M [5]. The cohomology class of σ_θ depends only on the de Rham cohomology class $[\theta] \in H^1_{dR}(M)$. A construction of the central extension of the group of hamiltonian diffeomorphisms of a surface of genus ≥ 2 integrating the central Lie algebra extension defined by σ_θ is given in [13]. The integrability in the case of a surface of genus 1 (a torus) is an open question.

On the flat 2-torus $\mathbb{T}^2$ with $\omega = dx_1 \wedge dx_2$, the hamiltonian vector field with hamiltonian function f is $H_f = (\partial_{x_2} f)\partial_{x_1} - (\partial_{x_1} f)\partial_{x_2}$. The Roger cocycle defined by a 1-form $\theta = \alpha dx_1 + \beta dx_2$ with constant coefficients $\alpha, \beta \in \mathbb{R}$ is

$$\sigma_\theta(H_f, H_g) = \int_{\mathbb{T}^2} f(\alpha\partial_{x_2} g - \beta\partial_{x_1} g) dx_1 \wedge dx_2. \tag{2.2}$$

The hamiltonian vector fields with hamiltonian functions $ie^{in\cdot x}$, $n \in \mathbb{Z}^2$, namely

$$l_n = e^{i(n_1x_1+n_2x_2)}(-n_2\partial_{x_1} + n_1\partial_{x_2}), \quad n \in \mathbb{Z}^2,$$

form a basis for the Lie algebra of hamiltonian vector fields on $\mathbb{T}^2$ with Lie bracket $[l_n, l_m] = i(n_1m_2 - n_2m_1)l_{n+m}$. The Roger cocycle (2.2) evaluated at two elements of this basis is

$$\sigma_\theta(l_n, l_m) = i(\beta m_1 - \alpha m_2)\delta(n + m),$$

hence the corresponding Lie algebra extension is the one from [4]:

$$[l_n, l_m] = i(n_1m_2 - n_2m_1)l_{n+m} + i(\beta m_1 - \alpha m_2)\delta(n + m)l_0,$$

with l_0 the central element.

Given a $2k$-dimensional compact symplectic manifold (M, ω), let $(b_1, b_2) = \int_M b_1 \wedge b_2 \wedge [\omega]^{k-1}$ denote the symplectic pairing on $H^1_{dR}(M)$ and $\mathrm{Vol}(M) = \int_M \omega^k$ the symplectic volume of M.

Theorem 1. *[6] The Lie algebra cocycle σ_θ on the Lie algebra of hamiltonian vector fields can be extended to a Lie algebra cocycle on the Lie algebra of symplectic vector fields if and only if*

$$(k-1)\,\mathrm{Vol}(M) \int_M [\theta] \wedge b_1 \wedge b_2 \wedge b_3 \wedge [\omega]^{k-2} = k \sum_{cycl} ([\theta], b_1)(b_2, b_3)$$

for all $b_1, b_2, b_3 \in H^1_{dR}(M)$.

On a surface M, the previous condition becomes $\sum_{cycl}([\theta], b_1)(b_2, b_3) = 0$ for all $b_1, b_2, b_3 \in H^1_{dR}(M)$. This condition is always satisfied on surfaces of genus one (the torus) and never satisfied on surfaces of genus ≥ 2. For the flat 2-torus with $\omega = dx_1 \wedge dx_2$, the extension $\bar{\sigma}_\theta$ of the cocycle σ_θ to the Lie algebra of symplectic vector fields exists and is uniquely determined by the conditions [7]:

$$\bar{\sigma}_\theta(\partial_{x_1}, \partial_{x_2}) = \bar{\sigma}_\theta(\partial_{x_1}, H_f) = \bar{\sigma}_\theta(\partial_{x_2}, H_f) = 0. \tag{2.3}$$

3 Ideal fluid flow and stream functions

For a Lie group G with right invariant metric, the geodesic equation written for the right logarithmic derivative $u = c'c^{-1}$ of a geodesic c is

$$u' = -\operatorname{ad}(u)^\top u, \tag{3.1}$$

where $\operatorname{ad}(u)^\top$ denotes the adjoint of $\operatorname{ad}(u)$ with respect to the scalar product $\langle , \rangle$ on $\mathfrak{g}$ given by the Riemannian metric. It is a first order equation on the Lie algebra $\mathfrak{g}$, called the (generalized) Euler equation.

Euler equation of motion of a perfect fluid (1.6) is a geodesic equation on the group $\operatorname{Diff}_\mu(M)$ of volume preserving diffeomorphisms of a compact Riemannian manifold M of dimension at least two and with volume form μ, for the right invariant L^2 metric [8][9]. In this case $\operatorname{ad}(X)^\top X = P(\nabla_X X)$ for all $X \in \mathfrak{X}_\mu(M)$, with P denoting the orthogonal projection on the space of divergence free vector fields in the decomposition $\mathfrak{X}(M) = \mathfrak{X}_\mu(M) \oplus \operatorname{Im}\operatorname{grad}$.

A Lie subgroup H of a Lie group G with right invariant Riemannian metric is totally geodesic if any geodesic c, starting at the identity e in a direction of the Lie algebra $\mathfrak{h}$ of H, stays in H. From Euler equation (3.1) we see that this is the case if

$$\operatorname{ad}(X)^\top X \in \mathfrak{h} \text{ for all } X \in \mathfrak{h}. \tag{3.2}$$

If there is a geodesic in G in any direction of $\mathfrak{h}$, then this condition is necessary and sufficient, so by definition we say that the Lie subalgebra $\mathfrak{h}$ is totally geodesic in $\mathfrak{g}$ if (3.2) holds.

The kernel of the flux homomorphism $\operatorname{flux}_\mu : X \in \mathfrak{X}_\mu(M) \mapsto [i_X\mu] \in H^{n-1}_{dR}(M)$ is the Lie algebra $\mathfrak{X}^{ex}_\mu(M)$ of exact divergence free vector fields. The Lie algebra homomorphism flux_μ integrates to the flux homomorphism $\operatorname{Flux}_\mu : \operatorname{Diff}_\mu(M)_0 \to H^{n-1}_{dR}(M)/\Gamma$ on the identity component of the group of volume preserving diffeomorphisms, with Γ a discrete subgroup of $H^{n-1}_{dR}(M)$. By definition the kernel of Flux_μ is the Lie group $\operatorname{Diff}^{ex}_\mu(M)$ of exact volume preserving diffeomorphisms. If M is a surface, then $\mathfrak{X}^{ex}_\mu(M)$ is the Lie algebra of hamiltonian vector fields, hence it consists of vector fields possessing stream functions, and $\operatorname{Diff}^{ex}_\mu(M)$ is the group of hamiltonian vector fields.

Theorem 2. *[10] The Riemannian manifolds M with the property that $\operatorname{Diff}^{ex}_\mu(M)$ is a totally geodesic subgroup of $\operatorname{Diff}_\mu(M)$ with the right invariant L^2 metric are twisted products $M = \mathbb{R}^k \times_\Lambda F$ of a flat torus $\mathbb{T}^k = \mathbb{R}^k/\Lambda$ and a manifold F with $H^1_{dR}(F) = 0$.*

In particular the ideal fluid flow (1.6) on the flat 2-torus preserves the property of having a stream function [11] and the evolution equation for the stream function ψ of the fluid velocity u becomes (1.7). Indeed, for $\omega = dx_1 \wedge dx_2$ and $u = H_\psi$, denoting by $\flat$ the inverse of $\sharp$, the following relations hold:

$$du^\flat = (\Delta\psi)\omega \text{ and } d(\nabla_u u)^\flat = L_u(du^\flat) = \{\Delta\psi, \psi\}\omega. \tag{3.3}$$

4 Quasigeostrophic motion

Let $\hat{G}$ be a 1-dimensional central Lie group extension of G with right invariant metric determined by the scalar product $\langle (X,a),(Y,b)\rangle_{\hat{\mathfrak{g}}} = \langle X,Y\rangle_{\mathfrak{g}} + ab$ on its Lie algebra $\hat{\mathfrak{g}} = \mathfrak{g} \times_\sigma \mathbb{R}$. The geodesic equation is

$$u' = -\operatorname{ad}(u)^\top u - ak(u), \quad a \in \mathbb{R}, \tag{4.1}$$

where u is a curve in $\mathfrak{g}$ and $k \in L_{skew}(\mathfrak{g})$ is defined by the Lie algebra cocycle σ via

$$\langle k(X), Y\rangle_{\mathfrak{g}} = \sigma(X,Y), \quad \forall X, Y \in \mathfrak{g}.$$

Indeed, $\operatorname{ad}(X,a)^\top(Y,b) = (\operatorname{ad}(X)^\top Y + bk(X), 0)$ because

$$\langle \operatorname{ad}(X,a)^\top(Y,b),(Z,c)\rangle_{\hat{\mathfrak{g}}} = \langle Y,[X,Z]\rangle_{\mathfrak{g}} + b\sigma(X,Z) = \langle \operatorname{ad}(X)^\top Y + bk(X), Z\rangle_{\mathfrak{g}}.$$

To a divergence free vector field X on the 2-torus one can assign a smooth zero integral function ψ_X, uniquely determined by X through $d\psi_X = i_X\omega - \langle i_X\omega\rangle$. Here $\langle\rangle$ denotes the average of a 1-form on the torus: $\langle \alpha dx_1 + \beta dx_2\rangle = (\int_{\mathbb{T}^2}\alpha\omega)dx_1 + (\int_{\mathbb{T}^2}\beta\omega)dx_2$. In particular $\psi_{H_f} = f$ whenever f has zero integral.

Proposition 1. *Let $\bar\sigma_\theta$ be the 2-cocycle extending (2.2) and satisfying (2.3). Euler equation for the L^2 scalar product on $\mathfrak{X}_\omega(\mathbb{T}^2) \times_{\bar\sigma_\theta} \mathbb{R}$ is*

$$\partial_t u = -\nabla_u u - \psi_u \theta^\sharp - \operatorname{grad} p. \tag{4.2}$$

Proof. We compute the map k corresponding to the cocycle $\bar\sigma_\theta$ and we apply equation (4.1). Using the fact that $\bar\sigma_\theta(\partial_{x_1}, X) = \bar\sigma_\theta(\partial_{x_2}, X) = 0$ for all $X \in \mathfrak{X}_\omega(\mathbb{T}^2)$, we get

$$\bar\sigma_\theta(u,X) = \bar\sigma_\theta(H_{\psi_u}, X) = \int_{\mathbb{T}^2}\psi_u\theta(X)\omega = \int_{\mathbb{T}^2} g(\psi_u\theta^\sharp, X)\omega = \langle P(\psi_u\theta^\sharp), X\rangle,$$

hence $k(u) = P(\psi_u\theta^\sharp)$. Knowing also that $\operatorname{ad}(u)^\top u = P(\nabla_u u)$, we get (4.2) as the Euler equation (4.1) on $\mathfrak{X}_\omega(\mathbb{T}^2) \times_{\bar\sigma_\theta} \mathbb{R}$ written for $a = 1$. ■

Proposition 2. *If the two coefficients of the 1-form θ on $\mathbb{T}^2$ are constant, then equation (4.2) preserves the property of having a stream function, i.e. $\mathfrak{X}^{ex}_\omega(\mathbb{T}^2) \times_{\sigma_\theta} \mathbb{R}$ is totally geodesic in $\mathfrak{X}_\omega(\mathbb{T}^2) \times_{\bar\sigma_\theta} \mathbb{R}$.*

Proof. By Theorem 2 for the flat 2-torus, $P(\nabla_X X)$ is hamiltonian for X a hamiltonian vector field, hence the totally geodesicity condition (3.2) in this case is equivalent to the fact that $P(\psi_X\theta^\sharp)$ is hamiltonian for X hamiltonian vector field. By Hodge decomposition this means $\psi_X\theta^\sharp$ is orthogonal to the space of harmonic vector fields, so

$$\langle P(\psi_X\theta^\sharp), Y\rangle = \int_{\mathbb{T}^2} g(\psi_X\theta^\sharp, Y)\omega = \int_{\mathbb{T}^2}\theta(Y)\psi_X\omega = 0, \quad \forall Y \text{ harmonic}.$$

On the flat torus the harmonic vector fields Y are the vector fields with constant coefficients. The 1-form θ has constant coefficients and the functions ψ_X have vanishing integral by definition, so the expression above vanishes for all harmonic vector fields Y and the totally geodesicity condition holds. ■

Corollary 1. *For $\theta = \beta dx_2$, $\beta \in \mathbb{R}$, equation (4.2) written for the stream function ψ of u becomes equation (1.1) for quasigeostrophic motion in β-plane approximation, with β the gradient of the Coriolis parameter.*

Proof. One uses (3.3) and the fact that $d(\psi\theta^\sharp)^\flat = d\psi \wedge \beta dx_2 = \beta\partial_{x_1}\psi dx_1 \wedge dx_2$. ■

This corollary recovers the result from [4] that quasigeostrophic motion is Euler equation on the central extension (1.3).

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