

SET-VALUED STOCHASTIC LEBESGUE INTEGRAL AND REPRESENTATION THEOREMS

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In this paper, we shall firstly illustrate why we should introduce set-valued stochastic integrals, and then we shall discuss some properties of set-valued stochastic processes and the relation between a set-valued stochastic process and its selection set. After recalling the Aumann type definition of stochastic integral, we shall introduce a new definition of Lebesgue integral of a set-valued stochastic process with respect to the time t . Finally we shall prove the presentation theorem of set-valued stochastic integral and discuss further properties that will be useful to study set-valued stochastic differential equations with their applications.

Keywords: set-valued stochastic process, selection process, set-valued Lebesgue integral, representation theorem.

1. Introduction

It is well-known that classical stochastic differential equations have widely been used in optimal control problems¹, mathematical finance^{2,3} and so on. Since the dynamical systems concerning practical uses are complex, the dynamical systems having velocities are usually not determined uniquely by the state of the systems. The investigations of this kind systems led to replacement of the differential equation $\dot{x}(t) = f(t, x(t))$ with the differential inclusion $\dot{x}(t) \in F(t, x(t))$, where F is a set-valued function. This kind of situation appears in studying the evolution of macro-systems in economic, social or biological sciences, where very often it is difficult to determine velocities uniquely.

On the other hand, we have to consider the systems in which there are random disturbances in

the real world. In this case, some stochastic optimal control problems can be described by set-valued stochastic differential inclusions. Indeed, assume that $f = \{(f_t(z, p))_{t \in I} : (z, p) \in \mathbb{R}^d \times U\}$, $g = \{(g_t(z, p))_{t \in I} : (z, p) \in \mathbb{R}^d \times U\}$ are d -dimensional measurable and adapted stochastic processes depending on parameters $z \in \mathbb{R}^d$, $p \in U$, where U is a fixed set and I is the set of time, for examples, $I = [0, T]$ or $I = [0, \infty)$, then the control equation is

$$x_t = \xi + \int_0^t f_s(x_s, u_s) ds + \int_0^t g_s(x_s, v_s) dB_s, \quad (1.1)$$

for any $t \in I$ a.e., where $B = (B_t)_{t \in I}$ is a Brownian motion. The stochastic process $((u_t)_{t \in I}, (v_t)_{t \in I})$ is called a strategy or a control taking values in U . If \mathcal{C} , \mathcal{U} , \mathcal{V} are given sets of constraints and strategies respectively, we shall look for the triples

Jungang Li and Shoumei Li

$(x^{u,v}, u, v)$ such that $x^{u,v} = (x_t^{u,v})_{t \in I}$ is the solution of (1.1), and $(x^{u,v}, u, v) \in \mathcal{C} \times \mathcal{U} \times \mathcal{V}$. If for any fixed $t \in I$, $z \in R^d$, put $F_t(z) = \{f_t(z, u_t) : u \in \mathcal{U}\}$; $G_t(z) = \{g_t(z, v_t) : v \in \mathcal{V}\}$. Then to look for the solutions of (1.1) becomes to determine the solution set of the following set-valued stochastic differential inclusion:

$$dx_t \in F_t(x_t)dt + G_t(x_t)dB_t, \quad x_0 = \xi, \quad (1.2)$$

or stochastic integral form

$$x_t - x_s \in \text{cl} \left(\int_s^t F_\tau(x_\tau)dt + G_\tau(x_\tau)dB_\tau \right), \quad s, t \in I \quad (1.2')$$

The purpose of stochastic optimal control is to minimize the expectation value of a given function $c : \mathcal{C} \times \mathcal{U} \times \mathcal{V} \rightarrow R$, where c characterizes the cost of the loss or the errors related to the choice of a given control strategy and $\bar{c}(u, v) = E[c(x^{u,v}, u, v)]$ is called the cost of control.

In (1.2), there are two parts: one is the part $F_t(x_t)dt$ which is related to the integral of a set-valued stochastic process with respect to time t , i.e. $\int_0^t F_s(x_s)ds$, and the other is the part $G_t(x_t)dB_t$ which is related to the Ito integral of a set-valued stochastic process with respect to B_t . How to define these two integrals suitably is the first problem in the theory of set-valued stochastic analysis. What properties do they have? These problems are what we shall consider.

There are many good works in this area. In 1994, Ahmed⁴ introduced set-valued differential inclusion with the special case that the second part G of (1.2) is a real-valued function. Kisielewicz⁵⁻¹⁰ discussed set-valued stochastic integral and solutions problems of general stochastic differential inclusion (1.2). In 1998, Aubin and Prato¹¹ obtained a viability theorem for stochastic differential inclusion, and Motyl¹² discussed stability problem for stochastic inclusion. We also would like to show our thanks to Polish mathematicians for their telling us the nice summary in this area^{13,14}.

However, there are only a few papers to discuss stochastic integrals of set-valued stochastic process. Kim¹⁵ used the definition of stochastic integral of set-valued stochastic process introduced by Kisielewicz⁶ and discussed its properties. We called it Aumann type integral since the idea came from

Aumann integral of a set-valued random variable¹⁶. We may consider the concept of Ito integral of a set-valued stochastic process by another way, because the Ito integral of a set-valued stochastic process with respect to a Brownian motion in R^d should be a set-valued stochastic processes in R^d rather than in $L^2[\Omega, R^d]$. So is the set-valued Lebesgue integral. Jung and Kim¹⁷ gave a new definition with basic space being R by taking fixed time t . It is a quite nice work. But we still think that this new definition is necessary to correct again since it may be more suitable to treat a set-valued stochastic process as a whole. Li and Ren¹⁸ introduced a new way to define the Ito integral of set-valued stochastic processes and discussed their properties. In this paper, we should consider the Lebesgue integral of set-valued stochastic processes and their properties.

We organize our paper as following: in section 2, we shall introduce some necessary notations, definitions and results about set-valued stochastic processes. In section 3, we shall give a new definition of Lebesgue integral of a set-valued process with respect to the time t . We shall also discuss some properties of set-valued stochastic integral, especially the presentation theorem of set-valued stochastic Lebesgue integral.

2. Representation theorems of set-valued L^p -bounded progressively measurable processes

Throughout this paper, assume that R is the set of all real numbers, $I = [0, T]$, N is the set of all natural numbers, R^d is the d -dimensional Euclidean space with usual norm $\|\cdot\|$, $\mathcal{B}(E)$ is the Borel field of the space E , $(\Omega, \mathcal{A}, \mu)$ is a complete atomless probability space, the σ -field filtration $\{\mathcal{A}_t : t \in I\}$ satisfies the usual conditions (i.e. complete, non-decreasing and right continuous). Let $L^p[\Omega, \mathcal{A}_t, \mu; R^d]$ be the set of R^d -valued \mathcal{A}_t -measurable random variables ξ with the expectation $E[\|\xi\|^p] < \infty$ ($p \geq 1$). When \mathcal{A}_t is replaced by \mathcal{A} , $L^p[\Omega, \mathcal{A}, \mu; R^d]$ can be written as $L^p[\Omega, R^d]$

Let $f = \{f(t), \mathcal{A}_t : t \in I\}$ be a R^d -valued adaptive stochastic process. It is said that f is progressively measurable if for any $t \in I$, the mapping $(s, \omega) \mapsto f(s, \omega)$ from $[0, t] \times \Omega$ to R^d is $\mathcal{B}([0, t]) \times \mathcal{A}_t$ -measurable. If denote

$$\{A \subset I \times \Omega : \forall t \in I, A \cap ([0, t] \times \Omega) \in \mathcal{B}([0, t]) \times \mathcal{A}_t\},$$

as \mathcal{C} , then f is progressively measurable if and only if f is \mathcal{C} -measurable. Each right continuous (left continuous) adaptive process is progressively measurable.

Assume that $\mathcal{L}^p(R^d)$ denotes the set of R^d -valued stochastic processes $f = \{f(t), \mathcal{A}_t : t \in I\}$ such that f satisfying (a) f is progressive; and (b)

$$\|f\|_p = \left[E \left(\int_0^T \|f(t, \omega)\|^p ds \right) \right]^{1/p} < \infty, \quad (2.1)$$

Let $f, f' \in \mathcal{L}^p(R^d)$, $f = f'$ if and only if $\|f - f'\|_p = 0$. Then $(\mathcal{L}^p(R^d), \|\cdot\|_p)$ is complete.

Now we review notation and concepts of set-valued stochastic processes.

Assume that $\mathbf{K}(R^d)$ is the family of all nonempty, closed subsets of R^d with usual norm $\|\cdot\|$, and $\mathbf{K}_c(R^d)$ (resp. $\mathbf{K}_k(R^d)$, $\mathbf{K}_{kc}(R^d)$) is the family of all nonempty closed convex (resp. compact, compact convex) subsets of R^d .

For any $x \in R^d$, A is a nonempty subset of R^d , define the distance of x and A

$$d(x, A) = \inf_{y \in A} \|x - y\|.$$

The Hausdorff metric on $\mathbf{K}(R^d)$ is defined as:

$$d_H(A, B) = \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right\} \quad (2.2)$$

for $A, B \in \mathbf{K}(R^d)$. Note that the Hausdorff metric between two closed sets A, B may take infinite when they are unbounded. But it is known (Theorem 1.1.2¹⁹) that the family of all bounded elements in $\mathbf{K}(R^d)$ is a complete separable space with respect to the Hausdorff metric d_H , and $\mathbf{K}_k(R^d)$ and $\mathbf{K}_{kc}(R^d)$ are its closed subsets. For $B \in \mathbf{K}(R^d)$, define $\|B\|_{\mathbf{K}} = d_H(\{0\}, B) = \sup_{a \in B} \|a\|$.

For a set-valued random variable F ,^{19 20} define the set

$$S_F^p = \{f \in L^p[\Omega, R^d] : f(\omega) \in F(\omega) \text{ a.e.}(\mu)\}.$$

where $L^p[\Omega, R^d]$ is the set of all R^d -valued random variables f such that $\|f\|_p = [E(\|f\|^p)]^{1/p} < \infty$, and constant $p \geq 1$. The expectation of F is defined as

$$E[F] = \{E[f] : f \in S_F^1\}.$$

It is called the Aumann integral introduced by Aumann¹⁶ in 1965. A set-valued random variable $F : \Omega \rightarrow \mathbf{K}(R^d)$ is called *integrable* if S_F^1 is non-empty. F is called *integrable bounded* if $\int_{\Omega} \|F(\omega)\|_{\mathbf{K}} d\mu < \infty$. Let $L^p[\Omega, \mathcal{A}, \mu; \mathbf{K}(R^d)]$ (resp. $L^p[\Omega, \mathcal{A}, \mu; \mathbf{K}_c(R^d)]$, $L^p[\Omega, \mathcal{A}, \mu; \mathbf{K}_{kc}(R^d)]$) denote the family of $\mathbf{K}(R^d)$ -valued (resp. $\mathbf{K}_c(R^d)$, $\mathbf{K}_{kc}(R^d)$ -valued) L^p -bounded random variables F such that $\|F(\omega)\|_{\mathbf{K}} \in L^p[\Omega, R]$. Concerning more definitions and results of set-valued random variables, readers could refer to the book¹⁹ or the excellent paper.²⁰

$F = \{F(t) : t \in I\}$ is called a *set-valued stochastic process* if $F : I \times \Omega \rightarrow \mathbf{K}(R^d)$ is a set-valued function such that for any fixed $t \in I$, $F(t, \cdot)$ is a set-valued random variable.

A set-valued process $F = \{F(t) : t \in I\}$ is called *adapted with respect to the filtration* $\{\mathcal{A}_t : t \in I\}$, if $F(t)$ is measurable with respect to \mathcal{A}_t for each $t \in I$, and denoted by $\{F(t), \mathcal{A}_t : t \in I\}$. F is called *measurable* if F is $I \times \Omega$ measurable, i.e. $\{(t, \omega) \in I \times \Omega : F(t, \omega) \cap A \neq \emptyset\} \in \mathcal{B}(I) \times \mathcal{A}$ for $A \in \mathcal{B}(R^d)$.

Definition 2.1 A set-valued stochastic process $F = \{F(t) : t \in I\}$ is called to be *progressively measurable*, if it is \mathcal{C} -measurable, i.e. for any $A \in \mathcal{B}(R^d)$, $\{(s, \omega) \in I \times \Omega : F(s, \omega) \cap A \neq \emptyset\} \in \mathcal{C}$.

If F is progressively measurable then F is adapted and measurable.

Definition 2.2 A progressively measurable set-valued stochastic process $F = \{F(t), \mathcal{A}_t : t \in I\}$ is called \mathcal{L}^p -bounded, if the real stochastic process $\{\|F(t)\|_{\mathbf{K}}, \mathcal{A}_t : t \in I\} \in \mathcal{L}^p(R)$.

Definition 2.3 A R^d -valued process $\{f(t), \mathcal{A}_t : t \in I\} \in \mathcal{L}^p(R^d)$ is called an \mathcal{L}^p -selection of $F = \{F(t), \mathcal{A}_t : t \in I\}$ if $f(t, \omega) \in F(t, \omega)$ a.e. $(t, \omega) \in I \times \Omega$.

Let $S^p(\{F(\cdot)\})$ or $S^p(F)$ denote the family of all $\mathcal{L}^p(R^d)$ -selections of $F = \{F(t), \mathcal{A}_t : t \in I\}$, i.e.

$$S^p(F) = \left\{ \{f(t)\} \in \mathcal{L}^p(R^d) : f(t, \omega) \in F(t, \omega), \text{ a.e. } (t, \omega) \in I \times \Omega \right\}.$$

Please note the difference between $S^p(\{F(\cdot)\})$ and $S_{F(t)}^p(\mathcal{A}_t)$, the later means the selection set in $L^p[\Omega, \mathcal{A}_t, \mu; R^d]$ of the set-valued random variable $F(t)$, $t \in I$.

Theorem 2.4^{18,21} If $S^p(F) \neq \emptyset$, then $S^p(F)$ is a closed set of $(\mathcal{L}^p(R^d), \|\cdot\|_p)$.

Jungang Li and Shoumei Li

It is natural to ask under what conditions $S^p(F) \neq \emptyset$ and whether the set-valued stochastic process can be represented by a sequence in $S^p(F)$. The next theorem will answer these questions.

Theorem 2.5^{19, 21} Assume that $F = \{F(t) : t \in I\}$ is an \mathcal{L}^p -bounded progressively measurable process. Then there exists a sequence of progressively measurable stochastic process $f_n : I \times \Omega \rightarrow R^d, n \geq 1$ in $S^p(F)$ such that for any $(t, \omega) \in I \times \Omega$,

$$F(t, \omega) = \text{cl}\{f_n(t, \omega) : n \geq 1\}. \quad (2.3)$$

Furthermore, if F is an \mathcal{L}^p -bounded closed convex progressively measurable set-valued stochastic process and satisfies

(C₁) for any $\omega \in \Omega, t \mapsto F(t, \omega)$ is lower semi-continuous (l.s.c.),

then there exists a sequence of Caratheodory selections $f_n : I \times \Omega, n \geq 1$ of F such that for any $(t, \omega) \in I \times \Omega, (2.3)$ exists.

Remark 2.6 For the definition of l.s.c. of a set-valued function, readers may refer to²⁹ or²¹). A set-valued stochastic process is said to be trajectory lower semi-continuous (l.s.c.) if it satisfies (C₁). Similarly, we may also have trajectory upper semi-continuous (u.s.c.), trajectory continuous, trajectory upper semi-continuous with respect to Hausdorff metric d_H (h.u.s.c.), trajectory lower semi-continuous with respect to d_H (h.l.s.c.) and trajectory continuous with respect to d_H .

Theorem 2.7 Assume that $F = \{F(t) : t \in I\}$ is an \mathcal{L}^p -bounded progressively measurable set-valued stochastic process. Let $f_n : I \times \Omega \rightarrow R^d, n \geq 1$ be a sequence of progressively measurable stochastic processes in $S^p(F)$ such that for any $(t, \omega) \in I \times \Omega$,

$$F(t, \omega) = \text{cl}\{f_n(t, \omega) : n \geq 1\}.$$

Then, for any $f \in S^p(F)$ and $\varepsilon > 0$, there exists a measurable partition $\{A_1, \dots, A_n\}$ of $I \times \Omega$ such that

$$\|f - \sum_{i=1}^n I_{A_i} f_i\|_p < \varepsilon.$$

Proof Assume that for any $(t, \omega) \in I \times \Omega, f(t, \omega) \in F(t, \omega)$ with $f \in S^p(F)$. Assume $\rho \in \mathcal{L}^1(R^d)$ is a positive stochastic process satisfying

$$\int_{I \times \Omega} \rho(t, \omega) d(\lambda \times \mu) < \frac{\varepsilon^p}{3}.$$

Let

$$B_1 = \{(t, \omega) : \|f(t, \omega) - f_1(t, \omega)\|^p < \rho(t, \omega)\}$$

Denote

$$\{(t, \omega) : \|f(t, \omega) - f_n(t, \omega)\|^p < \rho(t, \omega)\} \setminus (\cup_{i=1}^{n-1} B_i)$$

as $B_n, n \geq 2$. Then $\{B_i\}$ is a countable measurable partition of $I \times \Omega$. Since $f, f_1 \in \mathcal{L}^p(R^d)$, there exists an integer n such that

$$\sum_{i=n+1}^{\infty} \int_{B_i} \|f(t, \omega)\|^p d(\lambda \times \mu) < \frac{(\varepsilon/2)^p}{3},$$

and

$$\sum_{i=n+1}^{\infty} \int_{B_i} \|f_1(t, \omega)\|^p d(\lambda \times \mu) < \frac{(\varepsilon/2)^p}{3}.$$

Let $A_1 = B_1 \cup (\cup_{i=n+1}^{\infty} B_i)$ and $A_j = B_j, 2 \leq j \leq n$, then

$$\begin{aligned} & \|f - \sum_{i=1}^n I_{A_i} f_i\|_p^p \\ &= E \left(\int_0^T \|f(s, \omega) - \sum_{i=1}^n I_{A_i} f_i(s, \omega)\|^p ds \right) \\ &\leq \sum_{i=1}^n \int_{B_i} \|f(s, \omega) - f_i(s, \omega)\|^p d(\lambda \times \mu) \\ &\quad + \sum_{i=n+1}^{\infty} \int_{B_i} \|f(s, \omega) - f_i(s, \omega)\|^p d(\lambda \times \mu) \\ &\leq \int_{I \times \Omega} \rho(s, \omega) d(\lambda \times \mu) + \sum_{i=n+1}^{\infty} 2^p \int_{B_i} (\|f(s, \omega)\|^p \\ &\quad + \|f_1(s, \omega)\|^p) d(\lambda \times \mu) \\ &< \varepsilon^p. \end{aligned}$$

Thus $\|f - \sum_{i=1}^n I_{A_i} f_i\|_p < \varepsilon. \square$

Theorem 2.8 Assume that $F = \{F(t) : t \in I\}$ is a progressively measurable set-valued stochastic process, then F is \mathcal{L}^p -bounded if and only if $S^p(F)$ is bounded in $\mathcal{L}^p(R^d)$.

Proof Assume that F is \mathcal{L}^p -bounded, then $\bar{F} = \{\|F(t)\|_{\mathbf{K}}, A_t : t \in I\} \in \mathcal{L}^p(R)$. We have $\|f\|_p \leq \|\bar{F}\|_p < \infty$ for any $f \in S^p(F)$, i.e. $S^p(F)$ is bounded in $\mathcal{L}^p(R^d)$.

Now we consider opposite part. Assume that $S^p(F)$ is bounded in $\mathcal{L}^p(R^d)$. Since F is a progressively measurable set-valued stochastic process, there exists a sequence of progressively measurable

selections $f_n \in S^p(F)$, $n \geq 1$ of F such that for any $(t, \omega) \in I \times \Omega$,

$$F(t, \omega) = \text{cl}\{f_n(t, \omega) : n \geq 1\},$$

by Theorem 2.7. Thus by using Lemma 1.3.12¹⁹ (or²⁰) on product space $I \times \Omega$, we have

$$E \left[\int_0^T \|F(s)\|_{\mathbf{K}}^p ds \right] = \sup_{f \in S^p(F)} E \left[\int_0^T \|f(s)\|^p ds \right] < \infty,$$

Thus F is \mathcal{L}^p -bounded. \square

Theorem 2.9 Assume that $F = \{F(t) : t \in I\}$ is a progressively measurable closed convex set-valued stochastic process (i.e. for any $t \in I, \omega \in \Omega, F(t, \omega) \in \mathbf{K}_c(\mathbf{R}^d)$), and $S^p(F) \neq \emptyset$, then $S^p(F)$ is convex.

Proof Let $\{f_i(t) : t \in I\} \in S^p(F)$, $i = 1, 2, a, b \geq 0$, and $a + b = 1$. Since $F(t, \omega)$ is convex, for any $t \in I, (af_1 + bf_2)(t, \omega) \in F(t, \omega), a.e..$ By Minkowski inequality on product space $I \times \Omega$, we have that

$$\begin{aligned} & \|af_1 + bf_2\|_p \\ &= \left[E \left(\int_0^T \|af_1(s) + bf_2(s)\|^p ds \right) \right]^{1/p} \\ &\leq a \left[E \left(\int_0^T \|f_1(s)\|^p ds \right) \right]^{1/p} \\ &\quad + b \left[E \left(\int_0^T \|f_2(s)\|^p ds \right) \right]^{1/p} \\ &< \infty, \end{aligned}$$

i.e. $af_1 + bf_2 \in S^p(F)$, the proof is completed. \square

Let $\mathcal{L}^p(\mathbf{K}(\mathbf{R}^d))$ denote the set of all \mathcal{L}^p -bounded progressively measurable $\mathbf{K}(\mathbf{R}^d)$ -valued stochastic process. Similarly, we have notations $\mathcal{L}^p(\mathbf{K}_c(\mathbf{R}^d))$, $\mathcal{L}^p(\mathbf{K}_k(\mathbf{R}^d))$ and $\mathcal{L}^p(\mathbf{K}_{kc}(\mathbf{R}^d))$.

Take $F_i = \{F_i(t) : t \in I\} \in \mathcal{L}^p(\mathbf{K}(\mathbf{R}^d))$, $i = 1, 2$. Since for any $t \in I$,

$$\begin{aligned} d_H(F_1(t, \omega), F_2(t, \omega)) &\leq \|F_1(t, \omega)\|_{\mathbf{K}} + \|F_2(t, \omega)\|_{\mathbf{K}} \\ &< \infty, \end{aligned}$$

the real-valued stochastic process $\{d_H(F_1(t), F_2(t)) : t \in I\}$ belongs to $\mathcal{L}^p(\mathbf{R})$. Thus, define

$$\Delta_p(F_1, F_2) = \left[E \left(\int_0^T d_H^p(F_1(s, \omega), F_2(s, \omega)) ds \right) \right]^{1/p}.$$

F_1 and F_2 are said to be *equivalent*, if for any $t \in I, \Delta_p(F_1, F_2) = 0$, denoted by $F_1 = F_2$. Then Δ_p

is a metric on $\mathcal{L}^p(\mathbf{K}(\mathbf{R}^d))$. In fact, for any $F_i \in \mathcal{L}^p(\mathbf{K}(\mathbf{R}^d)), i = 1, 2, 3$, by Minkowski inequality, we have that

$$\begin{aligned} & \Delta_p(F_1, F_2) \\ &= \left[E \left(\int_0^T d_H^p(F_1(s, \omega), F_2(s, \omega)) ds \right) \right]^{1/p} \\ &\leq \left\{ E \left[\left(\int_0^T \left(d_H(F_1(s, \omega), F_3(s, \omega)) + \right. \right. \right. \right. \\ &\quad \left. \left. \left. d_H(F_3(s, \omega), F_2(s, \omega)) \right)^p ds \right) \right] \right\}^{1/p} \\ &\leq \left[E \left(\int_0^T d_H^p(F_1(s, \omega), F_3(s, \omega)) ds \right) \right]^{1/p} + \\ &\quad \left[E \left(\int_0^T d_H^p(F_3(s, \omega), F_2(s, \omega)) ds \right) \right]^{1/p} \\ &= \Delta_p(F_1, F_3) + \Delta_p(F_3, F_2). \end{aligned}$$

It is easy to prove that $(\mathcal{L}^p(\mathbf{K}(\mathbf{R}^d)), \Delta_p)$ is complete, $\mathcal{L}^p(\mathbf{K}_c(\mathbf{R}^d))$, $\mathcal{L}^p(\mathbf{K}_k(\mathbf{R}^d))$ and $\mathcal{L}^p(\mathbf{K}_{kc}(\mathbf{R}^d))$ are closed subsets of $(\mathcal{L}^p(\mathbf{K}(\mathbf{R}^d)), \Delta_p)$. Denote

$$\|F\|_p = \Delta_p(F, \{0\}) = \left[E \left(\int_0^T \|F(s)\|_{\mathbf{K}}^p ds \right) \right]^{1/p}.$$

Definition 2.10 A non-empty set $\Gamma \subset \mathcal{L}^p(\mathbf{R}^d)$ of \mathbf{R}^d -valued progressively measurable stochastic processes is called decomposable with respect to the progressively measurable σ -field \mathcal{C} , if for any $f, g \in \Gamma$, any $U \in \mathcal{C}$, we have $I_U f + I_{U^c} g \in \Gamma$.

Firstly, we know that for any progressively measurable stochastic process $F \in \mathcal{L}^p(\mathbf{K}(\mathbf{R}^d))$, $S^p(F)$ is decomposable with respect to σ -field \mathcal{C} . Furthermore we have the following Theorem.

Theorem 2.11 Assume that $\Gamma \subset \mathcal{L}^p(\mathbf{R}^d)$ is a non-empty closed set of \mathbf{R}^d -valued progressively measurable stochastic processes, then Γ is decomposable with respect to progressively measurable σ -field \mathcal{C} if and only if there exists a progressively measurable set-valued stochastic process $F \in \mathcal{L}^p(\mathbf{K}(\mathbf{R}^d))$ such that $\Gamma = S^p(F)$. Furthermore, Γ is convex if and only if $F \in \mathcal{L}^p(\mathbf{K}_c(\mathbf{R}^d))$.

Proof If there exists a progressively measurable set-valued stochastic process $F \in \mathcal{L}^p(\mathbf{K}(\mathbf{R}^d))$ such that $\Gamma = S^p(F)$, then Γ is decomposable with respect to σ -field \mathcal{C} since $S^p(F)$ is decomposable, i.e. for any $f_1, f_2 \in S^p(F)$, $U \in \mathcal{C}$ we have $I_U f_1 + I_{U^c} f_2 \in \Gamma$.

Now let Γ be a non-empty closed set of \mathbf{R}^d -valued progressively measurable stochastic processes in $\mathcal{L}^p(\mathbf{R}^d)$, and Γ is decomposable with respect to σ -field \mathcal{C} , then there exists a sequence of

Jungang Li and Shoumei Li

$\{f_i : i \in N\} \subset \mathcal{L}^p(\mathbf{R}^d)$ such that for any $t \in I, \omega \in \Omega, \{f_i(t, \omega) : i \in N\}$ is dense in \mathbf{R}^d .

For every i , let

$$\alpha_i = \inf\{\|f_i - g\|_p : g \in \Gamma\},$$

then there exists a sequence $\{g_{ij} : j \geq 1\} \subset \Gamma$, such that

$$\lim_{j \rightarrow \infty} \|f_i - g_{ij}\|_p = \alpha_i.$$

Define

$$F(t, \omega) = \text{cl}\{g_{ij}(t, \omega) : i, j \in N\}, (t, \omega) \in I \times \Omega,$$

then $F = \{F(t) : t \in I\} \in \mathcal{L}^p(\mathbf{K}(\mathbf{R}^d))$.

Next we will prove $\Gamma = S^p(F)$. For any $f \in S^p(F), \varepsilon > 0$, by Theorem 2.7, there exists a finite \mathcal{C} -measurable partition $\{A_1, \dots, A_n\}$ of $I \times \Omega$ and $\{h_1, \dots, h_n\} \subset \{g_{ij}\}$ such that

$$\|f - \sum_{k=1}^n I_{A_k} h_k\|_p < \varepsilon.$$

Since $\sum_{k=1}^n I_{A_k} h_k \in \Gamma$ and Γ is a closed set, $f \in \Gamma$. So $S^p(F) \subseteq \Gamma$.

Suppose that $S^p(F) \subsetneq \Gamma$. Then there exist $f \in \Gamma, A \in \mathcal{C}$, and $\delta > 0$ such that

$$\inf_{i,j} \|f(t, \omega) - g_{ij}(t, \omega)\| \geq \delta, \quad \forall (t, \omega) \in A$$

and $(\lambda \times \mu)(A) > 0$. Thus, there exists f_i such that

$$B = A \cap \left\{ (t, \omega) \in I \times \Omega : \|f(t, \omega) - f_i(t, \omega)\| \leq \frac{\delta}{3} \right\}$$

has a positive measure, and let

$$g'_j = I_B f + I_{\Omega \setminus B} g_{ij}, \quad j \in N,$$

then $\{g'_j\} \subset \Gamma$ with $\|f_i - g'_j\|_p \geq \alpha_i \geq 0$. Since

$$\begin{aligned} & \|f_i(t, \omega) - g_{ij}(t, \omega)\| \\ & \geq \|f(t, \omega) - g_{ij}(t, \omega)\| - \|f(t, \omega) - f_i(t, \omega)\| \\ & \geq \frac{2\delta}{3}, \end{aligned}$$

$$\begin{aligned} \|f_i - g_{ij}\|_p - \alpha_i & \geq \|f_i - g_{ij}\|_p - \|f_i - g'_j\|_p \\ & \geq T^{\frac{1}{p}} \left(\frac{2\delta}{3} - \frac{\delta}{3} \right) \mu(B) > 0. \end{aligned}$$

It contracts $\|f_i - g_{ij}\|_p \rightarrow \alpha_i$ when $j \rightarrow \infty$. Then we have $\Gamma = S^p(F)$.

Suppose that Γ is convex, and $\Gamma = S^p(F)$. Let $G = \overline{\text{co}}F$, then $S^p(G)$ is a closed convex set in $\mathcal{L}^p(\mathbf{R}^d)$. Since $S^p(F) \subset S^p(G), \overline{\text{co}}S^p(F) \subset S^p(G)$. Next we will prove $\overline{\text{co}}S^p(F) \supset S^p(G)$. By Theorem 2.5, there exists a sequence $\{f_i\} \subset S^p(F)$ such that $F(t, \omega) = \text{cl}\{f_i(t, \omega), i \geq 1\}$, let

$$W = \left\{ g : g = \sum_{i=1}^m \alpha_i f_i, \alpha_i \in Q^+, \sum_{i=1}^m \alpha_i = 1, m \geq 1 \right\}$$

then W is a subset of $S^p(G)$, and $G(t, \omega) = \text{cl}\{g(t, \omega) : g \in W\}$. For any $f \in S^p(G), \varepsilon > 0$, by Theorem 2.7, there exist a finite measurable partition $\{A_1, \dots, A_n\}$ of $I \times \Omega$ and stochastic processes g_1, \dots, g_n in W such that

$$\|f - \sum_{k=1}^n I_{A_k} g_k\|_p < \varepsilon.$$

Thus there exists an integer m , for $1 \leq k \leq n$, we have $g_k = \sum_{i=1}^m \alpha_{k_i} f_i$, where $\alpha_{k_i} \geq 0$ and $\sum_{i=1}^m \alpha_{k_i} = 1$.

Therefore

$$\sum_{k=1}^n I_{A_k} g_k = \sum_{k=1}^n I_{A_k} \left(\sum_{i=1}^m \alpha_{k_i} f_i \right) = \sum_{k=1}^n \sum_{i=1}^m \alpha_{k_i} I_{A_k} f_i.$$

Since $I_{A_k} f_i \in S^p(F), \sum_{k=1}^n I_{A_k} g_k$ is a convex combination of the elements in $S^p(F)$, which means $f \in \overline{\text{co}}S^p(F)$. Thus, we have $\overline{\text{co}}S^p(F) = S^p(G)$. Hence, F is convex. The opposite part is obvious, which completes the proof of Theorem. \square .

3. The Lebesgue integral of set-valued stochastic process.

Definition 3.1 Let a set-valued stochastic process $F = \{F(t) : t \in I\} \in \mathcal{L}^p(\mathbf{K}(\mathbf{R}^d)), 1 \leq p < +\infty$. For any $\omega \in \Omega, t \in I$, define

$$(A) \int_0^t F(s, \omega) ds := \left\{ \int_0^t f(s, \omega) ds : f \in S^p(F) \right\},$$

where $\int_0^t f(s, \omega) ds$ is the Lebesgue integral, $(A) \int_0^t F(s, \omega) ds$ is called the Aumann type Lebesgue integral of set-valued stochastic process F with respect to time t . For any $0 \leq u < t < \infty$,

$$(A) \int_u^t F(s, \omega) ds := (A) \int_0^t I_{[u,t]}(s) F(s, \omega) ds.$$

Remark 3.2 In the definition 3.1, the set of selections is $S^p(F)$, $1 \leq p < +\infty$. As a matter of fact, if we only consider the Lebesgue integral, we can use $S^1(F)$. But we often consider the sum of integral a set-valued stochastic process with respect to time t and integral of a set-valued stochastic process with respect to Brown motion, where we have to use $S^2(F)$. Thus we here use $S^p(F)$ for more general case.

Theorem 3.3 Let a set-valued stochastic process $F = \{F(t) : t \in I\} \in \mathcal{L}^p(\mathbf{K}(R^d))$, then for any $t \in I$, $(A) \int_0^t F(s)ds$ is a non-empty subset of $L^p[\Omega, \mathcal{A}_t, \mu; R^d]$. Furthermore, if $F \in \mathcal{L}^p(\mathbf{K}_c(R^d))$, then for any $t \in I$, $(A) \int_0^t F(s)ds$ is a non-empty convex subset of $L^p[\Omega, \mathcal{A}_t, \mu; R^d]$.

Proof Since $S^p(F)$ is non-empty and by the Jensen inequality of integral, it is easy to know that $(A) \int_0^t F(s)ds$ is a non-empty subset of $L^p[\Omega, \mathcal{A}_t, \mu; R^d]$. If $F \in \mathcal{L}^p(\mathbf{K}_c(R^d))$, then by Theorem 2.9, $S^p(F)$ is a convex subset of $\mathcal{L}^p(R^d)$. Thus $(A) \int_0^t F(s)ds$ is convex. \square

Remark 3.4 For any $t > 0$, it is natural to hope that the result of integral is a set-valued stochastic process taking values in $\mathbf{K}(R^d)$ rather than in $L^p[\Omega, \mathcal{A}_t, \mu; R^d]$. So it is necessary to give a new definition. However, $(A) \int_0^t F(s)ds$ is not decomposable in general. Hence, we firstly give the definition of decomposable closure.

Definition 3.5 For any non-empty subset $\Gamma \subset L^p[I \times \Omega, \mathcal{C}, \lambda \times \mu; R^d]$, define the decomposable closure of Γ with respect to \mathcal{C}

$$\overline{de}\Gamma = \{g = \{g(t, \omega) : t \in I\} : \text{for any } \varepsilon > 0, \text{ there}$$

exists a \mathcal{C} -measurable finite partition

$$\{A_1, \dots, A_n\} \text{ of } I \times \Omega \text{ and } f_1, \dots, f_n \in \Gamma$$

$$\text{such that } \|\|g - \sum_{i=1}^n I_{A_i} f_i\|\|_p < \varepsilon \}.$$

Theorem 3.6 Let $F = \{F(t) : t \in I\} \in \mathcal{L}^p(\mathbf{K}(R^d))$, $\Gamma(t) = (A) \int_0^t F(s)ds$, then there exists a \mathcal{C} -measurable set-valued stochastic process $L(F) = \{L_t(F) : t \in I\} \in \mathcal{L}^p(\mathbf{K}(R^d))$ such that $S^p(L(F)) = \overline{de}\{\Gamma(t) : t \in I\}$. Furthermore, if $F \in \mathcal{L}^p(\mathbf{K}_c(R^d))$, then $\{L_t(F) : t \in I\} \in \mathcal{L}^p(\mathbf{K}_c(R^d))$.

Proof For any $t \in I$, $\Gamma(t)$ is a non-empty subset of $L^p[\Omega, \mathcal{A}_t, \mu; R^d]$ from Theorem 3.3. For any $x(t) \in \Gamma(t)$, there exists $f \in S^p(F)$ such that $x(t)(\omega) =$

$$\int_0^t f(s, \omega)ds, \forall \omega \in \Omega. \text{ Let}$$

$$\begin{aligned} M &= \overline{de}\{\Gamma(t) : t \in I\} \\ &= \overline{de}\left\{g = \{g(t) : t \in I\} : g(t)(\omega) = \int_0^t f(s, \omega)ds, f \in S^p(F)\right\}, \end{aligned}$$

then M is a closed convex subset of $L^p[I \times \Omega, \mathcal{C}, \lambda \times \mu; R^d]$ and it is decomposable with respect to \mathcal{C} . By Theorem 2.11, there exists a set-valued stochastic process $L(F) = \{L_t(F) : t \in I\} \in \mathcal{L}^p(\mathbf{K}(R^d))$ such that $S^p(L(F)) = M$.

If $F \in \mathcal{L}^p(\mathbf{K}_c(R^d))$, then $\Gamma(t)$ is convex by Theorem 3.3. To finish the proof of Theorem, we only need to prove that $M = \overline{de}\{\Gamma(t) : t \in I\}$ is convex from Theorem 2.11.

For any $\phi, \psi \in M$, any $\varepsilon > 0$, there exists two \mathcal{C} -measurable partitions $\{A_i : i = 1, 2, \dots, n\}, \{B_j : j = 1, 2, \dots, m\}$ of $I \times \Omega$ and $\{\phi_i : i = 1, 2, \dots, n\}, \{\psi_j : j = 1, 2, \dots, m\} \subset U := \{g = \{g(t) : t \in I\} : g(t) = \int_0^t f(s)ds, \{f(\cdot)\} \in S^p(F)\}$ such that

$$\|\|\phi - \sum_{i=1}^n I_{A_i} \phi_i\|\|_p < \varepsilon,$$

$$\|\|\psi - \sum_{j=1}^m I_{B_j} \psi_j\|\|_p < \varepsilon.$$

For any $\alpha \in [0, 1]$, we have that

$$\begin{aligned} &\|\|\alpha\phi + (1-\alpha)\psi - \alpha \sum_{i=1}^n I_{A_i} \phi_i - (1-\alpha) \sum_{j=1}^m I_{B_j} \psi_j\|\|_p \\ &\leq \alpha \|\|\phi - \sum_{i=1}^n I_{A_i} \phi_i\|\|_p + (1-\alpha) \|\|\psi - \sum_{j=1}^m I_{B_j} \psi_j\|\|_p \\ &\leq \alpha\varepsilon + (1-\alpha)\varepsilon = \varepsilon, \end{aligned}$$

and

$$\begin{aligned} &\alpha \sum_{i=1}^n I_{A_i} \phi_i + (1-\alpha) \sum_{j=1}^m I_{B_j} \psi_j \\ &= \sum_{i=1}^n \sum_{j=1}^m I_{A_i \cap B_j} (\alpha\phi_i + (1-\alpha)\psi_j). \end{aligned}$$

Since $S^p(F)$ is convex, $\{\alpha\phi_i + (1-\alpha)\psi_j : i = 1, \dots, n; j = 1, \dots, m\} \subset U$. This with $\{A_i \cap B_j : i = 1, \dots, n; j = 1, \dots, m\}$ being also a \mathcal{C} -measurable partition of $I \times \Omega$ implies $\alpha\phi + (1-\alpha)\psi \in \overline{de}U = M$, the proof is completed. \square

Jungang Li and Shoumei Li

Remark 3.7 We can prove that the decomposable closure of $\Gamma(t) = (A) \int_0^t F(s)ds$ is bounded in \mathcal{L}^p . That is,

$$\begin{aligned} & \left\| \sum_{i=1}^n I_{A_i} \int_0^t f_i(s, \omega) ds \right\|_p \\ &= [E \int_0^T \left\| \sum_{i=1}^n I_{A_i} \int_0^t f_i(s, \omega) ds \right\|^p dt]^{\frac{1}{p}} \\ &\leq [E \int_0^T (\sum_{i=1}^n \|I_{A_i} \int_0^t f_i(s, \omega) ds\|)^p dt]^{\frac{1}{p}} \\ &\leq [E \int_0^T (\sum_{i=1}^n \|I_{A_i}\| \left\| \int_0^t f_i(s, \omega) ds \right\|)^p dt]^{\frac{1}{p}} \\ &\leq [E \int_0^T (\sum_{i=1}^n I_{A_i} \int_0^t \|f_i(s, \omega)\| ds)^p dt]^{\frac{1}{p}} \\ &\leq [E \int_0^T (\sum_{i=1}^n I_{A_i} \int_0^t \|F(s, \omega)\|_{\mathbf{K}} ds)^p dt]^{\frac{1}{p}} \\ &\leq [E \int_0^T (\sum_{i=1}^n I_{A_i} \int_0^T \|F(s, \omega)\|_{\mathbf{K}} ds)^p dt]^{\frac{1}{p}} \\ &= [E \int_0^T (\int_0^T \|F(s, \omega)\|_{\mathbf{K}} ds)^p (\sum_{i=1}^n I_{A_i}(t))^p dt]^{\frac{1}{p}} \\ &= [E \int_0^T (\int_0^T \|F(s, \omega)\|_{\mathbf{K}} ds)^p dt]^{\frac{1}{p}} \\ &< C \end{aligned}$$

where C is a constant and is not relative to n .

Definition 3.8 The set-valued stochastic process $L(F) = \{L_t(F) : t \in I\}$ defined in Theorem 3.6 is called the Lebesgue integral of a set-valued stochastic process $F = \{F_t : t \in I\} \in \mathcal{L}_T^p(\mathbf{K}(R^d))$ with respect to the time t , and denoted as $L_t(F) = (L) \int_0^t F(s)ds$.

Now we state the representation theorem of Lebesgue integral of the set-valued stochastic process.

Theorem 3.9 Let $F = \{F_t : t \in I\} \in \mathcal{L}^p(\mathbf{K}(R^d))$, then there exists a sequence of R^d -valued stochastic processes $\{f^i = \{f^i(t) : t \in I\} : i \geq 1\} \subset S^p(F)$ such that

$$F(t, \omega) = \text{cl}\{f^i(t, \omega) : i \geq 1\}, \quad \text{a.e. } (t, \omega) \in I \times \Omega,$$

and

$$L_t(F) = \text{cl}\left\{ \int_0^t f^i(s, \omega) ds : i \geq 1 \right\} \quad \text{a.e. } (t, \omega) \in I \times \Omega.$$

Proof For any $t \in I$, $\{L_t(F) : t \in I\} \in \mathcal{L}^p(\mathbf{K}(R^d))$ from Theorem 3.6. By virtue of Theorem 2.5, there exists a sequence of $\{\Phi_n = \{\phi_n(t) : t \in I\} : n \geq 1\} \subset S^p(L(F))$ such that

$$L_t(F)(\omega) = \text{cl}\{\phi_n(t, \omega) : n \geq 1\}, \quad \text{a.e. } (t, \omega) \in I \times \Omega.$$

Since

$$\begin{aligned} & S^p(L(F)) \\ &= \overline{\text{de}}\{\Gamma(t) : t \in I\} \\ &= \overline{\text{de}}\left\{g = \{g(t) : t \in I\} : g(t) = \int_0^t f(s)ds, \right. \\ &\quad \left. \{f(\cdot)\} \in S^p(F)\right\} \\ &= \text{cl}\left\{h = \{h(t) : t \in I\} : h(t) = \sum_{k=1}^l I_{A_k} \int_0^t f_k(s)ds, \right. \\ &\quad \left. \{A_k : k = 1, 2, \dots, l\} \subset \mathcal{C} \text{ is a finite partition of } \right. \\ &\quad \left. I \times \Omega \text{ and } \{\{f_k(\cdot)\} : k = 1, \dots, l\} \subset S^p(F), \right. \\ &\quad \left. l \geq 1\right\}, \end{aligned}$$

then for any $n \geq 1$, there exists $\{h_n^i : i \geq 1\}$ such that $\|\phi_n(t) - h_n^i(t)\|_p \rightarrow 0 \quad (i \rightarrow \infty)$, and

$$h_n^i(t) = \sum_{k=1}^{l(i,n)} I_{A_k^{(i,n)}} \int_0^t f_k^{(i,n)}(s)ds,$$

where $\{A_k^{(i,n)} : k = 1, 2, \dots, l(i,n)\} \subset \mathcal{C}$ is a finite partition of Ω and $\{\{f_k^{(i,n)}(t) : t \in I\} : k = 1, 2, \dots, l(i,n)\} \subset S^p(F)$. Hence, there exists a subsequence $\{i_j : j \geq 1\}$ of $\{1, 2, \dots\}$ such that $\|\phi_n(t, \omega) - h_n^{i_j}(t, \omega)\|_p \rightarrow 0$ a.e. $(j \rightarrow \infty)$. Thus, for a.e. $(t, \omega) \in I \times \Omega$, we have that

$$\begin{aligned} L_t(F)(\omega) &= \text{cl}\{h_n^{i_j}(t, \omega) : n, j \geq 1\} \\ &\subset \text{cl}\left\{ \int_0^t f_k^{(i_j, n)}(s, \omega) ds : n, j \geq 1, \right. \\ &\quad \left. k = 1, \dots, l(i_j, n) \right\} \\ &\subset L_t(F)(\omega). \end{aligned}$$

This implies that for a.e. $(t, \omega) \in I \times \Omega$, we have

$$L_t(F)(\omega) = \text{cl}\left\{ \int_0^t f_k^{(i_j, n)}(s, \omega) ds : n, j \geq 1, \right. \\ \left. k = 1, \dots, l(i_j, n) \right\}. \quad (3.1)$$

Since $F \in \mathcal{L}^p(\mathbf{K}(R^d))$ and from Theorem 2.5, there exists a sequence $\{\xi_d(t) : t \in I\} \in S^p(F)$, $d \geq 1$ such that for any $(t, \omega) \in I \times \Omega$,

$$F(t, \omega) = \text{cl}\{\xi_d(t, \omega) : d \geq 1\}. \quad (3.2)$$

We rewrite the countable set $\{\{\xi_d(t) : t \in I\}, \{f_k^{(i,j,n)}(t) : t \in I\} : n, j, d \geq 1, k = 1, \dots, l(i, j, n)\}$, as $\{\{f^i(t) : t \in I\} : i \geq 1\}$. Then from (3.1) and (3.2), $\{\{f^i(t) : t \in I\} : i \geq 1\}$ satisfies the Theorem. The proof is completed. \square

Example 3.10 Let $f = \{f_t : t \in I\}$, $g = \{g_t : t \in I\}$ be real valued stochastic processes satisfying that for a.e. $(t, \omega) \in I \times \Omega$, $f(t, \omega) < g(t, \omega)$, and $f, g \in \mathcal{L}^p(R)$. Define $F_t(\omega) = [f_t(\omega), g_t(\omega)]$, then it is easy to have that $F = \{F_t, \mathcal{A}_t : t \geq 0\}$ is a L^p -bounded set-valued stochastic process.

If $h = \{h_t : t \in I\} \in \mathcal{L}^p(R)$ satisfying $f_t(\omega) \leq h_t(\omega) \leq g_t(\omega)$ for a.e. $(t, \omega) \in I \times \Omega$, then $h \in S^p(F)$ and $S^p(F) = \{h = \{h_t : t \in I\} \in \mathcal{L}^p(R) : f_t(\omega) \leq h_t(\omega) \leq g_t(\omega) \text{ for a.e. } (t, \omega) \in I \times \Omega\}$. Thus, there exists a sequence of real valued stochastic process $\{h^i = \{h^i(t) : t \in I\} : i \geq 1\} \subset S^p(F)$ such that

$$F(t, \omega) = \text{cl}\{h^i(t, \omega) : i \geq 1\}, \quad \text{a.e. } (t, \omega) \in I \times \Omega,$$

and

$$\begin{aligned} L_t(F) &= \text{cl}\left\{\int_0^t h^i(s, \omega) ds : i \geq 1\right\} \\ &= \left[\int_0^t f(s, \omega) ds, \int_0^t g(s, \omega) ds\right], \quad \text{a.e. } (t, \omega) \in I \times \Omega. \end{aligned}$$

Theorem 3.11 Let set-valued stochastic processes $F^{(i)} = \{F^{(i)}(t), \mathcal{A}_t : t \in I\} \in \mathcal{L}^p(\mathbf{K}(R^d))$, $i = 1, 2$, then for any $t \in I$,

$$\begin{aligned} (L) \int_0^t \text{cl}(F^{(1)}(s, \omega) + F^{(2)}(s, \omega)) ds \\ = \text{cl}\left\{(L) \int_0^t F^{(1)}(s) ds + (L) \int_0^t F^{(2)}(s, \omega) ds\right\}. \end{aligned}$$

Proof It only needs to prove the equality

$$\begin{aligned} S^p\left(L(\text{cl}(F^{(1)} + F^{(2)}))\right) \\ = \text{cl}\left(S^p(L(F^{(1)})) + S^p(L(F^{(2)}))\right). \end{aligned}$$

In fact,

$$\begin{aligned} S^p\left(L(\text{cl}(F^{(1)} + F^{(2)}))\right) &= \overline{\text{de}}\left\{g = \{g(t) : t \in I\} : g(t) = \int_0^t f(s) ds, \right. \\ &\quad \left. \{f(\cdot)\} \in S^p\left(\text{cl}(F^{(1)} + F^{(2)})\right)\right\} \\ &= \overline{\text{de}}\left\{g = \{g(t) : t \in I\} : g(t) = \int_0^t f(s) ds, \right. \\ &\quad \left. \{f(\cdot)\} \in \text{cl}\left(S^p(F^{(1)}) + S^p(F^{(2)})\right)\right\} \\ &= \overline{\text{de}}\left\{h = \{h(t) : t \in I\} : h(t) := g^{(1)}(t) + g^{(2)}(t) \right. \\ &\quad \left. := \int_0^t f^{(1)}(s) ds + \int_0^t f^{(2)}(s) ds, \right. \\ &\quad \left. \{f^{(1)}(\cdot)\} \in S^p(F^{(1)}), \{f^{(2)}(\cdot)\} \in S^p(F^{(2)})\right\} \\ &= \text{cl}\left\{\overline{\text{de}}\left\{g = \{g^{(1)}(t) : t \in I\} : g^{(1)}(t) = \right. \right. \\ &\quad \left. \int_0^t f^{(1)}(s) ds, \{f^{(1)}(\cdot)\} \in S^p(F^{(1)})\right\} \\ &\quad \left. + \overline{\text{de}}\left\{g = \{g^{(2)}(t) : t \in I\} : g^{(2)}(t) = \right. \right. \\ &\quad \left. \int_0^t f^{(2)}(s) ds, \{f^{(2)}(s)\} \in S^p(F^{(2)})\right\}\right\} \\ &= \text{cl}\left(S^p(L(F^{(1)})) + S^p(L(F^{(2)}))\right). \quad \square \end{aligned}$$

4. Conclusions

In this paper, we firstly stated the necessities of studying set-valued stochastic integrals. It may be useful in the area of stochastic control and mathematical finance. Secondly, we defined a new type Lebesgue integral of a set-valued stochastic process with respect to time t based on the nice works such as Kisielewicz ⁶, Kim ¹⁵, M. Kisielewicz, M. Michta and J. Motyl ^{13, 14}. And then we discussed some properties of set-valued Lebesgue integral, especially we proved the presentation theorem of set-valued stochastic integral. Those results will be useful to study set-valued stochastic differential equations and their applications.

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Jungang Li and Shoumei Li

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Set-Valued Stochastic Lebesgue Integral and Representation Theorems

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