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Superposition formulas for integrable vector evolutionary equations on a Sphere

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Abstract

The superposition formulas for solutions of integrable vector evolutionary equations on a sphere are constructed by means of auto-Bäcklund transformation. The equations under consideration were obtained earlier by Sokolov and Meshkov in the frame of the symmetry approach.

1 Introduction

In work [1], a classification of vector integrable evolution equations of the form

$$\boldsymbol{z}_t = \boldsymbol{z}_{xxx} + \frac{3}{2}\left(\ln f_2\right)_x \boldsymbol{z}_{xx} + f_1\,\boldsymbol{z}_x + f_0\boldsymbol{z},\ \ \boldsymbol{z}^2 = 1 \tag{1.1}$$

was presented. Here $\boldsymbol{z}(x,t)$ is N-component vector of Euclidean space V with the standard scalar product $(\cdot,\cdot)$ and $f_2 = f_2(\boldsymbol{z},\boldsymbol{z}_x)$, $f_1 = f_1(\boldsymbol{z},\boldsymbol{z}_x,\boldsymbol{z}_{xx})$, $f_0 = f_0(\boldsymbol{z},\boldsymbol{z}_x,\boldsymbol{z}_{xx})$ are scalar functions. The special form of the second coefficient in the right hand side of (1.1) is followed from the integrability conditions (for more details see [1]).

If functions f_i depend on scalar variables that are constructed by product $(\cdot,\cdot)$ only, then eq.(1.1) is called **isotropic**. If we extend the set of dynamical variables by adding the variables $\langle \boldsymbol{x},\boldsymbol{y}\rangle = (\boldsymbol{x}, R\,\boldsymbol{y})$, where R is a constant symmetric matrix, then eq.(1.1) is called **anisotropic**.

According to [1] there exist four different isotropic equations of form (1.1):

$$\boldsymbol{z}_t = \boldsymbol{z}_{xxx} - 3\,\frac{(\boldsymbol{z}_x,\boldsymbol{z}_{xx})}{\boldsymbol{z}_x^2}\boldsymbol{z}_{xx} + \frac{3}{2}\left(\frac{\boldsymbol{z}_{xx}^2}{\boldsymbol{z}_x^2} + \frac{(\boldsymbol{z}_x,\boldsymbol{z}_{xx})^2}{\boldsymbol{z}_x^4\,(1+a\,\boldsymbol{z}_x^2)}\right)\boldsymbol{z}_x, \tag{1.2}$$

$$\boldsymbol{z}_t = \boldsymbol{z}_{xxx} + \frac{3}{2}\left(\frac{a^2\,(\boldsymbol{z}_x,\boldsymbol{z}_{xx})^2}{1+a\,\boldsymbol{z}_x^2} - a\,(\boldsymbol{z}_{xx}^2 - \boldsymbol{z}_x^4) + \boldsymbol{z}_x^2\right)\boldsymbol{z}_x + 3\,(\boldsymbol{z}_x,\boldsymbol{z}_{xx})\,\boldsymbol{z}, \tag{1.3}$$

$$\begin{aligned}\boldsymbol{z}_t = \boldsymbol{z}_{xxx} &+ \frac{3}{2}\,D_x\left(\ln\frac{1+z}{\boldsymbol{z}_x^2}\right)\boldsymbol{z}_{xx} - \frac{3}{2}\,\frac{(1-z)\,(\boldsymbol{z}_x,\boldsymbol{z}_{xx})}{z}\,\boldsymbol{z} + \\ &+ \frac{3}{2}\left(\frac{(1+z)\,\boldsymbol{z}_{xx}^2}{\boldsymbol{z}_x^2} - \frac{a\,(1+z)\,(\boldsymbol{z}_x,\boldsymbol{z}_{xx})^2}{z^2\,\boldsymbol{z}_x^2} + \boldsymbol{z}_x^2\,(1-z)\right)\boldsymbol{z}_x,\ \ z^2 = 1 + a\,\boldsymbol{z}_x^2,\end{aligned} \tag{1.4}$$

$$\boldsymbol{z}_t = \boldsymbol{z}_{xxx} - 3\,\frac{(\boldsymbol{z}_x,\boldsymbol{z}_{xx})}{\boldsymbol{z}_x^2}\boldsymbol{z}_{xx} + \frac{3}{2}\,\frac{\boldsymbol{z}_{xx}^2}{\boldsymbol{z}_x^2}\,\boldsymbol{z}_x. \tag{1.5}$$

There are some other articles devoted to the vector equation in $\mathbb{R}^n$ having the form of conservation laws were presented in [2], and a special case was considered in [3]. Integrable anisotropic evolutionary equations on the n-dimensional sphere were classified in [4]. However, superposition formulas were found only for a few equations, namely for the vector Schwarz-KdV [2], two generalizations of the mKdV [5], and the Landau-Lifshitz generalization [6].

In this paper we construct superposition formulas for eqs.(1.2), (1.3), for a special case of (1.4), and for the anisotropic Schwartz-KdV equation. In section 2 the constructing method of a superposition formulas is described in detail by example of an anisotropic equation. It will be shown that this equation is connected with the vector generalization of the Landau-Lifshitz equation. Section 3 is dedicated to eqs.(1.2) – (1.4). The superposition formula for the anisotropic Schwartz-KdV equation and some properties of the vector Schwartz-KdV (1.5) are studied in section 4.

2 Algorithm of calculations

Proposition 1. If an integrable vector equation of the form (1.1) has the auto-Bäcklund transformation (**ABT**)

$$\boldsymbol{y}_x = f\,\boldsymbol{z}_x + g\,\boldsymbol{y} + h\,\boldsymbol{z}, \tag{2.1}$$

where f, g, and h are scalar functions of variables $\boldsymbol{y}, \boldsymbol{z}, \boldsymbol{y}_x$, and $\boldsymbol{z}_x$, then

$$f = \sqrt{f_2(\boldsymbol{z},\, \boldsymbol{z}_x)\; f_2(\boldsymbol{y},\, \boldsymbol{y}_x)^{-1}}.$$

The proposition may be checked by a direct calculation. Firstly, we denote two equations of the form (1.1) as $\boldsymbol{z}_t = F(\boldsymbol{z})$ and $\boldsymbol{y}_t = F(\boldsymbol{y})$. Secondly, we differentiate (2.1) with respect to t in virtue of these equations. Then excluding vector $\boldsymbol{y}_x$ and all its x-derivatives, we obtain an identity with respect to vector variables $\boldsymbol{y}, \boldsymbol{z}_x$, $\boldsymbol{z}_{xx}$ and $\boldsymbol{z}_{xxx}$. By equating the terms with $\boldsymbol{z}_{xxx}$ to zero one can find that $f = \sqrt{f_2(\boldsymbol{z},\, \boldsymbol{z}_x)\; f_2(\boldsymbol{y},\, \boldsymbol{y}_x)^{-1}}$.

Below we demonstrate how to construct a superposition formula (**SF**) for the anisotropic integrable equation

$$\boldsymbol{z}_t = \boldsymbol{z}_{xxx} - 3\,\frac{(\boldsymbol{z}_x, \boldsymbol{z}_{xx})}{\boldsymbol{z}_x^2}\boldsymbol{z}_{xx} + \frac{3}{2}\left(\frac{\boldsymbol{z}_{xx}^2}{\boldsymbol{z}_x^2} + \frac{(\boldsymbol{z}_x, \boldsymbol{z}_{xx})^2}{\boldsymbol{z}_x^4} + \frac{\langle \boldsymbol{z}_x\rangle^2}{\boldsymbol{z}_x^2}\right)\boldsymbol{z}_x, \tag{2.2}$$

and its auto-Bäcklund transformation [1]:

$$\boldsymbol{y}_x = \left(\sqrt{\frac{\mu\,(1 + (\boldsymbol{y}, \boldsymbol{z})) - \langle \boldsymbol{y} + \boldsymbol{z}\rangle^2}{\boldsymbol{z}_x^2}} - 1\right)\left(\boldsymbol{z}_x - \frac{(\boldsymbol{y}, \boldsymbol{z}_x)}{1 + (\boldsymbol{y}, \boldsymbol{z})}(\boldsymbol{y} + \boldsymbol{z})\right). \tag{2.3}$$

Here and hereafter the abbreviations $\langle \boldsymbol{w}, \boldsymbol{w}\rangle = \langle \boldsymbol{w}\rangle^2$ and $(\boldsymbol{w}, \boldsymbol{w}) = \boldsymbol{w}^2$ are used.

According to **Prop. 1**, if $\boldsymbol{y}$ and $\boldsymbol{z}$ are solutions of (2.2) associated with transformation (2.3), then the following relation

$$\sqrt{\mu(1 + (\boldsymbol{y}, \boldsymbol{z})) - \langle \boldsymbol{y} + \boldsymbol{z}\rangle^2} = \sqrt{\boldsymbol{y}_x^2} + \sqrt{\boldsymbol{z}_x^2} \tag{2.4}$$

is satisfied.

By a direct calculation we find that the equation

$$\frac{\mu\,(1+(\boldsymbol{y},\boldsymbol{z}))-\langle \boldsymbol{y}+\boldsymbol{z}\rangle^2}{\boldsymbol{z}_x^2}=\left(\frac{(\tilde{y}-\tilde{z})(1+(\boldsymbol{y},\boldsymbol{z}))}{(\boldsymbol{y},\boldsymbol{z}_x)}\right)^2, \tag{2.5}$$

where

$$\tilde{z}=\frac{1}{2}\,D_x^{-1}\left(\frac{\boldsymbol{z}_{xx}^2}{\boldsymbol{z}_x^2}-\frac{(\boldsymbol{z}_x,\boldsymbol{z}_{xx})^2}{\boldsymbol{z}_x^4}+\frac{\langle \boldsymbol{z}_x\rangle^2}{\boldsymbol{z}_x^2}\right),\ \tilde{y}=\frac{1}{2}\,D_x^{-1}\left(\frac{\boldsymbol{y}_{xx}^2}{\boldsymbol{y}_x^2}-\frac{(\boldsymbol{y}_x,\boldsymbol{y}_{xx})^2}{\boldsymbol{y}_x^4}+\frac{\langle \boldsymbol{y}_x\rangle^2}{\boldsymbol{y}_x^2}\right),$$

is compatible with eqs.(2.2) and (2.3). In other words, if we differentiate (2.5) with respect to t in virtue of (2.2) or if we take the total x-derivative of (2.5), then using (2.3), (2.5), and its differential consequences we get an identity.

We start from the usual assumption that the diagram

$$\begin{array}{ccc} \boldsymbol{u} & \xrightarrow{\ \nu\ } & \boldsymbol{q} \\ {\scriptstyle\mu}\uparrow & & \uparrow{\scriptstyle\mu} \\ \boldsymbol{p} & \xrightarrow[\ \nu\ ]{} & \boldsymbol{v} \end{array} \tag{2.6}$$

for the ABT (2.3) is commutative [7]. Here $\boldsymbol{p},\boldsymbol{u},\boldsymbol{v},\boldsymbol{q}$ are vector solutions of (2.2) and μ, ν are parameters. This means that the four vector equations

$$\begin{aligned}
\boldsymbol{u}_x&=\left(\sqrt{\frac{\mu\,(1+(\boldsymbol{u},\boldsymbol{p}))-\langle \boldsymbol{u}+\boldsymbol{p}\rangle^2}{\boldsymbol{p}_x^2}}-1\right)\left(\boldsymbol{p}_x-\frac{(\boldsymbol{u},\boldsymbol{p}_x)}{1+(\boldsymbol{u},\boldsymbol{p})}(\boldsymbol{u}+\boldsymbol{p})\right),\\
\boldsymbol{v}_x&=\left(\sqrt{\frac{\nu\,(1+(\boldsymbol{v},\boldsymbol{p}))-\langle \boldsymbol{v}+\boldsymbol{p}\rangle^2}{\boldsymbol{p}_x^2}}-1\right)\left(\boldsymbol{p}_x-\frac{(\boldsymbol{v},\boldsymbol{p}_x)}{1+(\boldsymbol{v},\boldsymbol{p})}(\boldsymbol{v}+\boldsymbol{p})\right),\\
\boldsymbol{q}_x&=\left(\sqrt{\frac{\nu\,(1+(\boldsymbol{q},\boldsymbol{u}))-\langle \boldsymbol{q}+\boldsymbol{u}\rangle^2}{\boldsymbol{u}_x^2}}-1\right)\left(\boldsymbol{u}_x-\frac{(\boldsymbol{q},\boldsymbol{u}_x)}{1+(\boldsymbol{q},\boldsymbol{u})}(\boldsymbol{q}+\boldsymbol{u})\right),\\
\boldsymbol{q}_x&=\left(\sqrt{\frac{\mu\,(1+(\boldsymbol{q},\boldsymbol{v}))-\langle \boldsymbol{q}+\boldsymbol{v}\rangle^2}{\boldsymbol{v}_x^2}}-1\right)\left(\boldsymbol{v}_x-\frac{(\boldsymbol{q},\boldsymbol{v}_x)}{1+(\boldsymbol{q},\boldsymbol{v})}(\boldsymbol{q}+\boldsymbol{v})\right),
\end{aligned} \tag{2.7}$$

and the eight scalar equations

$$\begin{aligned}
\sqrt{\boldsymbol{u}_x^2}+\sqrt{\boldsymbol{p}_x^2}&=\sqrt{\mu(1+(\boldsymbol{u},\boldsymbol{p}))-\langle \boldsymbol{u}+\boldsymbol{p}\rangle^2},\\
\sqrt{\boldsymbol{v}_x^2}+\sqrt{\boldsymbol{p}_x^2}&=\sqrt{\nu(1+(\boldsymbol{v},\boldsymbol{p}))-\langle \boldsymbol{v}+\boldsymbol{p}\rangle^2},\\
\sqrt{\boldsymbol{q}_x^2}+\sqrt{\boldsymbol{u}_x^2}&=\sqrt{\nu(1+(\boldsymbol{q},\boldsymbol{u}))-\langle \boldsymbol{q}+\boldsymbol{u}\rangle^2},\\
\sqrt{\boldsymbol{q}_x^2}+\sqrt{\boldsymbol{v}_x^2}&=\sqrt{\mu(1+(\boldsymbol{q},\boldsymbol{v}))-\langle \boldsymbol{q}+\boldsymbol{v}\rangle^2},
\end{aligned} \tag{2.8}$$

$$\begin{aligned}
\frac{\mu\,(1+(\boldsymbol{u},\boldsymbol{p}))-\langle\boldsymbol{u}+\boldsymbol{p}\rangle^2}{\boldsymbol{p}_x^2} &= \left(\frac{(\tilde u-\tilde p)(1+(\boldsymbol{u},\boldsymbol{p}))}{(\boldsymbol{u},\boldsymbol{p}_x)}\right)^2,\\
\frac{\nu\,(1+(\boldsymbol{v},\boldsymbol{p}))-\langle\boldsymbol{v}+\boldsymbol{p}\rangle^2}{\boldsymbol{p}_x^2} &= \left(\frac{(\tilde v-\tilde p)(1+(\boldsymbol{v},\boldsymbol{p}))}{(\boldsymbol{v},\boldsymbol{p}_x)}\right)^2,\\
\frac{\nu\,(1+(\boldsymbol{q},\boldsymbol{u}))-\langle\boldsymbol{q}+\boldsymbol{u}\rangle^2}{\boldsymbol{u}_x^2} &= \left(\frac{(\tilde q-\tilde u)(1+(\boldsymbol{q},\boldsymbol{u}))}{(\boldsymbol{q},\boldsymbol{u}_x)}\right)^2,\\
\frac{\mu\,(1+(\boldsymbol{q},\boldsymbol{v}))-\langle\boldsymbol{q}+\boldsymbol{v}\rangle^2}{\boldsymbol{v}_x^2} &= \left(\frac{(\tilde q-\tilde v)(1+(\boldsymbol{q},\boldsymbol{v}))}{(\boldsymbol{q},\boldsymbol{v}_x)}\right)^2,
\end{aligned} \tag{2.9}$$

are satisfied.

From (2.7) we find:

$$\varphi\,\boldsymbol{p}_x+\varphi_1\,\boldsymbol{q}+\varphi_2\,\boldsymbol{u}+\varphi_3\,\boldsymbol{v}+\varphi_4\,\boldsymbol{p}=0. \tag{2.10}$$

Taking into account (2.8), we have $\varphi\equiv 0$ and φ_i are some functions of scalar variables contained in eqs.(2.7). From (2.9) we find variables $(\boldsymbol{q},\boldsymbol{v}_x)$, $(\boldsymbol{q},\boldsymbol{u}_x)$, $(\boldsymbol{u},\boldsymbol{p}_x)$, $(\boldsymbol{v},\boldsymbol{p}_x)$ and substitute them into (2.10). Then we multiply this equation by the vectors $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{p}$, $\boldsymbol{q}$ and obtain eight equations: four equations that are constructed by product $(\cdot,\cdot)$ and the other four ones are constructed by product $\langle\cdot,\cdot\rangle$. Solving these equations and eqs.(2.8) we find all zero order scalar products with $\boldsymbol{q}$ and $\tilde q$, $\boldsymbol{q}_x^2$. Substituting all obtained quantities in (2.10) we have the SF for solutions of (2.2)

$$\boldsymbol{q}=\frac{(\boldsymbol{u}-\boldsymbol{v})^2\left(\xi\,\eta\,\boldsymbol{p}+(\nu'-\mu')\left(\eta\,(\boldsymbol{v}+\boldsymbol{p})^2\,\boldsymbol{u}-\xi\,(\boldsymbol{u}+\boldsymbol{p})^2\,\boldsymbol{v}\right)\right)}{\left(\xi\,(\boldsymbol{u}+\boldsymbol{p})^2-\eta\,(\boldsymbol{v}+\boldsymbol{p}^2)\right)^2}, \tag{2.11}$$

where

$$\xi=\langle\boldsymbol{u}-\boldsymbol{v}\rangle^2-\mu'\,(\boldsymbol{u}-\boldsymbol{v})^2+\psi^2,\ \ \eta=\langle\boldsymbol{u}-\boldsymbol{v}\rangle^2-\nu'\,(\boldsymbol{u}-\boldsymbol{v})^2+\psi^2,$$
$$\psi=\sqrt{\mu'\,(\boldsymbol{u}+\boldsymbol{p})^2-\langle\boldsymbol{u}+\boldsymbol{p}\rangle^2}-\sqrt{\nu'\,(\boldsymbol{v}+\boldsymbol{p})^2-\langle\boldsymbol{v}+\boldsymbol{p}\rangle^2},\ \ \mu=2\,\mu',\ \nu=2\,\nu'.$$

In the last step to check the validity of (2.11) we differentiate it with respect to t in virtue of (2.2). With the help of (2.11) we exclude $\boldsymbol{q}$ from this equation as well as all its x-derivatives and all scalar products containing them. Then using the first two equations (2.7) we can see that (2.11) is identically satisfied. All these tedious calculations were performed on the computer [8]. In detail these steps are described in [5].

Expression (2.11) coincides with the SF of the vector generalization of the Landau-Lifshitz equation [6]. It was proved in [4] that (2.2) is integrable in $\mathbb{R}^n$. The divergent form of (2.2) in $\mathbb{R}^n$ is

$$\boldsymbol{z}_t=D_x\left(\boldsymbol{z}_{xx}-3\,\frac{(\boldsymbol{z},\boldsymbol{z}_x)}{\boldsymbol{z}^2}\boldsymbol{z}_x+\frac{3}{2}\left(\frac{\boldsymbol{z}_x^2}{\boldsymbol{z}^2}+\frac{(\boldsymbol{z},\boldsymbol{z}_x)^2}{\boldsymbol{z}^4}+\frac{\langle\boldsymbol{z}\rangle^2}{\boldsymbol{z}^2}\right)\boldsymbol{z}\right). \tag{2.12}$$

If we pass on to the coordinates $\{\boldsymbol{y}=\boldsymbol{z}\,r^{-1},\ r=\sqrt{\boldsymbol{z}^2}\}$ in (2.12) then the equation is equivalent to the system of one vector and one scalar equation, where the vector equation for $\boldsymbol{y}$ is the generalization of the Landau-Lifshitz equation

$$\boldsymbol{y}_t=D_x\left(\boldsymbol{y}_{xx}+\frac{3}{2}\,(\boldsymbol{y}_x,\boldsymbol{y}_x)\,\boldsymbol{y}\right)+\frac{3}{2}\,\langle\boldsymbol{y}\rangle^2\,\boldsymbol{y}_x,\quad \boldsymbol{y}^2=1.$$

3 Isotropic case

The ABT for (1.2) has the form [1]:

$$\begin{aligned} \boldsymbol{y}_x &= f\left(\boldsymbol{z}_x - \frac{(\boldsymbol{y},\boldsymbol{z}_x)}{\varphi}(\boldsymbol{y}+\boldsymbol{z})\right), \\ f &= \sqrt{\mu\,\varphi\left(1+(a\,\boldsymbol{z}_x^2)^{-1}\right)\left(\mu\,\varphi+2\right)} - \left(1+\mu\,\varphi\right), \quad \varphi = 1+(\boldsymbol{y},\boldsymbol{z}). \end{aligned} \tag{3.1}$$

From **Prop. 1** it follows that the relation

$$\frac{\boldsymbol{y}_x^2}{\boldsymbol{z}_x^2} = f^2 \tag{3.2}$$

holds. By a direct calculation we establish that the equation

$$\tilde{y} - \tilde{z} = -2\,(\boldsymbol{y},\boldsymbol{z}_x)\,\frac{1+f}{\varphi}, \tag{3.3}$$

where

$$\tilde{y} = D_x^{-1}\left(\frac{\boldsymbol{y}_{xx}^2}{\boldsymbol{y}_x^2} - \frac{(\boldsymbol{y}_x,\boldsymbol{y}_{xx})^2\,(1+2\,a\,\boldsymbol{y}_x^2)}{\boldsymbol{y}_x^4\,(1+a\,\boldsymbol{y}_x^2)}\right),\ \tilde{z} = D_x^{-1}\left(\frac{\boldsymbol{z}_{xx}^2}{\boldsymbol{z}_x^2} - \frac{(\boldsymbol{z}_x,\boldsymbol{z}_{xx})^2\,(1+2\,a\,\boldsymbol{z}_x^2)}{\boldsymbol{z}_x^4\,(1+a\,\boldsymbol{z}_x^2)}\right)$$

is compatible with eqs.(1.2) and (3.1).

Then the ABT (1.2) can be written in the symmetric form as

$$\frac{\boldsymbol{y}_x}{\sqrt{\boldsymbol{y}_x^2}} - \frac{\boldsymbol{z}_x}{\sqrt{\boldsymbol{z}_x^2}} = \frac{\tilde{y}-\tilde{z}}{2(\sqrt{\boldsymbol{z}_x^2}+\sqrt{\boldsymbol{y}_x^2})}(\boldsymbol{y}+\boldsymbol{z}).$$

We denote $\sinh^2 y = a\,\boldsymbol{y}_x^2$ and $\sinh^2 z = a\,\boldsymbol{z}_x^2$ for simplicity. With the new notations, eqs.(3.2) and (3.3) become

$$\varphi = \frac{\cosh(y+z)-1}{\mu}, \quad \tilde{y}-\tilde{z} = 2\,\mu\,(\boldsymbol{y},\boldsymbol{z}_x)\left(1 - \frac{\sinh(y+z)}{\tanh z\,(\cosh(y+z)-1)}\right).$$

The algorithm of the further calculations is similar to the one presented above. The SF for solutions of (1.2) reads

$$\boldsymbol{q} = \frac{(\boldsymbol{u}-\boldsymbol{v})^4}{\zeta_1\,\zeta_2}\Big(\mu\,\nu\,\xi\,\eta\,\boldsymbol{p} + 4\,(\mu-\nu)\left(\nu\,\xi\,\sinh^2(u+p)\,\boldsymbol{v} - \mu\,\eta\,\sinh^2(v+p)\,\boldsymbol{u}\right)\Big),$$

where

$$\begin{aligned} &\xi = \mu\,(\boldsymbol{u}-\boldsymbol{v})^2 + 4\,\sinh^2(u-v), \quad \eta = \nu\,(\boldsymbol{u}-\boldsymbol{v})^2 + 4\,\sinh^2(u-v), \\ &\zeta_1 = \xi+\eta+\xi\,\eta - \xi\,e^{u-v} - \eta\,e^{v-u} + e^{-p}\,(e^{-v}-e^{-u})\,(\xi-\eta), \\ &\zeta_2 = \xi+\eta+\xi\,\eta - \xi\,e^{v-u} - \eta\,e^{u-v} + e^{p}\,(e^{v}-e^{u})\,(\xi-\eta), \\ &\sinh^2 u = a\,\boldsymbol{u}_x^2, \quad \sinh^2 v = a\,\boldsymbol{v}_x^2, \quad \sinh^2 p = a\,\boldsymbol{p}_x^2. \end{aligned}$$

In the last expression it is possible to exclude variables u, v, and p using relations

$$\cosh(u+p) = 1+\mu\,(1+(\boldsymbol{u},\boldsymbol{p})), \quad \cosh(v+p) = 1+\nu\,(1+(\boldsymbol{v},\boldsymbol{p})).$$

Now we consider the ABT for (1.3) from [1]:

$$
\begin{aligned}
&\boldsymbol{y}_x = -\boldsymbol{z}_x + \left(\frac{(\boldsymbol{y}, \boldsymbol{z}_x)\,(1 - \mu\, a\, (\boldsymbol{y}, \boldsymbol{z}))}{1 + (\boldsymbol{y}, \boldsymbol{z})} - (\boldsymbol{y}, \boldsymbol{z})\, f \right) \boldsymbol{y} + \left(\frac{(\boldsymbol{y}, \boldsymbol{z}_x)(1 + \mu\, a)}{1 + (\boldsymbol{y}, \boldsymbol{z})} + f \right) \boldsymbol{z}, \\
&f^2 = \frac{\mu \left(4 + \mu\, a\, (\boldsymbol{y} - \boldsymbol{z})^2\right) \left(1 + a\, \boldsymbol{z}_x^2\right)}{(\boldsymbol{y} + \boldsymbol{z})^2}.
\end{aligned}
\tag{3.4}
$$

By a direct calculation, we have found that the expression

$$
(\boldsymbol{y}, \boldsymbol{z}_x) = -\frac{\tilde{y} - \tilde{z} + 2\,(1 + (\boldsymbol{y}, \boldsymbol{z}))\, f}{2\, \mu\, a} \tag{3.5}
$$

is compatible with eqs.(1.3) and (3.4). Here

$$
\begin{aligned}
\tilde{y} &= D_x^{-1} \left(a\, (\boldsymbol{y}_{xx}^2 - \boldsymbol{y}_x^4) - \frac{a^2 (\boldsymbol{y}_x, \boldsymbol{y}_{xx})^2}{1 + a\, \boldsymbol{y}_x^2} - \boldsymbol{y}_x^2 \right), \\
\tilde{z} &= D_x^{-1} \left(a\, (\boldsymbol{z}_{xx}^2 - \boldsymbol{z}_x^4) - \frac{a^2 (\boldsymbol{z}_x, \boldsymbol{z}_{xx})^2}{1 + a\, \boldsymbol{z}_x^2} - \boldsymbol{z}_x^2 \right).
\end{aligned}
$$

Further, we substitute (3.5) in (3.4) and take the square of both sides of the equation. Using $\boldsymbol{y}^2 = \boldsymbol{z}^2 = 1$ and $(\boldsymbol{y}, \boldsymbol{y}_x) = 0$ we find f. The result is the ABT (3.4) in the symmetric form

$$
\boldsymbol{y}_x + \boldsymbol{z}_x = \frac{\tilde{y} - \tilde{z}}{4} (\boldsymbol{y} - \boldsymbol{z}) - \frac{\boldsymbol{y}_x^2 - \boldsymbol{z}_x^2}{\tilde{y} - \tilde{z}} (\boldsymbol{y} + \boldsymbol{z}). \tag{3.6}
$$

Substituting the obtained f into (3.5) we have the following expressions

$$
(\boldsymbol{y}, \boldsymbol{z}_x) = \frac{1}{4} (2 - \varphi)\, (\tilde{y} - \tilde{z}) - \frac{\varphi}{\tilde{y} - \tilde{z}} (\boldsymbol{y}_x^2 - \boldsymbol{z}_x^2),
$$

and

$$
\varphi = \frac{2\, (\tilde{y} - \tilde{z})^2\, (1 + \mu\, a)}{\mu\, (4\, (y - z)^2 + a\, (\tilde{y} - \tilde{z})^2)},
$$

where $\varphi = 1 + (\boldsymbol{y}, \boldsymbol{z})$, $y^2 = 1 + a\, \boldsymbol{y}_x^2$, and $z^2 = 1 + a\, \boldsymbol{z}_x^2$.

The SF for solutions of (1.3) is

$$
\boldsymbol{q} = \frac{\xi - \eta}{F - G}\, \boldsymbol{p} + \frac{G - \eta}{F - G}\, \boldsymbol{u} - \frac{F - \xi}{F - G}\, \boldsymbol{v}\,,
$$

where

$$
\begin{aligned}
F &= \xi - \frac{2\, (\nu - \mu)\, (\xi - \eta)\, (a + \xi^2)}{\mu \left(\nu\, (1 - (\boldsymbol{u}, \boldsymbol{v}))\, (a + \eta^2)\, (a + \xi^2) - 2\, (\xi - \eta)^2 (1 + a\, \nu) \right)}, \\
G &= \eta - \frac{2\, (\nu - \mu)\, (\xi - \eta)\, (a + \eta^2)}{\nu \left(\mu\, (1 - (\boldsymbol{u}, \boldsymbol{v}))\, (a + \eta^2)\, (a + \xi^2) - 2\, (\xi - \eta)^2 (1 + a\, \mu) \right)}, \\
\eta^2 &= \frac{2 + a\, \mu\, (1 - (\boldsymbol{u}, \boldsymbol{p})\,)}{4\, \mu\, (1 + (\boldsymbol{u}, \boldsymbol{p})\,)}, \ \xi^2 = \frac{2 + a\, \nu\, (1 - (\boldsymbol{v}, \boldsymbol{p})\,)}{4\, \nu\, (1 + (\boldsymbol{v}, \boldsymbol{p})\,)}.
\end{aligned}
$$

Let us turn to eq. (1.4). The ABT for (1.4) has the form [1]:

$$\boldsymbol{y}_x = \psi \left(\boldsymbol{z}_x - (\boldsymbol{y}, \boldsymbol{z}_x)\, \boldsymbol{y} + \mu \left(z - 1 + \frac{(\boldsymbol{y}, \boldsymbol{z}_x)\,(\varphi + (\boldsymbol{y}, \boldsymbol{z}))}{\mu\,(1 - (\boldsymbol{y}, \boldsymbol{z})^2)} \right) \Big(\boldsymbol{z} - (\boldsymbol{y}, \boldsymbol{z})\, \boldsymbol{y} \Big) \right),$$
$$\psi = \frac{a\,\mu\,(\boldsymbol{y}, \boldsymbol{z}_x)}{z-1} + \varphi,\ \varphi^2 = 1 + a\,\mu^2\,(1 - (\boldsymbol{y}, \boldsymbol{z})^2). \tag{3.7}$$

It follows from **Prop. 1** that

$$\frac{y-1}{z-1} = \psi^2, \ \text{ where } y^2 = 1 + a\,\boldsymbol{y}_x^2,\ z^2 = 1 + a\,\boldsymbol{z}_x^2,$$

or in the equivalent form

$$(\boldsymbol{y}, \boldsymbol{z}_x) = \frac{Z\,(Y - \varphi\,Z)}{a\,\mu}, \ \text{ where } Y^2 = y - 1,\ Z^2 = z - 1. \tag{3.8}$$

A direct calculation shows that

$$(\boldsymbol{y}, \boldsymbol{z}) = \frac{(1-\varphi^2)\,(\tilde{y} - \tilde{z}) - 2\,\mu\,(Y^2 + Z^2 - 2\,\varphi\,Y\,Z)}{2\,\mu\,(\varphi\,(Y^2 + Z^2) - 2\,Y\,Z)} \tag{3.9}$$

is compatible with eqs.(1.4) and (3.7). Here

$$\tilde{y} = D_x^{-1} \left(\frac{a\,\boldsymbol{y}_{xx}^2}{1-y} + \frac{1}{2}\,\frac{a^2\,(2\,y-1)\,(\boldsymbol{y}_x, \boldsymbol{y}_{xx})^2}{y^2\,(1-y)^2} + \frac{(1-y)^2\,(1+y)}{a} \right),$$
$$\tilde{z} = D_x^{-1} \left(\frac{a\,\boldsymbol{z}_{xx}^2}{1-z} + \frac{1}{2}\,\frac{a^2\,(2\,z-1)\,(\boldsymbol{z}_x, \boldsymbol{z}_{xx})^2}{z^2\,(1-z)^2} + \frac{(1-z)^2\,(1+z)}{a} \right), \tag{3.10}$$

Then (3.7) can be rewritten in the symmetric form

$$\frac{\boldsymbol{y}_x}{Y} - \frac{\boldsymbol{z}_x}{Z} = \frac{\mu\,\big(Z^2 - Y^2\big)\,\big(Z\,\boldsymbol{y} - Y\,\boldsymbol{z}\big)}{\varphi\,(Y^2 + Z^2) - 2\,Y\,Z} + \frac{\big(\tilde{y} - \tilde{z}\big)\,\big(\varphi\,(Z\,\boldsymbol{y} + Y\,\boldsymbol{z}) - Z\,\boldsymbol{y} - Y\,\boldsymbol{z}\big)}{2\,\big(\varphi\,(Y^2 + Z^2) - 2\,Y\,Z\big)}. \tag{3.11}$$

To simplify the expression for φ we introduce the following notations

$$\sinh\zeta = b^{-1}\,\big(Y^2 - Z^2\big)\,\big(\tilde{y} - \tilde{z}\big)^{-1}, \quad k = 2\,\mu,\ a = 4\,b^2$$

and obtain

$$\varphi = \frac{k\,b\,\sinh\zeta\,\big(Z^2 - Y^2\big) + Y^2 + Z^2 + 2\,\cosh\zeta\,Y\,Z}{\cosh\zeta\,\big(Y^2 + Z^2\big) + 2\,Y\,Z}.$$

Because (3.11) is cumbersome, we can not construct the SF for (1.4). Now we consider the simpler cases of equation (1.4)

$$\boldsymbol{z}_t = \boldsymbol{z}_{xxx} + 3\,\boldsymbol{z}_x^2\,\boldsymbol{z}_x + 3\,(\boldsymbol{z}_x, \boldsymbol{z}_{xx})\,\boldsymbol{z}, \tag{3.12}$$

$$\boldsymbol{z}_t = \boldsymbol{z}_{xxx} - 3\,\frac{(\boldsymbol{z}_x, \boldsymbol{z}_{xx})}{\boldsymbol{z}_x^2}\,\boldsymbol{z}_{xx} + 3\,\frac{\boldsymbol{z}_{xx}^2}{\boldsymbol{z}_x^2}\,\boldsymbol{z}_x, \tag{3.13}$$

which were pointed out in [1].

Equation (3.13) is integrable on $\mathbb{R}^n$ [2] and has the divergent form

$$\boldsymbol{z}_t = D_x \left(\boldsymbol{z}_{xx} - 3\,\frac{(\boldsymbol{z}, \boldsymbol{z}_x)}{\boldsymbol{z}^2}\,\boldsymbol{z}_x + 3\,\frac{\boldsymbol{z}_x^2}{\boldsymbol{z}^2}\,\boldsymbol{z} \right). \tag{3.14}$$

If we pass to the coordinates $\{\boldsymbol{y} = \boldsymbol{z}\,r^{-1},\ r = \sqrt{\boldsymbol{z}^2}\}$ in (3.14), then the vector $\boldsymbol{y}$ satisfies (3.12). Therefore, we consider only (3.12).

The ABT for (3.12) was presented in [1]:

$$\boldsymbol{y}_x = \boldsymbol{z}_x - (\boldsymbol{y}, \boldsymbol{z}_x)\,\boldsymbol{y} - 2\left(\mu - \frac{(\boldsymbol{y}, \boldsymbol{z}_x)}{(\boldsymbol{y} - \boldsymbol{z})^2}\right)\big(\boldsymbol{z} - (\boldsymbol{y}, \boldsymbol{z})\,\boldsymbol{y}\big). \tag{3.15}$$

We square left and right hand sides of (3.15) and find $(\boldsymbol{y}, \boldsymbol{z}_x)$. Substituting the result into (3.15) we receive that

$$\boldsymbol{y}_x = \boldsymbol{z}_x - \frac{\mu\big(\boldsymbol{y} - \boldsymbol{z}\big)^2}{2}\big(\boldsymbol{y} + \boldsymbol{z}\big) + \frac{\boldsymbol{y}_x^2 - \boldsymbol{z}_x^2}{2\,\mu\,\big(\boldsymbol{y} - \boldsymbol{z}\big)^2}\big(\boldsymbol{y} - \boldsymbol{z}\big). \tag{3.16}$$

The SF for solutions of (3.12) is

$$\boldsymbol{q} = \zeta^{-1}\,\big(\boldsymbol{u} - \boldsymbol{v}\big)^2\Big(\xi\,\eta\,\boldsymbol{p} + \big(\mu + \nu\big)\big(\xi\,(\boldsymbol{v} - \boldsymbol{p})^2\,\boldsymbol{u} + \eta\,(\boldsymbol{u} - \boldsymbol{p})^2\,\boldsymbol{v}\big)\Big), \tag{3.17}$$

where

$$\begin{aligned} \xi &= 2\,\nu\,(\boldsymbol{u} - \boldsymbol{p}, \boldsymbol{v}) - \mu\,\big(\boldsymbol{u} - \boldsymbol{p}\big)^2, \quad \eta = 2\,\mu\,(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{p}) - \nu\,\big(\boldsymbol{v} - \boldsymbol{p}\big)^2, \\ \zeta &= -\,\big(\xi\,(\boldsymbol{v} - \boldsymbol{p}) - (\boldsymbol{u} - \boldsymbol{p})\,\eta\big)^2. \end{aligned}$$

In addition, it can be verified that the expression

$$\tilde{y} - \tilde{z} = \mu\,\big(\boldsymbol{y} - \boldsymbol{z}\big)^2, \tag{3.18}$$

where $\tilde{y} = D_x^{-1}(\boldsymbol{y}_x, \boldsymbol{y}_x)$, $\tilde{z} = D_x^{-1}(\boldsymbol{z}_x, \boldsymbol{z}_x)$, is compatible with eqs.(3.12) and (3.15). According to (3.18) the transformation (3.15) can be rewritten as

$$\boldsymbol{y}_x = \boldsymbol{z}_x - \frac{\tilde{y} - \tilde{z}}{2}\big(\boldsymbol{y} + \boldsymbol{z}\big) + \frac{\boldsymbol{y}_x^2 - \boldsymbol{z}_x^2}{2\,(\tilde{y} - \tilde{z})}\big(\boldsymbol{y} - \boldsymbol{z}\big).$$

As the obtained ABT depends on the quasi local variables $\tilde{y}$ and $\tilde{z}$, then the SF can also depends on quasi local variables $\tilde{u}$, $\tilde{v}$, and $\tilde{p}$. To introduce these variables explicitly one must perform the substitutions $(\boldsymbol{u}, \boldsymbol{p}) = 1 - (\tilde{u} - \tilde{p})/(2\,\mu)$, $(\boldsymbol{v}, \boldsymbol{p}) = 1 - (\tilde{v} - \tilde{p})/(2\,\nu)$ into (3.17).

Also if we write (3.18) for each pair of solutions $\{\boldsymbol{p},\, \boldsymbol{u}\}$, $\{\boldsymbol{p},\, \boldsymbol{v}\}$, $\{\boldsymbol{q},\, \boldsymbol{u}\}$, $\{\boldsymbol{q},\, \boldsymbol{v}\}$ and exclude the quasi local variables, then the following scalar SF for (3.12) is obtained:

$$\mu\,\big(\boldsymbol{u} - \boldsymbol{p}\big)^2 - \nu\,\big(\boldsymbol{v} - \boldsymbol{p}\big)^2 = \mu\,\big(\boldsymbol{q} - \boldsymbol{v}\big)^2 - \nu\,\big(\boldsymbol{q} - \boldsymbol{u}\big)^2.$$

4 Anisotropic Schwartz-KdV equation

Let us now consider the anisotropic Schwartz-KdV equation

$$\begin{aligned} \boldsymbol{z}_t =&\boldsymbol{z}_{xxx} + \frac{3}{2}\,(\ln\, f)_x\,(\boldsymbol{z}_{xx} + \boldsymbol{z}_x^2\boldsymbol{z}) + 3\,(\boldsymbol{z}_x,\boldsymbol{z}_{xx})\,\boldsymbol{z}+ \\ &+\frac{3}{2}\,f\left(\langle\boldsymbol{z}_{xx}\rangle^2 - \frac{(\langle\boldsymbol{z}\rangle^2\,\langle\boldsymbol{z},\boldsymbol{z}_x\rangle_x - \langle\boldsymbol{z},\boldsymbol{z}_x\rangle^2)^2}{\langle\boldsymbol{z}\rangle^6}\right)\boldsymbol{z}_x, \quad \boldsymbol{z}^2 = 1, \end{aligned} \tag{4.1}$$

and its ABT [4]

$$\begin{aligned} \boldsymbol{y}_x =&\mu\, f\,(\langle\boldsymbol{y},\boldsymbol{z}\rangle + g\langle\boldsymbol{z}\rangle^2)(\boldsymbol{z}_x - (\boldsymbol{y},\boldsymbol{z}_x)\,\boldsymbol{y})+ \\ &+\mu\, f\,(\langle\boldsymbol{y},\boldsymbol{z}_x\rangle + g\langle\boldsymbol{z},\boldsymbol{z}_x\rangle)(\langle\boldsymbol{y},\boldsymbol{z}\rangle\,\boldsymbol{y} - \boldsymbol{z}), \\ f =&\frac{\langle\boldsymbol{z}\rangle^2}{\langle\boldsymbol{z}\rangle^2\,\langle\boldsymbol{z}_x\rangle^2 - \langle\boldsymbol{z},\boldsymbol{z}_x\rangle^2} =: f\{\boldsymbol{z}\}, \quad g^2 = \frac{\langle\boldsymbol{y}\rangle^2}{\langle\boldsymbol{z}\rangle^2}. \end{aligned} \tag{4.2}$$

According to **Prop. 1** we have

$$\left(\sqrt{f\{\boldsymbol{y}\}\,f\{\boldsymbol{z}\}}\,\right)^{-1} = \mu\,(\langle\boldsymbol{y},\boldsymbol{z}\rangle + \sqrt{\langle\boldsymbol{y}\rangle^2\,\langle\boldsymbol{z}\rangle^2}). \tag{4.3}$$

Using the algorithm described above we have found that

$$\begin{aligned} (\boldsymbol{y},\boldsymbol{z}_x) =&\frac{1}{2}\,\frac{\left(\tilde{y} - \tilde{z}\right)\left(y\,(\boldsymbol{y},\boldsymbol{z}) + z\right)\sqrt{f\{\boldsymbol{y}\}}}{y\,\sqrt{f\{\boldsymbol{y}\}} + z\,\sqrt{f\{\boldsymbol{z}\}}} \\ &-\frac{\langle\boldsymbol{z},\boldsymbol{z}_x\rangle\Big((\boldsymbol{y},\boldsymbol{z})\,(\sqrt{f\{\boldsymbol{y}\}}\,y - \sqrt{f\{\boldsymbol{z}\}}\,z) + 2\,\sqrt{f\{\boldsymbol{y}\}}\,z\Big)}{z^2\left(\sqrt{f\{\boldsymbol{y}\}}\,y + \sqrt{f\{\boldsymbol{z}\}}\,z\right)} \\ &-\frac{\langle\boldsymbol{y},\boldsymbol{y}_x\rangle\Big(\sqrt{f\{\boldsymbol{y}\}}\,y - \sqrt{f\{\boldsymbol{z}\}}\,z - 2\,(\boldsymbol{y},\boldsymbol{z})\,\sqrt{f\{\boldsymbol{z}\}}\,y\Big)\sqrt{f\{\boldsymbol{y}\}}}{y^2\left(\sqrt{f\{\boldsymbol{y}\}}\,y + \sqrt{f\{\boldsymbol{z}\}}\,z\right)\sqrt{f\{\boldsymbol{z}\}}}, \end{aligned} \tag{4.4}$$

is compatible with eqs.(4.1) and (4.2). Here $y^2 = \langle\boldsymbol{y}\rangle^2$, $z^2 = \langle\boldsymbol{z}\rangle^2$ and

$$\begin{aligned} &D_x\,\tilde{y} = \langle\boldsymbol{y}_{xx}\rangle^2 f - \frac{2\,f^2\left(\langle\boldsymbol{y}\rangle^2\,\langle\boldsymbol{y}_x,\boldsymbol{y}_{xx}\rangle - \langle\boldsymbol{y},\boldsymbol{y}_x\rangle\langle\boldsymbol{y},\boldsymbol{y}_{xx}\rangle\right)^2}{\langle\boldsymbol{y}\rangle^4} \\ &+\frac{4f^2\langle\boldsymbol{y},\boldsymbol{y}_x\rangle^4 + 8f\langle\boldsymbol{y}\rangle^2\langle\boldsymbol{y},\boldsymbol{y}_x\rangle^2 + \langle\boldsymbol{y}\rangle^4 + 4f^2\langle\boldsymbol{y},\boldsymbol{y}_x\rangle\langle\boldsymbol{y}_x,\boldsymbol{y}_{xx}\rangle\langle\boldsymbol{y}\rangle^4}{f\langle\boldsymbol{y}\rangle^6}, \\ &-\frac{\left(2\,f^2\langle\boldsymbol{y},\boldsymbol{y}_x\rangle^2 + 2\langle\boldsymbol{y}\rangle^2 + f^2\langle\boldsymbol{y}\rangle^2\langle\boldsymbol{y},\boldsymbol{y}_{xx}\rangle\right)^2}{f\langle\boldsymbol{y}\rangle^6} =: \varphi\{\boldsymbol{y}\}, \quad f := f\{\boldsymbol{y}\}; \\ &D_x\,\tilde{z} = \varphi\{\boldsymbol{z}\}. \end{aligned}$$

Taking into account (4.3) and (4.4) the ABT for (4.2) can be transformed into the symmetric form

$$\begin{aligned}\sqrt{f\{\boldsymbol{y}\}}\,\boldsymbol{y}_x-\sqrt{f\{\boldsymbol{z}\}}\,\boldsymbol{z}_x&=-\frac{(\tilde{y}-\tilde{z})\sqrt{f\{\boldsymbol{y}\}f\{\boldsymbol{z}\}}}{2\big(\sqrt{f\{\boldsymbol{y}\}}\,y+\sqrt{f\{\boldsymbol{z}\}}\,z\big)}(y\,\boldsymbol{z}+z\,\boldsymbol{y})+\\&+\frac{\langle\boldsymbol{z},\boldsymbol{z}_x\rangle\Big(\boldsymbol{z}\left(\sqrt{f\{\boldsymbol{y}\}}\,y-\sqrt{f\{\boldsymbol{z}\}}\,z\right)+2\sqrt{f\{\boldsymbol{y}\}}\,z\,\boldsymbol{y}\Big)\sqrt{f\{\boldsymbol{z}\}}}{z^2\left(\sqrt{f\{\boldsymbol{y}\}}\,y+\sqrt{f\{\boldsymbol{z}\}}\,z\right)}-\\&-\frac{\langle\boldsymbol{y},\boldsymbol{y}_x\rangle\Big(\boldsymbol{y}\left(\sqrt{f\{\boldsymbol{z}\}}\,z-\sqrt{f\{\boldsymbol{y}\}}\,y\right)+2\sqrt{f\{\boldsymbol{z}\}}\,y\,\boldsymbol{z}\Big)\sqrt{f\{\boldsymbol{y}\}}}{y^2\left(\sqrt{f\{\boldsymbol{y}\}}\,y+\sqrt{f\{\boldsymbol{z}\}}\,z\right)}.\end{aligned}\tag{4.5}$$

For solutions of (4.1) we have constructed the following SF

$$\boldsymbol{q}=\frac{\mu\,\nu\,\xi\,\boldsymbol{p}+(\mu-\nu)\,(\nu\,\zeta\,\boldsymbol{u}-\mu\,\eta\,\boldsymbol{v})}{\sqrt{\big(\mu\,\nu\,\xi\,\boldsymbol{p}+(\mu-\nu)\,(\nu\,\zeta\,\boldsymbol{u}-\mu\,\eta\,\boldsymbol{v})\big)^2}},\tag{4.6}$$

where

$$\xi=u\,v-\langle\boldsymbol{u},\,\boldsymbol{v}\rangle,\ \eta=u\,p+\langle\boldsymbol{u},\,\boldsymbol{p}\rangle,\ \zeta=v\,p+\langle\boldsymbol{v},\,\boldsymbol{p}\rangle,$$
$$p^2=\langle\boldsymbol{p}\rangle^2,\ u^2=\langle\boldsymbol{u}\rangle^2,\ v^2=\langle\boldsymbol{v}\rangle^2.$$

The scalar SF can be derived from diagram (2.6) and equations of form (4.3):

$$\frac{\nu\,(\langle\boldsymbol{q},\boldsymbol{u}\rangle+\sqrt{\langle\boldsymbol{q}\rangle^2\,\langle\boldsymbol{u}\rangle^2})}{\mu\,(\langle\boldsymbol{q},\boldsymbol{v}\rangle+\sqrt{\langle\boldsymbol{q}\rangle^2\,\langle\boldsymbol{v}\rangle^2})}=\frac{\mu\,(\langle\boldsymbol{u},\boldsymbol{p}\rangle+\sqrt{\langle\boldsymbol{u}\rangle^2\,\langle\boldsymbol{p}\rangle^2})}{\nu\,(\langle\boldsymbol{v},\boldsymbol{p}\rangle+\sqrt{\langle\boldsymbol{v}\rangle^2\,\langle\boldsymbol{p}\rangle^2})}.\tag{4.7}$$

If we make the reduction $\langle\cdot,\cdot\rangle=(\cdot,\cdot)$ in (4.6), then we get the SF for the vector Schwartz-KdV equation on a sphere (cf. [2])

$$\boldsymbol{q}=\frac{-\mu\,\nu\,(\boldsymbol{u}-\boldsymbol{v})^2\boldsymbol{p}+(\nu-\mu)\,(\,\nu\,(\boldsymbol{v}+\boldsymbol{p})^2\boldsymbol{u}-\mu\,(\boldsymbol{u}+\boldsymbol{p})^2\boldsymbol{v})}{(\mu\,(\boldsymbol{u}+\boldsymbol{p})-(\boldsymbol{v}+\boldsymbol{p})\,\nu)^2},$$

and hence (4.7) can be rewritten as

$$\nu^2\,(\boldsymbol{q}+\boldsymbol{u})^2\,(\boldsymbol{v}+\boldsymbol{p})^2=\mu^2\,(\boldsymbol{u}+\boldsymbol{p})^2\,(\boldsymbol{q}+\boldsymbol{v})^2.$$

This relation is an analogue of the well known integrable discrete model in [9]:

$$\frac{u(n+1,m)-u(n,m)}{u(n,m+1)-u(n,m)}\,\frac{u(n+1,m+1)-u(n,m+1)}{u(n+1,m+1)-u(n+1,m)}=\frac{\nu}{\mu}.$$

The isotropic Schwartz-KdV equation on sphere $\mathbb{S}^n$ was obtained in [1]. Also we can take the limit transition $a\to\infty$ in (1.2) and get

$$\boldsymbol{z}_t=\boldsymbol{z}_{xxx}-3\,\frac{(\boldsymbol{z}_x,\boldsymbol{z}_{xx})}{\boldsymbol{z}_x^2}\boldsymbol{z}_{xx}+\frac{3}{2}\,\frac{\boldsymbol{z}_{xx}^2}{\boldsymbol{z}_x^2}\,\boldsymbol{z}_x,\quad \boldsymbol{z}^2=1.\tag{4.8}$$

Next, if $a\to\infty$ in ABT (3.1) then as a result we obtain the transformation

$$\boldsymbol{y}_x=\left(\sqrt{\mu'\,(1+(\boldsymbol{y},\boldsymbol{z})\,)}-\sqrt{\mu'\,(1+(\boldsymbol{y},\boldsymbol{z})\,)-1}\right)^2\,\left(\boldsymbol{z}_x-\frac{(\boldsymbol{y},\boldsymbol{z}_x)}{1+(\boldsymbol{y},\boldsymbol{z})}(\boldsymbol{y}+\boldsymbol{z})\right).\tag{4.9}$$

The ABT of the isotropic Schwartz-KdV on $\mathbb{R}^n$ was constructed in [2]:

$$\boldsymbol{y}_x = \mu\, \boldsymbol{z}_x^{-2} \big(2\,(\boldsymbol{y}-\boldsymbol{z}, \boldsymbol{z}_x)\,(\boldsymbol{y}-\boldsymbol{z}) - (\boldsymbol{y}-\boldsymbol{z})^2 \boldsymbol{z}_x\big). \tag{4.10}$$

The reduction $\boldsymbol{y}^2 = \boldsymbol{z}^2 = 1$ in (4.10) coincides with the result of the reduction $\langle\cdot,\cdot\rangle = (\cdot,\cdot)$ in (4.2), but it does not coincide with transformation (4.9). Transformation (4.9) is compatible with (4.8), hence it is also the ABT for (4.8). We have constructed the SF for (4.8) with the help of (4.9).

Firstly, according to **Prop. 1** we obtain

$$\sqrt{\boldsymbol{y}_x^2\, \boldsymbol{z}_x^{-2}} = \Big(\sqrt{\mu'\,\big(1+(\boldsymbol{y},\boldsymbol{z})\big)} - \sqrt{\mu'\,\big(1+(\boldsymbol{y},\boldsymbol{z})\big)-1}\Big)^2. \tag{4.11}$$

Secondly, we have established that the following equation

$$\tilde{y} - \tilde{z} = 2\,\frac{(\boldsymbol{y},\boldsymbol{z}_x)}{1+(\boldsymbol{y},\boldsymbol{z})}\,\left(1+\Big(\sqrt{\mu'\,\big(1+(\boldsymbol{y},\boldsymbol{z})\big)} - \sqrt{\mu'\,\big(1+(\boldsymbol{y},\boldsymbol{z})\big)-1}\Big)^2\right), \tag{4.12}$$

where

$$\tilde{y} = D_x^{-1}\left(\frac{\boldsymbol{y}_{xx}^2}{\boldsymbol{y}_x^2} - 2\,\frac{(\boldsymbol{y}_x,\boldsymbol{y}_{xx})^2}{\boldsymbol{y}_x^4}\right), \quad \tilde{z} = D_x^{-1}\left(\frac{\boldsymbol{z}_{xx}^2}{\boldsymbol{z}_x^2} - 2\,\frac{(\boldsymbol{z}_x,\boldsymbol{z}_{xx})^2}{\boldsymbol{z}_x^4}\right),$$

is compatible with eqs.(4.8) and (4.9).

Using (4.11) and (4.12), equation (4.9) can be transformed to the form

$$\frac{\boldsymbol{y}_x}{\sqrt{\boldsymbol{y}_x^2}} - \frac{\boldsymbol{z}_x}{\sqrt{\boldsymbol{z}_x^2}} = -\frac{1}{2}\,\frac{\tilde{y}-\tilde{z}}{\sqrt{\boldsymbol{y}_x^2}+\sqrt{\boldsymbol{z}_x^2}}\,(\boldsymbol{y}+\boldsymbol{z}) \tag{4.13}$$

that precisely coincides with the result of the reduction $\langle\cdot,\cdot\rangle = (\cdot,\cdot)$ in (4.5).

For simplicity we denote

$$\cosh^2\varphi_1 = \mu\,\big(1+(\boldsymbol{u},\boldsymbol{p})\big),\ \cosh^2\varphi_2 = \nu\,\big(1+(\boldsymbol{v},\boldsymbol{p})\big),$$
$$\cosh^2\varphi_3 = \nu\,\big(1+(\boldsymbol{q},\boldsymbol{u})\big),\ \cosh^2\varphi_4 = \mu\,\big(1+(\boldsymbol{q},\boldsymbol{v})\big).$$

If one write four equations (4.11) for the pairs $\{\boldsymbol{p},\,\boldsymbol{u}\}$, $\{\boldsymbol{p},\,\boldsymbol{v}\}$, $\{\boldsymbol{q},\,\boldsymbol{u}\}$, $\{\boldsymbol{q},\,\boldsymbol{v}\}$ and assume commutativity of diagram (2.6), then the scalar SF for solutions of (4.8) follows

$$\varphi_1 + \varphi_3 = \varphi_2 + \varphi_4$$

with above defined φ_i.

The SF for pairs of solutions of (4.8) connected by transformation (4.9) has the form

$$\boldsymbol{q} = \psi\,\Big(\mu'\,\nu'\,\xi\,\eta\,(\boldsymbol{u}-\boldsymbol{v})^2\,\zeta_1\,\zeta_2\,\boldsymbol{p} + (\nu'-\mu')\,\eta\,(\xi+1)^2\zeta_1\big(\,(\xi-\eta)^2-\zeta_2\big)\,\boldsymbol{u} +$$
$$+ (\mu'-\nu')\,\xi\,(\eta+1)^2\zeta_2\big(\,(\xi-\eta)^2-\zeta_1\big)\,\boldsymbol{v}\Big),$$

where

$$\psi = \frac{\xi\,\eta\,(\boldsymbol{u}-\boldsymbol{v})^2}{\Big(\,(\xi-\eta)\,\big(\xi\,\zeta_2-\eta\,\zeta_1+\eta\,\xi(\zeta_2-\zeta_1)\,\big)-\zeta_1\,\zeta_2\Big)\,\Big(\,(\xi-\eta)\,\big(\xi\,\zeta_1-\eta\,\zeta_2+\zeta_1-\zeta_2\big)-\zeta_1\,\zeta_2\Big)},$$
$$\xi = \frac{1}{2}\Big(\sqrt{\nu'\,\big(1+(\boldsymbol{v},\boldsymbol{p})\big)} - \sqrt{\nu'\,\big(1+(\boldsymbol{v},\boldsymbol{p})\big)-2}\Big)^2,\ \zeta_1 = (\xi-\eta)^2 + \nu'\,\xi\,\eta\,(\boldsymbol{u}-\boldsymbol{v})^2,$$
$$\eta = \frac{1}{2}\Big(\sqrt{\mu'\,\big(1+(\boldsymbol{u},\boldsymbol{p})\big)} - \sqrt{\mu'\,\big(1+(\boldsymbol{u},\boldsymbol{p})\big)-2}\Big)^2,\ \zeta_2 = (\xi-\eta)^2 + \mu'\,\xi\,\eta\,(\boldsymbol{u}-\boldsymbol{v})^2.$$

Proposition 2. Let $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{p}$ be solutions of (4.8). If solutions $\boldsymbol{v}$ and $\boldsymbol{p}$ are connected by ABT of the form

$$\boldsymbol{v}_x = \mu_1 \boldsymbol{p}_x^{-2} \Big((\boldsymbol{v}, \boldsymbol{p}_x) \, (\boldsymbol{v} - \boldsymbol{p}) - \big(1 - (\boldsymbol{v}, \boldsymbol{p})\big) \, \boldsymbol{p}_x \Big), \tag{4.14}$$

and $\boldsymbol{u}$, $\boldsymbol{p}$ are connected by the following ABT

$$\boldsymbol{u}_x = \left(\sqrt{\mu_2 \, (1 + (\boldsymbol{u}, \boldsymbol{p}) \,)} - \sqrt{\mu_2 \, (1 + (\boldsymbol{u}, \boldsymbol{p}) \,) - 1} \right)^2 \left(\boldsymbol{p}_x - \frac{(\boldsymbol{u}, \boldsymbol{p}_x)}{1 + (\boldsymbol{u}, \boldsymbol{p})} (\boldsymbol{u} + \boldsymbol{p}) \right), \tag{4.15}$$

then

$$\left(\sqrt{\mu_2 \, (1 + (\boldsymbol{u}, \boldsymbol{p}) \,)} - \sqrt{\mu_2 \, (1 + (\boldsymbol{u}, \boldsymbol{p}) \,) - 1} \right)^4 \mu_1^2 \, \big(1 - (\boldsymbol{v}, \boldsymbol{p})\big)^2 - \boldsymbol{u}_x^2 \, \boldsymbol{v}_x^2 = 0 \tag{4.16}$$

is compatible with eqs.(4.8), (4.14), and (4.15).

In the scalar limit, eqs.(4.8), (4.14), and (4.15) take the forms

$$u_t = u_{xxx} - \frac{3}{2} \frac{u_{xx}^2}{u_x}, \tag{4.17}$$

$$\mu_1 = \frac{2 \, v_x \, p_x}{(v - p)^2}, \tag{4.18}$$

$$\mu_2 = \frac{\big(p_x \, (1 + u^2) + (1 + p^2) \, u_x\big)^2}{8 \, u_x \, p_x \, (1 + u \, p)^2} \tag{4.19}$$

correspondingly. In the same limit, (4.16) becomes an identity according to eqs.(4.18) and (4.19).

The Schwarz-KdV equation (4.17) is a special case of the Krichever-Novikov equation. The ABT (4.18) can be obtained from the ABT for KN equation [10]. Equation (4.17) is invariant under the linear-fractional transformation of u. It can de verified that ABTs (4.19) and (4.18) are not connected by the linear-fractional transformation of u, v, and p. However, each pair of solutions $\{v, \, p\}$ and $\{u, \, p\}$, connected by transformations (4.18) and (4.19) accordingly, satisfy the scalar limit of (4.13):

$$y_x \, z - z_x \, y = \frac{1}{4} (\tilde{y} - \tilde{z}) \, (1 + y \, z),$$

where $\tilde{y} = D_x^{-1} \, (y_{xx}/y_x)^2$, $\tilde{z} = D_x^{-1} \, (z_{xx}/z_x)^2$.

We have also established that equation (4.8) does not change under the transformation $\boldsymbol{z}_x = \boldsymbol{y}_x / \boldsymbol{y}_x^2$. Obviously, (4.17) admits the transformation $z_x = y_x^{-1}$. Probably, this fact explains the existence of two ABT (4.18) and (4.19) for (4.17).

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References

[1] A.G. Meshkov, V.V. Sokolov, Integrable evolution equations on the N-dimensional sphere, *Comm. Math. Phys.* **232** (2002), 1-18.

[2] A.G. Meshkov, V.V. Sokolov, Classification of integrable divergent N-component evolution systems, *Theoret. and Math. Phys.* **139** (2004), 609-622.

[3] M.Yu. Balakhnev, A class of integrable evolutionary vector equations, *Theoret. and Math. Phys.* **142** (2005) 8-14.

[4] A.G. Meshkov, M.Yu. Balakhnev, Integrable anisotropic evolution equations on a sphere, *Symmetry, Integrability and Geometry: Meth. and Appl.* V.1 (2005), Paper 027.

[5] M.Yu. Balakhnev, Superposition formulas for vector generalizations of mKdV equation, *Math. Notes* **82** (2007), 448-450.

[6] M.Yu. Balakhnev, Superposition formulas for vector evolution equations, *Theoret. and Math. Phys.* **154** (2008), 220-226.

[7] L. Bianchi, Lezioni di Geometria Differenziale, Pisa, (1902) V. II, P. 418.

[8] A.G. Meshkov. Computer package for investigation of the completele integrability, *Proc. of the Third Int. Conf. Symmetry in Nonlinear Math. Phys*, Kyiv, (2000), V. I, 35-46.

[9] A.I. Bobenko, Yu.B. Suris, Integrable systems on quad-graphs, *Internat. Math. Res. Notices* **11** (2002), 573-611.

[10] V.E. Adler, Bäcklund transformation for the Krichever-Novikov equation, *Internat. Math. Res. Notices* **1** (1998), 1-4.