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On nonlocal symmetries of integrable three-field evolutionary systems

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Abstract

Nonlocal symmetries for exactly integrable three-field evolutionary systems have been computed. Differentiation the nonlocal symmetries with respect to x gives a few hyperbolic systems for each evolution system. Zero curvature representations for all nonlocal systems and for some of the hyperbolic systems are constructed.

1 Introduction

It is well known that the modified Korteweg-de Vries equation (mKdV) is the symmetry of the sine-Gordon equation [1]. And vice versa, the sine-Gordon equation considered as the nonlocal evolution equation $u_t = \partial_x^{-1} \sin u$ is the symmetry of the mKdV equation.

In recent years interrelations between evolution and hyperbolic systems have been studied in terms of the Lax representations. Hyperbolic systems appear in this approach as negative flows in hierarchies of integrable evolution systems. Hereafter we call a system integrable for brevity if it possesses a nontrivial zero curvature representation or Lax representation. There are a lot of papers where Lax representations are constructed for several equations. Most important among others was [2] where the general construction for Lax representation of KdV type equations was presented. These results were published in detail in [3] where the proofs of all theorems and many examples were presented. We also point out [4], where it was shown how one can obtain some popular equations using the results of [2].

This investigation has been motivated by the fact that the symmetry analysis of hyperbolic systems is an extremely difficult task as compared with the evolutionary systems. Now a lot of integrable evolution systems are known (see [5], for example), but the list of the known integrable hyperbolic systems is much shorter. See for example [6] – [9] and references cited therein. Having [1] in a mind, it may be reasonable to compute nonlocal symmetries directly for each known evolution integrable system. Some of these symmetries, considered as evolution equations, will belong to the hierarchy of corresponding evolution integrable system as negative flows. Differentiating any nonlocal symmetry few times with respect to x one can obtain a local nonevolution system. By doing so one can find several nonevolution systems belonging to hierarchy of each integrable evolution system (see (3.3), (3.4), (3.6), (3.8) and (3.10) as an example).

Some of nonlocal symmetries may not belong to the hierarchy of the system under consideration. Such irrelevant symmetries are not commutative with the higher flows of the hierarchy. Therefore one ought to verify commutativity of nonlocal symmetries with the higher flows of the hierarchy. The higher flows can be found by a recursion operator or by constructing a zero curvature representation. Another and simpler way to prove the integrability of any symmetry $u_\tau = \sigma$ is to construct a nontrivial zero curvature representation for it

$$(U_\tau - V_x + [U,V])\big|_{u_\tau = \sigma} = 0, \tag{1.1}$$

where $[U,V] = UV - VU$. This computation is sufficiently simple, because the matrix U is common for all members of the hierarchy. If an evolution system is integrable and its matrix U is known, then equation (1.1) gives a linear differential system for elements of V. This system can be easily solved and the zero curvature representation can be obtain. If the symmetry $u_\tau = \sigma$ does not belong to the hierarchy, then system (1.1) is incompatible.

In this article we apply the discussed scheme to the integrable three-field evolutionary systems that were obtained in our previous articles [9, 10, 11]. These results make it possible to find new hyperbolic systems. The zero curvature representations for some of the new systems are found.

Section 2 introduces notation and basic notions. Section 3 is devoted to the nonlocal symmetries and corresponding hyperbolic systems. Section 4 contains zero curvature representations for all nonlocal systems and selected hyperbolic systems.

2 Notation

Consider an evolution system with two independent t, x and m dependent variables u^α

$$u_t = K(t, x, u, u_x, \ldots, u_n), \tag{2.1}$$

where $K = \{K^\alpha,\ \alpha = 1, \ldots, m\}$ and $u = \{u^\alpha,\ \alpha = 1, \ldots, m\}$ are infinitely differentiable vector functions; $u \equiv u_0$, $u_x \equiv u_1$, $u_k = \partial^k u/\partial x^k$. The set of the variables u_i^α is usually denoted as u for brevity.

Definition 1. (see [5, 12, 13, 14]). A vector function $\sigma(t, x, u, u_x, \ldots, u_k)$ is said to be the generalized symmetry of system (2.1) if it satisfies the following equation

$$(D_t - K_*)\sigma = 0, \tag{2.2}$$

where

$$(K_*)^\alpha_\beta = \sum_{k\geqslant 0} \frac{\partial K^\alpha}{\partial u_k^\beta} D_x^k, , \quad D_x = \frac{\partial}{\partial x} + \sum_{\alpha,k\geqslant 0} u_{k+1}^\alpha \frac{\partial}{\partial u_k^\alpha}, \quad D_t = \frac{\partial}{\partial t} + \sum_{\alpha,k\geqslant 0} \left(D_x^k K^\alpha\right) \frac{\partial}{\partial u_k^\alpha}.$$

Operation $* : K \to K_*$ is called the linearization one, D_x is the total differentiation operator with respect to x, D_t is the evolutionary differentiation operator. The order of the differential operator f_* is called the order of the (vector-) function f.

A generalized symmetries σ are often written as the following evolutionary systems

$$u_\tau = \sigma(t, x, u), \tag{2.3}$$

where τ is another parameter of evolution. It is possible because the compatibility condition for equations (2.1) and (2.3) coinciding with (2.2).

Definition 2. (see [5, 12, 13, 14]). If the differentiable functions ρ and θ satisfy the following identity

$$D_t \rho(t, x, u) = D_x \theta(t, x, u), \tag{2.4}$$

for any solution u of system (2.1), then this identity is called the local conservation law of system (2.1). The functions ρ and θ are called the conserved density and flux correspondingly. The pair (ρ, θ) is called the conserved current.

As the operators D_t and D_x are commutative, the functions $\rho_0 = D_x f$ and $\theta_0 = D_t f$, where f is an arbitrary function, identically satisfy (2.4). Such currents are called trivial. Conserved densities are always defined by modulo trivial (divergent) densities.

Let (ρ, θ) be a conserved current of a finite order. Then the following system

$$D_x w = \rho(t, x, u), \quad D_t w = \theta(t, x, u), \tag{2.5}$$

where u is a solution of system (2.1), is compatible. The solution of this system is formally written in the form $w = D_x^{-1}\rho$. One may consider w as a new dynamical variable. It is called weakly nonlocal or quasi-local (see [15]). We will simply write "nonlocal variables" for brevity. We call the nonlocal variables $\{w_i^{(1)} = D_x^{-1}\rho_i^{(0)}\}$, that are constructed by means the local conserved currents $\{\rho_i^{(0)}, \theta_i^{(0)}\}$, the first order nonlocal variables.

To operate with the new variables one must prolong the operators D_t and D_x:

$$D_x^{(1)} = D_x + \rho_i^{(0)}\frac{\partial}{\partial w_i^{(1)}}, \quad D_t^{(1)} = D_t + \theta_i^{(0)}\frac{\partial}{\partial w_i^{(1)}}, \tag{2.6}$$

where summation over the repeated indeces is implied. The prolonged operators are commutative

$$\left[D_t^{(1)}, D_x^{(1)}\right] = [D_t, D_x] + \left(D_t\rho_i^{(0)} - D_x\theta_i^{(0)}\right)\frac{\partial}{\partial w_i^{(1)}} = 0$$

because of (2.4) and commutatuivity D_t and D_x.

Now one may search nonlocal conserved currents. If there exists a nontrivial conserved current $\{\rho^{(1)}, \theta^{(1)}\}$ depending on $t, x, u, u_x, \ldots, u_k$ and $w_i^{(1)}$, and $w_i^{(1)}$ cannot be removed by a gauge transformation $\rho \to \rho + D_x^{(1)}f$, $\theta \to \theta + D_t^{(1)}f$, then we call the nonlocal variable $\{w^{(2)} = D_x^{-1}\rho^{(1)}\}$ the second order nonlocal variable and so on. The new prolongation of D_t and D_x is constructed on each step

$$D_x^{(n+1)} = D_x^{(n)} + \rho_i^{(n)}\frac{\partial}{\partial w_i^{(n+1)}}, \quad D_t^{(n+1)} = D_t^{(n)} + \theta_i^{(n)}\frac{\partial}{\partial w_i^{(n+1)}} \tag{2.7}$$

and each prolongation gives the commutative operators because

$$\left[D_t^{(n+1)}, D_x^{(n+1)}\right] = [D_t^{(n)}, D_x^{(n)}] + \left(D_t^{(n)}\rho_i^{(n)} - D_x^{(n)}\theta_i^{(n)}\right)\frac{\partial}{\partial w_i^{(n+1)}},$$

(see also [16], for example).

If equation (2.2) with the prolonged operators D_t and D_x possesses a solution σ depending on the nonlocal variables, then σ is called the nonlocal symmetry. We stress that there exist nonintegrable nonlocal symmetries, and the best way to prove integrability a nonlocal symmetry is constructing the zero curvature representation for it.

3 Nonlocal symmetries and hyperbolic systems

We consider here the exactly integrable evolutionary systems that are found in paper [11]. We will cite these systems as "a", …, "g" in accordance with the mentioned article. For the reader's convenience we write the evolution system under consideration in the beginning of each subsection 2.1 – 2.7. Then we present the nonlocal variables and nonlocal symmetries in form (2.3). All symmetries linearly depend on arbitrary parameters c_i. Each nonlocal system $u_\tau = \sigma(x, u, w, c_i)$ can be reduced to a local form, but for some parameters we obtain very cumbersume local systems. We set such parameters to zero and consider simplest symmetries only.

Besides, the order of equations that may be interesting for applications is not too high. That is why to construct the nonlocal variables we find conserved densities of the zero and first orders only. The chains of nonlocal variables for the systems under consideration seems to be infinite. That is why only nonlocal variables of the first and second orders are considered here.

All nonlocal systems presented below possess zero curvature representations.

3.1 System "a"

It is an exceptional case when the third order system possesses the second order symmetry:

$$u_t = m_x n_x, \quad m_t = -m_{xx} - m_x^2 + 2m_x n_x + 2m_x u_x, \quad n_t = n_{xx} + n_x^2 + 2n_x u_x - 2m_x n_x.$$

We consider this system instead of the original system of the third order for simplicity. This system possesses the following 8-parametric nonlocal symmetry

$$\begin{aligned} u_\tau &= c_1 w_1 + c_2 w_2 - c_4 w_4 + c_5 w_6 + c_6(w_1 w_4 - w_2 w_6) - c_7 w_1 w_2 - c_8 w_1 w_4, \\ m_\tau &= c_1 w_1 + c_3 w_3 + (c_4 - c_6 w_1)(w_2 w_3 - w_4) + c_5 w_1 w_3 \\ &\qquad + c_7(w_5 - w_1 w_2) + c_8(w_3 w_5 - w_1 w_4), \\ n_\tau &= -c_2 w_2 + (c_3 + c_4 w_2 + c_5 w_1 - c_6 w_1 w_2 + c_8 w_5)(w_3 + e^{-m-n}) \\ &\qquad + w_6(c_6 w_2 - c_5) + c_7 w_5, \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} &w_1 = D_x^{-1} e^{2u+n}, \quad w_2 = D_x^{-1} e^{m-2u}, \quad w_3 = D_x^{-1} m_x e^{-m-n}, \\ &w_4 = D_x^{-1} w_3 e^{m-2u}, \quad w_5 = D_x^{-1} w_1 e^{m-2u}, \quad w_6 = D_x^{-1}(w_3 e^{2u+n} + e^{2u-m}). \end{aligned}$$

We found the following simple local systems from (3.1).

A.1. If we adopt $c_i = 0$, $i > 3$ in (3.1), then the following system follows:

$$u_\tau = c_1 w_1 + c_2 w_2, \quad m_\tau = c_1 w_1 + c_3 w_3, \quad n_\tau = -c_2 w_2 + c_3(w_3 + e^{-m-n}). \tag{3.2}$$

This implies $(u + n - m)_\tau = c_3 e^{-m-n}$, therefore we introduce the new variables:

$$p = \frac{1}{3}(u + n - m), \quad q = \frac{1}{3}(2u + m + 2n), \quad r = \frac{1}{3}(2u - 2m - n).$$

Rewriting system (3.2) in the new variables and differentiating two of the three equations, we obtain

$$\begin{aligned} q_{\tau x} &= c_1 e^{r+2q} + a(q_x + 2r_x - 6p_x)e^{r-q}, \quad p_\tau = a e^{r-q}, \\ r_{\tau x} &= c_2 e^{-2r-q} - a(2q_x + r_x - 6p_x)e^{r-q}, \end{aligned} \tag{3.3}$$

where $a = c_3/3$.

A.2. One may transform system (3.2) in another manner adopting $w_1 = p, w_2 = q, w_3 = r$ as the new variables. It follows from the definitions of the variables w_{1-3} that

$$n + 2u = \ln p_x, \quad m - 2u = \ln q_x, \quad n + m = \ln(p_x q_x), \quad m_x = p_x q_x r_x.$$

Using these formulas one can easily odtain

$$\begin{aligned} p_{\tau x} &= p_x(2c_1 p + c_2 q + c_3 r) + c_3 q_x^{-1}, \quad q_{\tau x} = q_x(c_3 r - c_1 p - 2c_2 q), \\ r_{\tau x} &= r_x(c_2 q - c_1 p - 2c_3 r) + c_1 q_x^{-1}. \end{aligned} \tag{3.4}$$

One can perform this transformation for the general system (3.1), but one has to add to the system four additional relations:

$$w_{4,x} = rq_x, \quad w_{5,x} = pq_x, \quad w_{6,x} = rp_x + q_x^{-1}, \quad w_{7,x} = qrp_x + qq_x^{-1}.$$

Now we consider the case when $c_i = 0$, $i > 4$ and $c_4 \neq 0$. The shifts $w_2 \to w_2 - c_3/c_4$ and $w_3 \to w_3 + c_2/c_4$ remove the parameters c_2 and c_3, hence (3.1) takes the following form:

$$u_\tau = c_1 w_1 - c_4 w_4, \quad m_\tau = c_1 w_1 + c_4(w_2 w_3 - w_4), \quad n_\tau = c_4 w_2 (w_3 + e^{-m-n}). \tag{3.5}$$

For (3.5) two further transformations have been found.

A.3. Setting $p = w_3$, $q = n + 2u$, $r = w_2$, and taking into account the identities $p_x = m_x e^{-m-n}, r_x = e^{m-2u}, w_{1,x} = e^q, w_{4,x} = pr_x, e^{m+n} = e^q r_x$, one can easily reduce equations (3.5) to the following form:

$$\begin{aligned} p_{\tau x} &= c_1 r_x^{-1} - \frac{1}{2} p_x q_\tau + \frac{1}{2} c_4 e^{-q} p_x r r_x^{-1} - \frac{3}{2} c_4 p r p_x, \\ q_{\tau x} &= 2c_1 e^q + c_4(r p_x - p r_x) + c_4(r r_x^{-1} e^{-q})_x, \\ r_{\tau x} &= \frac{3}{2} c_4 p r r_x - \frac{1}{2} r_x q_\tau + \frac{1}{2} c_4 r e^{-q}. \end{aligned} \tag{3.6}$$

A.4. Introducing the new variable $z = w_3 e^m$ one can find w_2 from the third equation of (3.5):

$$c_4 w_2 = e^{m+n} \frac{n_\tau}{z e^n + 1}.$$

This allows us to eliminate w_2 and reduce system (3.5) to the following form:

$$\begin{aligned} u_{\tau x} &= c_1 e^{n+2u} - c_4 z e^{-2u}, \quad m_{\tau x} = c_1 e^{n+2u} + \frac{n_\tau m_x}{z e^n + 1}, \\ n_{\tau x} &= c_4 e^{-n-2u}(z e^n + 1) - \frac{n_\tau n_x}{z e^n + 1}. \end{aligned} \tag{3.7}$$

Then differentiating the equation $z = w_3 e^m$ we obtain the relations

$$m_x = \frac{z_x}{z + e^{-n}}, \quad z_x = m_x(z + e^{-n}), \quad z_{\tau x} = m_{\tau x}(z + e^{-n}) + m_x(z_\tau - n_\tau e^{-n}),$$

which make it possible to to eliminate the function m from the system. To write the system in a symmetrical form, we introduce another substitution $n = \ln \bar{z}$. The result is a chiral-type system

$$u_{\tau x} = c_1 \bar{z} e^{2u} - c_4 z e^{-2u}, \quad z_{\tau x} = c_1 F e^{2u} + \bar{z} z_t z_x F^{-1}, \quad \bar{z}_{\tau x} = c_4 F e^{-2u} + z \bar{z}_t \bar{z}_x F^{-1}, \tag{3.8}$$

where $F = z\bar{z} + 1$. Exact integrability of this system has been proved in [9].

If we set in (3.1) $c_4 = c_6 = c_7 = c_8 = 0$, $c_5 \neq 0$, then the shift $w_1 \to w_1 - c_3/c_5$ removes c_3; the second shift $w_3 \to w_3 - c_1/c_5$ implies $w_6 \to w_6 - c_1/c_5 w_1$ and removes c_1. These transformations reduce system (3.1) to the following form:

$$u_\tau = c_2 w_2 + c_5 w_6, \quad m_\tau = c_5 w_1 w_3, \quad n_\tau = c_5(w_1 e^{-m-n} - w_6 + w_1 w_3) - c_2 w_2. \tag{3.9}$$

We found two transformations to the local form for this system.

A.5. If we set $p = w_3$, $q = m - 2u$, $r = w_1$, then using the relations $p_x = m_x e^{-m-n}, r_x = e^{2u+n}, w_{2,x} = e^q, w_{6,x} = pr_x + e^{-q}, e^{m+n} = e^q r_x$ we reduce equation (3.9) to the form

$$\begin{aligned} p_{\tau x} &= c_5 p e^{-q} - \frac{1}{2} p_x q_\tau - \frac{3}{2} c_5 p r p_x, \\ q_{\tau x} &= c_5(r p_x - p r_x) - 2c_2 e^q - 2c_5 e^{-q}, \\ r_{\tau x} &= \frac{3}{2} c_5 p r r_x - \frac{1}{2} r_x q_\tau + c_5 r e^{-q}, \end{aligned} \tag{3.10}$$

different from (3.6).

The second possibility consists in performing the substitution $z = -w_3 e^n - e^{-m}$, $\bar z = e^m$. Eliminating w_1 with the help of the second equation (3.9): $c_5 w_1 = -\bar z_\tau e^n (z\bar z + 1)^{-1}$ we obtain system (3.8).

3.2 System "b"

$$\begin{aligned}
u_t &= \frac{1}{4}\,u_{xxx} + \frac{3}{2}\,(m_x n_{xx} - n_x m_{xx}) + 3n_x^2 m_x - 3m_x^2 n_x + 3u_x n_x m_x - \frac{1}{2}\,u_x^3,\\
m_t &= m_{xxx} - 3m_{xx}(u_x + n_x - m_x) - \frac{3}{2}\,m_x u_{xx} - 3u_x m_x^2 + \frac{3}{2}\,u_x^2 m_x + 6u_x n_x m_x\\
&\quad - 6m_x^2 n_x + 3n_x^2 m_x + m_x^3,\\
n_t &= n_{xxx} + 3n_{xx}(u_x + n_x - m_x) + \frac{3}{2}u_{xx} n_x + 3u_x n_x^2 - 6n_x^2 m_x + \frac{3}{2} n_x u_x^2\\
&\quad - 6u_x n_x m_x + 3m_x^2 n_x + n_x^3.
\end{aligned}$$

This system possesses the following 8-parametric nonlocal symmetry

$$\begin{aligned}
u_\tau &= c_1 w_1 - c_2 w_2 + c_4 w_4 + c_5 w_5 + 2c_6 w_6 + c_7 w_7 + c_8(w_1 w_7 - w_2 w_8),\\
m_\tau &= c_1 w_1 + c_3 w_3 + c_4 w_4 + c_5(w_5 - n_x e^{2(n-m+u)}) + c_6(w_6 - w_2 w_3)\\
&\quad + c_7(w_7 - w_3 w_6) + c_8(w_1 w_7 - w_3 w_9),\\
n_\tau &= c_2 w_2 + (c_3 - c_6 w_2 - c_7 w_6 - c_8 w_9)(w_3 + e^{-m-n})\\
&\quad + c_4(m_x e^{2(m-n-u)} - w_4) - c_5 w_5 - c_6 w_6 + c_8 w_2 w_8,
\end{aligned} \tag{3.11}$$

where

$$\begin{aligned}
&w_1 = D_x^{-1} e^{2n+2u}, \quad w_2 = D_x^{-1} e^{2m-2u}, \quad w_3 = D_x^{-1} m_x e^{-m-n},\\
&w_4 = D_x^{-1} m_x^2 e^{2m-2n-2u}, \quad w_5 = D_x^{-1} n_x^2 e^{2n-2m+2u}, \quad w_6 = D_x^{-1} w_3 e^{2m-2u},\\
&w_7 = D_x^{-1} w_3^2 e^{2m-2u}, \quad w_8 = D_x^{-1}(e^{2u-2m} + 2w_3 e^{2u-m+n} + w_3^2 e^{2n+2u}),\\
&w_9 = D_x^{-1}(w_2 e^{2u-m+n} + w_2 w_3 e^{2n+2u} + w_1 w_3 e^{2m-2u}).
\end{aligned}$$

The simple local systems that follow from (3.18) are presented below.

B.1. Adopting $c_i = 0$, $i > 5$ in (3.11) we obtain by differentiation the following hyperbolic system:

$$\begin{aligned}
u_{\tau x} &= c_1 e^{2(n+u)} - c_2 e^{2(m-u)} + c_4 m_x^2 e^{2(m-n-u)} + c_5 n_x^2 e^{2(n-m+u)},\\
m_{\tau x} &= c_1 e^{2(n+u)} + c_3 m_x e^{-m-n} + c_4 m_x^2 e^{2(m-n-u)}\\
&\quad - c_5(n_{xx} - 2m_x n_x + n_x^2 + 2u_x n_x) e^{2(n-m+u)},\\
n_{\tau x} &= c_2 e^{2(m-u)} - c_3 n_x e^{-m-n} - c_5 n_x^2 e^{2(n-m+u)}\\
&\quad + c_4(m_{xx} + m_x^2 - 2m_x n_x - 2m_x u_x) e^{2(m-n-u)}.
\end{aligned} \tag{3.12}$$

If one supposes additionally $c_4 = c_5 = 0$, then (3.11) implies $(u - m + n)_\tau = c_3 e^{-m-n}$. Therefore the substitution $p = u - m + n$, $q = u - m$, $r = u + n$ gives a simpler system:

$$p_\tau = c_3 e^{q-r}, \quad r_{\tau x} = c_1 e^{2r} + c_3(q_x - p_x) e^{q-r}, \quad q_{\tau x} = -c_2 e^{-2q} + c_3(p_x - r_x) e^{q-r}. \tag{3.13}$$

B.2. We can rewrite system (3.13) choosing $p = w_1, q = w_2, r = w_3$ as new unknown functions:

$$\begin{aligned}
p_{\tau x} &= 2(c_1 p + c_3 r) p_x + 2c_3 \sqrt{p_x q_x^{-1}}, \quad q_{\tau x} = 2(c_3 r - c_2 q) q_x,\\
r_{\tau x} &= (c_2 q - c_1 p - 2c_3 r) r_x + c_1 \sqrt{p_x q_x^{-1}}.
\end{aligned} \tag{3.14}$$

Then we consider system (3.11), with $c_2 = c_4 = c_5 = c_8 = 0$ and $c_7 \neq 0$. The shift $w_3 \to w_3 + a$ implies $w_6 \to w_6 + aw_2, w_7 \to w_7 + 2aw_6 + a^2 w_2$, hence we obtain $c_6 = 0$ by setting $a = -c_6/c_7$. The second shift $w_6 \to w_6 + c_3/c_7$ removes c_3 and we obtain the simple system:

$$u_\tau = c_1 w_1 + c_7 w_7, \quad m_\tau = c_1 w_1 + c_7(w_7 - w_3 w_6), \quad n_\tau = -c_7 w_6(w_3 + e^{-m-n}). \tag{3.15}$$

The further transformations can be performed in two manners.

B.3. The substitution $z = w_3 e^m, \bar{z} = e^n$ (see point A.4.) reduces system (3.15) to the following form:

$$u_{\tau x} = c_1 \bar{z}^2 e^{2u} + c_7 z^2 e^{-2u}, \quad z_{\tau x} = F^{-1} \bar{z} z_\tau z_x + c_1 F \bar{z} e^{2u}, \quad \bar{z}_{\tau x} = F^{-1} z \bar{z}_\tau \bar{z}_x - c_7 F z e^{-2u}, \tag{3.16}$$

where $F = z\bar{z} + 1$. Exact integrability of this system has been proved in [11].

B.4. Substitution $p = w_3, q = n + u, r = w_6$ reduces system (3.15) to another form:

$$\begin{aligned} p_{\tau x} &= c_1 e^q \sqrt{p r_x^{-1}} - p_x q_t + c_7 p_x r (p - e^{-q} \sqrt{p r_x^{-1}}), \\ q_{\tau x} &= c_1 e^{2q} - c_7 p_x r - c_7 (r e^{-q} \sqrt{p r_x^{-1}})_x, \quad r_{\tau x} = r_x (p^{-1} p_x - 2 c_7 r p). \end{aligned} \tag{3.17}$$

3.3 System "c"

$$\begin{aligned} u_t &= 2m_{xx}n_x - m_x n_{xx} + m_x n_x(4u_x + 3m_x - 3n_x), \\ m_t &= m_{xxx} + u_{xx}m_x - 3m_{xx}(n_x - m_x - u_x) + 3m_x n_x(n_x - 2u_x) \\ &\quad - m_x^2(5n_x - 3u_x) + m_x(m_x^2 + 2u_x^2), \\ n_t &= n_{xxx} - 2u_{xx}n_x + 3n_{xx}(n_x - m_x - u_x) + 3m_x n_x(m_x + 2u_x) \\ &\quad - n_x^2(5m_x + 3u_x) + n_x(n_x^2 + 2u_x^2). \end{aligned}$$

This system possesses the following 10-parametric nonlocal symmetry:

$$\begin{aligned} u_\tau &= c_1 w_1 - c_2 w_2 + c_4 w_4 - c_5 w_5 + 2c_6 w_7 + c_7 w_{10} \\ &\quad + c_8\big(w_8 + 2w_2(w_5 + w_7) - 2w_1 w_5 - 4w_6\big) + 2c_9(w_2 w_8 - 2w_1 w_6) \\ &\quad - 2c_{10}(w_1 w_5^2 - w_2 w_5 w_7 + 2w_6 w_7 - w_5 w_8), \\ m_\tau &= c_2 w_2 + c_4(n_x e^{2n-2m-2u} - w_4) - c_6 w_7 - c_7 w_{10} + c_9(w_9 - 2w_2 w_8) \\ &\quad + \big(c_3 + c_5 w_2 + c_6 w_1 + c_7 w_7 + c_8 w_2(w_1 + w_2) + c_9 w_1 w_2^2\big)(e^{-m-n} + w_3) \\ &\quad + c_{10}(2w_1 w_2 w_5 - 2w_2 w_8 + w_2^2 w_7 - 2w_1 w_6 + w_9)(e^{-m-n} + w_3) \\ &\quad + c_8(2w_6 - 2w_2 w_5 - w_2 w_7) + 2c_{10} w_7(w_6 - w_2 w_5), \\ n_\tau &= c_1 w_1 + c_3 w_3 + c_4 w_4 + c_5(w_2 w_3 - w_5) + c_6(w_7 + w_1 w_3) + c_7 w_3 w_7 \\ &\quad + c_8(w_2^2 w_3 - 2w_1 w_5 - 2w_6 + w_1 w_2 w_3 + w_8) + c_9(w_9 - 4w_1 w_6 + w_1 w_2^2 w_3) \\ &\quad - c_{10}\big(2(w_2 w_3 - w_5)(w_8 - w_1 w_5) + 2w_6(w_7 + w_1 w_3) - w_2^2 w_3 w_7 - w_3 w_9\big). \end{aligned} \tag{3.18}$$

It is denoted here

$$\begin{aligned} w_1 &= D_x^{-1} e^{2(m+u)}, \quad w_2 = D_x^{-1} e^{n-u}, \quad w_3 = D_x^{-1} n_x e^{-m-n}, \\ w_4 &= D_x^{-1} n_x^2 e^{2(n-m-u)}, \quad w_5 = D_x^{-1} w_3 e^{n-u}, \\ w_6 &= D_x^{-1} w_2 w_3 e^{n-u}, \quad w_7 = D_x^{-1}\big(e^{m-n+2u} + e^{2(m+u)} w_3\big), \\ w_8 &= D_x^{-1}\big(w_2 e^{m-n+2u} + w_2 w_3 e^{2(m+u)} + w_1 w_3 e^{n-u}\big), \\ w_9 &= D_x^{-1} w_2\big(w_2 e^{m-n+2u} + w_2 w_3 e^{2(m+u)} + 2w_1 w_3 e^{n-u}\big), \\ w_{10} &= D_x^{-1}\big(e^{2(u-n)} + 2w_3 e^{m-n+2u} + w_3^2 e^{2(m+u)}\big). \end{aligned}$$

The simple local systems that follow from (3.18) are presented below.

C.1. Adopting $c_i = 0, i > 4$ in (3.18) we obtain by differentiation the following hyperbolic system:

$$\begin{aligned} u_{\tau x} &= c_1 e^{2(m+u)} - c_2 e^{n-u} + c_4 n_x^2 e^{2(n-m-u)}, \\ m_{\tau x} &= c_2 e^{n-u} - c_3 m_x e^{-m-n} + c_4 e^{2(n-m-u)}(n_x^2 - 2m_x n_x - 2n_x u_x + n_{xx}), \\ n_{\tau x} &= c_1 e^{2(m+u)} + c_3 n_x e^{-m-n} + c_4 n_x^2 e^{2(n-m-u)}. \end{aligned} \tag{3.19}$$

If additionally $c_4 = 0$ in (3.18), then $(m - n + u)_\tau = c_3 e^{-m-n}$. Therefore for the new unknown functions $p = m - n + u, q = m + u, r = n - u$ the system takes a simpler form:

$$p_\tau = c_3 e^{-q-r}, \quad q_{\tau x} = c_1 e^{2q} - c_3(p_x + r_x)e^{-q-r}, \quad r_{\tau x} = c_2 e^r + c_3(q_x - p_x)e^{-q-r}. \tag{3.20}$$

C.2. If we set $c_4 = 0, c_5 \neq 0$ and $c_i = 0$ for $i > 5$ in (3.18), then the shifts $w_3 \to w_3 - c_2/c_5, w_5 \to w_5 - c_2/c_5 w_2$ and $w_2 \to w_2 - c_3/c_5$ remove c_2 and c_3, thus system (3.18) takes the simple form:

$$u_\tau = c_1 w_1 - c_5 w_5, \quad m_\tau = c_5 w_2(e^{-m-n} + w_3), \quad n_\tau = c_1 w_1 + c_5(w_2 w_3 - w_5). \tag{3.21}$$

The substitution $z = w_3 e^n, \bar{z} = e^m$ (see point A.4.) reduces system (3.21) to the following form:

$$u_{\tau x} = c_1 \bar{z}^2 e^{2u} - c_5 z e^{-u}, \quad z_{\tau x} = c_1 F \bar{z} e^{2u} + F^{-1} \bar{z} z_t z_x, \quad \bar{z}_{\tau x} = c_5 F e^{-u} + F^{-1} z \bar{z}_t \bar{z}_x, \tag{3.22}$$

where $F = z\bar{z} + 1$. Exact integrability of this system has been proved in [11].

C.3. For the new unknown functions $p = w_2, q = w_3, r = m + u$ system (3.21) has another form:

$$\begin{aligned} &p_{\tau x} = c_5 p q p_x, \quad r_{\tau x} = c_1 e^{2r} + c_5 p q_x + c_5(e^{-r} p p_x^{-1})_x, \\ &q_{\tau x} = (c_1 e^r + c_5 e^{-r} p q_x) p_x^{-1} - q_x r_\tau - c_5 p q q_x. \end{aligned} \tag{3.23}$$

3.4 System "d"

$$\begin{aligned} u_t &= -\frac{1}{2} u_{xxx} + 3n_x^2 m_x + \frac{3}{2} m_x n_{xx} - \frac{3}{2} n_x m_{xx} - 3m_x^2 n_x + \frac{1}{4} u_x^3 + \frac{9}{2} u_x n_x m_x, \\ m_t &= m_{xxx} - 3m_{xx}(n_x + u_x - m_x) - \frac{3}{2} m_x u_{xx} + \frac{9}{4} u_x^2 m_x - \frac{9}{2} m_x^2 n_x - 3u_x m_x^2 \\ &\quad + 6u_x n_x m_x + 3n_x^2 m_x + m_x^3, \\ n_t &= n_{xxx} + 3n_{xx}(n_x + u_x - m_x) + \frac{3}{2} u_{xx} n_x + \frac{9}{4} u_x^2 n_x - \frac{9}{2} n_x^2 m_x + 3u_x n_x^2 \\ &\quad - 6u_x n_x m_x + 3m_x^2 n_x + n_x^3. \end{aligned}$$

This system possesses the following 11-parametric nonlocal symmetry:

$$\begin{aligned} u_\tau &= c_1 w_1 + c_2 w_2 - c_4 w_4 + 2c_5(w_1 w_4 - 2w_5) + c_6 w_6 + 2c_7(2w_7 - w_2 w_6) \\ &\quad + c_8(w_1 w_6 - w_2 w_4) + 2c_9(w_1 w_8 - w_2 w_{11}) + c_{10}(2w_5 w_2 - 2w_1 w_9 + w_{11}) \\ &\quad + c_{11}(2w_1 w_7 + w_8 - 2w_2 w_9), \\ m_\tau &= c_2 w_2 + c_3 w_3 + c_4(w_1 w_3 - w_4) + c_5(w_1^2 w_3 - 2w_5) + c_6 w_2 w_3 \\ &\quad + c_7(w_2^2 w_3 - 2w_2 w_6 + 2w_7) + c_9(w_1^2 w_2^2 w_3 + 2w_{10} - 2w_2 w_{11}) \\ &\quad + c_8 w_2(w_1 w_3 - w_4) + c_{10} w_2(2w_5 - w_1^2 w_3) + c_{11}(w_1 w_2^2 w_3 + w_8 - 2w_2 w_9), \\ n_\tau &= \big(c_3 + c_4 w_1 + c_5 w_1^2 + c_6 w_2 + c_7 w_2^2 + c_8 w_1 w_2 + c_9 w_1^2 w_2^2 - c_{10} w_1^2 w_2 \\ &\qquad + c_{11} w_1 w_2^2\big)(w_3 + e^{-m-n}) - c_1 w_1 + 2c_5(w_5 - w_1 w_4) - c_6 w_6 \\ &\quad - 2c_7 w_7 - c_8 w_1 w_6 + 2c_9(w_{10} - w_1 w_8) + c_{10}(2w_1 w_9 - w_{11}) - 2c_{11} w_1 w_7. \end{aligned} \tag{3.24}$$

It is denoted here

$$\begin{aligned}
&w_1 = D_x^{-1} e^{m-u}, \quad w_2 = D_x^{-1} e^{n+u}, \quad w_3 = D_x^{-1} m_x e^{-m-n}, \quad w_4 = D_x^{-1} w_3 e^{m-u},\\
&w_5 = D_x^{-1} w_1 w_3 e^{m-u}, \quad w_6 = D_x^{-1}\big(w_3 e^{n+u} + e^{u-m}\big), \quad w_7 = D_x^{-1} w_2\big(w_3 e^{n+u} + e^{u-m}\big),\\
&w_8 = D_x^{-1} w_2\big(2w_1 e^{u-m} + 2w_1 w_3 e^{n+u} + w_2 w_3 e^{m-u}\big),\\
&w_9 = D_x^{-1}\big(w_1 e^{u-m} + w_1 w_3 e^{n+u} + w_2 w_3 e^{m-u}\big),\\
&w_{10} = D_x^{-1} w_1 w_2\big(w_1 e^{u-m} + w_1 w_3 e^{n+u} + w_2 w_3 e^{m-u}\big),\\
&w_{11} = D_x^{-1} w_1\big(w_1 e^{u-m} + w_1 w_3 e^{n+u} + 2w_2 w_3 e^{m-u}\big)
\end{aligned}$$

The simple local systems that follow from (3.24) are presented below.

D.1. Adopting $c_i = 0, i > 3$ in (3.24) we obtain by differentiation the following hyperbolic system:

$$p_\tau = c_3 e^{q-r}, \quad q_{\tau x} = c_1 e^{-q} + c_3(p_x - r_x) e^{q-r}, \quad r_{\tau x} = c_2 e^r + c_3(q_x - p_x) e^{q-r}, \tag{3.25}$$

where $p = n - m + u, q = u - m, r = u + n$.

D.2. If $c_i = 0, i > 4$ and $c_4 \neq 0$ in (3.24), then the shifts $w_1 \to w_1 + c_3/c_4, w_3 \to w_3 - c_1/c_4, w_4 \to w_4 - c_1/c_4 w_1$ remove c_1 and c_3, and the system reads:

$$u_\tau = c_2 w_2 - c_4 w_4, \quad m_\tau = c_2 w_2 + c_4(w_1 w_3 - w_4), \quad n_\tau = c_4 w_1(w_3 + e^{-m-n}). \tag{3.26}$$

The substitution $z = w_3 e^m, \bar z = e^n$ (see point A.4.) reduces this system to the following form

$$u_{\tau x} = c_2 \bar z e^u - c_4 z e^{-u}, \quad z_{\tau x} = c_2 F e^u + F^{-1} \bar z z_t z_x, \quad \bar z_{\tau x} = c_4 F e^{-u} + F^{-1} z \bar z_t \bar z_x, \tag{3.27}$$

where $F = z\bar z + 1$. Exact integrability of this system has been proved in [10].

D.3. The substitution $p = w_1, q = w_3, r = m + n$ reduces system (3.26) to the following form:

$$\begin{aligned}
&p_{\tau x} = c_4 p q p_x, \quad q_{\tau x} = c_2 p_x^{-1} - q_x r_\tau + c_4 e^{-r} p q_x,\\
&r_{\tau x} = c_2 e^r p_x^{-1} + 2c_4 p q_x + c_4 q p_x + c_4 (p e^{-r})_x.
\end{aligned} \tag{3.28}$$

D.4. If $c_5 \neq 0$ and $c_i = 0$ for $i > 5$ in (3.24), then the shift $w_1 \to w_1 - c_4/(2c_5)$ removes c_4. Adopting also $c_2 = 0$ for simplicity, we obtain:

$$\begin{aligned}
&u_\tau = c_1 w_1 + 2c_5(w_1 w_4 - 2w_5), \quad m_\tau = c_3 w_3 + c_5(w_1^2 w_3 - 2w_5),\\
&n_\tau = -c_1 w_1 + (c_3 + c_5 w_1^2)(w_3 e^{-m-n}) + 2c_5(w_5 - w_1 w_4).
\end{aligned}$$

Introducing the new unknown functions $p = w_1, q = w_3, r = m + n$ one can reduce this system to the following form:

$$\begin{aligned}
&p_{\tau x} = p_x\big(r_\tau - c_3(q + e^{-r}) - c_5 p^2 e^{-r} + 2c_5 w\big), \quad w_x = p q p_x, \quad (w = w_5),\\
&q_{\tau x} = q_x(c_3 + c_5 p^2) e^{-r} - q_x r_\tau,\\
&r_{\tau x} = p^{-1} p_x r_\tau + 2c_5 p^2 q_x + c_5 p (p e^{-r})_x + c_3(2q_x - r_x e^{-r}) - c_3 p^{-1} p_x (2q + e^{-r}).
\end{aligned} \tag{3.29}$$

Remark. If $c_6 \neq 0$ and the other $c_i = 0$ for $i \geqslant 4$ in (3.24), then there exist shifts along w_3 and w_2 that reduce the system to the following form:

$$u_\tau = c_1 w_1 + c_6 w_6, \quad m_\tau = c_6 w_2 w_3, \quad n_\tau = -c_1 w_1 + c_6 w_2 (w_3 + e^{-m-n}) - c_6 w_6.$$

This system is reducible to the forms (3.27) or (3.28) by the substitutions $z = e^m, \bar z = -w_3 e^n - e^{-m}$ or $p = -w_2, q = -w_3 - e^{-m-n}, r = m + n$ correspondingly.

3.5 System "e"

$$u_t = u_{xxx} - 2u_x^3 - 6u_x n_x m_x,$$

$$m_t = m_{xxx} + 3m_{xx}(m_x - n_x) - 6m_x^2 n_x - 6u_x^2 m_x + 3n_x^2 m_x + m_x^3,$$

$$n_t = n_{xxx} - 3n_{xx}(m_x - n_x) - 6n_x^2 m_x - 6u_x^2 n_x + 3m_x^2 n_x + n_x^3.$$

This system possesses the following 12-parametric nonlocal symmetry:

$$\begin{aligned}
u_\tau &= c_1 w_1 + c_2 w_2 + c_4 m_x e^{m-n} + c_5 n_x e^{-m+n} + c_6 w_6 + c_7 w_7 + c_8 w_8 + c_9 w_9 \\
&\quad + c_{10}(w_1 w_7 - w_2 w_6) + c_{11}(w_1 w_9 - w_2 w_8) + c_{12}(w_6 w_7 - 4w_2 w_8), \\
m_\tau &= c_1 w_1 - c_2 w_2 + c_3 w_3 + 2c_4 w_4 + 2c_5(u_x e^{-m+n} - w_5) + 2c_6 w_1 w_3 \\
&\quad - 2c_7 w_2 w_3 + c_8(w_3 w_6 - w_8) + c_9(w_9 - w_3 w_7) \\
&\quad + c_{10}(w_2 w_6 - w_{10} + w_1 w_7 - 2w_3 w_1 w_2) + c_{11}(w_2 w_8 + w_1 w_9 - w_3 w_{10}) \\
&\quad + 2c_{12}(2w_2 w_8 - w_3 w_{10} - w_2 w_3 w_6 + w_1 w_3 w_7), \\
n_\tau &= c_1 w_1 - c_2 w_2 + 2c_4(u_x e^{m-n} - w_4) + 2c_5 w_5 - c_8 w_8 + w_9 c_9 \\
&\quad + \big(w_3 + e^{-m-n}\big)\big(c_3 + 2c_6 w_1 - 2c_7 w_2 + c_8 w_6 - c_9 w_7 \\
&\qquad\qquad - 2c_{10} w_1 w_2 - c_{11} w_{10} - 2c_{12}(w_{10} + w_2 w_6 - w_1 w_7)\big) \\
&\quad + c_{10}(w_2 w_6 - w_{10} + w_1 w_7) + c_{11}(w_2 w_8 + w_1 w_9) + 4c_{12} w_2 w_8.
\end{aligned} \tag{3.30}$$

It is denoted here

$$\begin{aligned}
&w_1 = D_x^{-1} e^{m+n+2u}, \quad w_2 = D_x^{-1} e^{m+n-2u}, \quad w_3 = D_x^{-1} m_x e^{-m-n}, \\
&w_4 = D_x^{-1} u_x m_x e^{m-n}, \quad w_5 = D_x^{-1} u_x n_x e^{n-m}, \\
&w_6 = D_x^{-1}\big(2w_3 e^{m+n+2u} + e^{2u}\big), \quad w_7 = D_x^{-1}\big(2w_3 e^{m+n-2u} + e^{-2u}\big), \\
&w_8 = D_x^{-1} w_3\big(w_3 e^{m+n+2u} + e^{2u}\big), \quad w_9 = D_x^{-1} w_3\big(w_3 e^{m+n-2u} + e^{-2u}\big), \\
&w_{10} = D_x^{-1}\big(w_2 e^{2u} + 2w_2 w_3 e^{m+n+2u} + w_1 e^{-2u} + 2w_1 w_3 e^{m+n-2u}\big).
\end{aligned}$$

The simple local systems that follow from (3.30) are presented below.

E.1. Adopting $c_i = 0, i > 5$ in (3.24) we obtain by differentiation the following hyperbolic system:

$$\begin{aligned}
u_{\tau x} &= c_1 e^{m+n+2u} + c_2 e^{m+n-2u} \\
&\quad + c_4 e^{m-n}(m_{xx} + m_x^2 - m_x n_x) + c_5 e^{n-m}(n_{xx} - m_x n_x + n_x^2), \\
m_{\tau x} &= c_1 e^{m+n+2u} - c_2 e^{m+n-2u} + c_3 m_x e^{-m-n} \\
&\quad + 2c_4 u_x m_x e^{m-n} + 2c_5 e^{n-m}(u_{xx} - u_x m_x), \\
n_{\tau x} &= c_1 e^{m+n+2u} - c_2 e^{m+n-2u} - c_3 n_x e^{-m-n} \\
&\quad + 2c_4 e^{m-n}(u_{xx} - u_x n_x) + 2c_5 u_x n_x e^{-m+n}.
\end{aligned} \tag{3.31}$$

E.2. If additionally $c_4 = c_5 = 0$ in (3.31), then $(n-m)_\tau = c_3 e^{-m-n}$. Therefore for the new unknown functions $p = n - m, q = u - \frac{1}{2}(m+n), r = u + \frac{1}{2}(m+n)$ the system takes the simpler form:

$$p_\tau = c_3 e^{q-r}, \quad q_{\tau x} = 2c_2 e^{-2q} + \frac{c_3}{2}\, p_x e^{q-r}, \quad r_{\tau x} = 2c_1 e^{2r} - \frac{c_3}{2}\, p_x e^{q-r}. \tag{3.32}$$

E.3. If $(c_6, c_7) \neq 0$ and the other $c_i = 0$ in (3.30), then we obtain:

$$u_\tau = c_6 w_6 + c_7 w_7, \quad m_\tau = 2w_3(c_6 w_1 - c_7 w_2), \quad n_\tau = 2(c_6 w_1 - c_7 w_2)\big(w_3 + e^{-m-n}\big).$$

The substitution $z = w_3 e^m, \bar{z} = e^n$ (see point A.4.) reduces this system to the following form:

$$\begin{aligned} u_{\tau x} &= (1 + 2z\bar{z})(c_6 e^{2u} + c_7 e^{-2u}), \\ z_{\tau x} &= F^{-1}\bar{z} z_\tau z_x + 2zF(c_6 e^{2u} - c_7 e^{-2u}), \\ \bar{z}_{\tau x} &= F^{-1} z \bar{z}_\tau \bar{z}_x + 2\bar{z} F(c_6 e^{2u} - c_7 e^{-2u}), \end{aligned} \tag{3.33}$$

where $F = z\bar{z} + 1$. Exact integrability of this system has been proved in [11].

3.6 System "f"

$$u_t = -\frac{1}{2} u_{xxx} + \frac{3}{2} (m_x n_x)_x - 3u_x n_x m_x + {u_x}^3,$$

$$m_t = m_{xxx} + 3m_{xx}(m_x - n_x) - 3u_{xx} m_x - 3{u_x}^2 m_x - 3m_x^2 n_x + 3n_x^2 m_x + m_x^3,$$

$$n_t = n_{xxx} - 3n_{xx}(m_x - n_x) - 3u_{xx} n_x - 3{u_x}^2 n_x - 3n_x^2 m_x + 3m_x^2 n_x + n_x^3.$$

This system possesses the following 11-parametric nonlocal symmetry:

$$\begin{aligned} u_\tau &= c_1 w_1 + c_2 w_2 + c_4 w_1 w_2 + c_5 w_5 + c_6(2w_6 + w_1^2 w_2 - 2w_1 w_4) \\ &\quad + c_7 w_7 + c_8 w_1 w_7 + c_9(w_9 - 2w_1 w_8) + c_{10} w_1 w_5 + c_{11}(w_{11} - 2w_1 w_{10}), \\ m_\tau &= c_2 w_2 + c_3 w_3 + c_4(w_1 w_2 - w_4) + 2c_5 w_2 w_3 + c_6(w_1^2 w_2 - 2w_6) \\ &\quad + c_7(w_3 w_5 - w_7) + c_8(w_3 w_8 - w_{10}) - 2c_9 w_3(w_1^2 w_2 - 2w_6) \\ &\quad + 2c_{10} w_3(w_1 w_2 - w_4) + c_{11}(w_{11} - w_3 w_9), \\ n_\tau &= c_2 w_2 + c_4(w_1 w_2 - w_4) + c_6(w_1^2 w_2 - 2w_6) - c_7 w_7 - c_8 w_{10} \\ &\quad + c_{11} w_{11} + \big(w_3 + e^{-m-n}\big)\big(c_3 + 2c_5 w_2 + c_7 w_5 + c_8 w_8 - c_{11} w_9 \\ &\qquad + 2c_9(2w_6 - w_1^2 w_2) + 2c_{10}(w_1 w_2 - w_4)\big). \end{aligned} \tag{3.34}$$

It is denoted here

$$\begin{aligned} & w_1 = D_x^{-1} e^{-2u}, \quad w_2 = D_x^{-1} e^{m+n+2u}, \quad w_3 = D_x^{-1} m_x e^{-m-n}, \quad w_4 = D_x^{-1} w_2 e^{-2u}, \\ & w_5 = D_x^{-1} e^{2u}\big(1 + 2w_3 e^{m+n}\big), \quad w_6 = D_x^{-1} w_1 w_2 e^{-2u}, \quad w_7 = D_x^{-1} w_3 e^{2u}\big(1 + w_3 e^{m+n}\big), \\ & w_8 = D_x^{-1} w_1 e^{2u}\big(1 + 2w_3 e^{m+n}\big), \quad w_9 = D_x^{-1} w_1^2 e^{2u}\big(1 + 2w_3 e^{m+n}\big), \\ & w_{10} = D_x^{-1} w_1 w_3 e^{2u}\big(1 + w_3 e^{m+n}\big), \quad w_{11} = D_x^{-1} w_1^2 w_3 e^{2u}\big(1 + w_3 e^{m+n}\big). \end{aligned}$$

The simple local systems that follow from (3.34) are presented below.

F.1. Adopting $c_i = 0, i > 3$ in (3.34) we obtain by differentiation the following hyperbolic system:

$$p_\tau = c_3 e^{-q}, \quad q_{\tau x} = 2c_2 e^{q+2u} - c_3 p_x e^{-q}, \quad u_{\tau x} = c_1 e^{-2u} + c_2 e^{q+2u}, \tag{3.35}$$

where $p = n - m,\ q = n + m$.

F.2. In the case $c_i = 0, i > 4$ we obtain the third order system:

$$\begin{aligned} & p_\tau = c_3 e^{-q}, \quad \big(e^{-q-2u} q_{\tau x}\big)_x = 2c_4 e^{-2u} - c_3\big(p_x e^{-2u-q}\big)_x, \\ & \big[e^{2u}(2u_{\tau x} - q_{\tau x})\big]_x = c_3\big(p_x e^{2u-q}\big)_x + 2c_4 e^{2u+q}. \end{aligned} \tag{3.36}$$

Here p and q are the same as in **F.1**.

F.3. If one introduces the new unknown functions $p = n - m, y = w_1, z = w_2$, then system (3.36) takes a simpler form:

$$p_\tau = c_3(y_x z_x)^{-1}, \quad y_{\tau x} = -2y_x(c_1 y + c_2 z + c_4 yz),$$
$$\left(z_x^{-1} z_{\tau x}\right)_x = 2c_1 y_x + 4c_2 z_x + 4c_4 y z_x + 2c_4 z y_x - p_x p_\tau. \tag{3.37}$$

F.4. If $c_4 = 0$, $c_i = 0$ for $i > 5$, and $c_5 \neq 0$ in (3.34), then two shifts along w_2 and w_3 remove c_2 and c_3 and we obtain the following system:

$$u_\tau = c_1 w_1 + c_5 w_5, \quad m_\tau = 2c_5 w_2 w_3, \quad n_\tau = 2c_5 w_2(w_3 + e^{-m-n}).$$

The substitution $z = w_3 e^m, \bar{z} = e^n$ (see point A.4.) reduces this system to the following form

$$u_{\tau x} = c_1 e^{-2u} + c_5(2z\bar{z} + 1)e^{2u},$$
$$z_{\tau x} = F^{-1}\bar{z} z_\tau z_x + 2c_5 z F e^{2u}, \tag{3.38}$$
$$\bar{z}_{\tau x} = F^{-1} z \bar{z}_\tau \bar{z}_x + 2c_5 \bar{z} F e^{2u},$$

where $F = z\bar{z} + 1$. Exact integrability of this system has been proved in [11].

3.7 System "g"

$$u_t = m_x n_{xx} + n_x m_{xx} - 4u_x n_x m_x,$$
$$m_t = m_{xxx} + 3m_{xx}(m_x - n_x) - 2m_x u_{xx} - 4{u_x}^2 m_x - 4m_x^2 n_x + 3n_x^2 m_x + m_x^3,$$
$$n_t = n_{xxx} - 3n_{xx}(m_x - n_x) - 2n_x u_{xx} - 4{u_x}^2 n_x - 4n_x^2 m_x + 3m_x^2 n_x + n_x^3.$$

This system possesses the following 10-parametric nonlocal symmetry:

$$\begin{aligned}
u_\tau &= c_1 w_1 + c_2 w_2 + c_4 w_4 + c_5(2w_2 w_4 - w_5) + c_6 w_6 + c_7(2w_2 w_6 - w_7)\\
&\quad + c_8(2w_1 w_2 - w_8) + c_9 w_1(w_1 w_2 - w_8)\\
&\quad + c_{10}(4w_1 w_2 w_6 - w_2 w_4^2 - 2w_1 w_7 + w_4 w_5 - 2w_6 w_8),\\
m_\tau &= c_1 w_1 + c_3 w_3 + 2c_4 w_1 w_3 + 2c_5 w_3 w_8 + c_6(w_3 w_4 - w_6) + c_7(w_3 w_5 - w_7)\\
&\quad + c_8 w_8 + c_9(2w_9 - w_1 w_8) + 2c_{10}(w_1 w_3 w_5 - w_3 w_4 w_8 + w_6 w_8 - w_1 w_7),\\
n_\tau &= c_1 w_1 - c_6 w_6 - c_7 w_7 + c_8 w_8 + c_9(2w_9 - w_1 w_8) + 2c_{10}(w_6 w_8 - w_1 w_7)\\
&\quad + \big(c_3 + 2c_4 w_1 + 2c_5 w_8 + c_6 w_4 + c_7 w_5 + 2c_{10}(w_1 w_5 - w_8 w_4)\big)\big(w_3 + e^{-m-n}\big).
\end{aligned} \tag{3.39}$$

It is denoted here

$$w_1 = D_x^{-1} e^{m+n+2u}, \quad w_2 = D_x^{-1} e^{-4u}, \quad w_4 = D_x^{-1} e^{2u}\big(1 + 2w_3 e^{m+n}\big),$$
$$w_3 = D_x^{-1} m_x e^{-m-n}, \quad w_5 = D_x^{-1} w_2 e^{2u}\big(1 + 2w_3 e^{m+n}\big), \quad w_6 = D_x^{-1} w_3 e^{2u}\big(1 + w_3 e^{m+n}\big),$$
$$w_7 = D_x^{-1} w_2 w_3 e^{2u}\big(1 + w_3 e^{m+n}\big), \quad w_8 = D_x^{-1} w_2 e^{m+n+2u}, \quad w_9 = D_x^{-1} w_1 w_2 e^{m+n+2u}.$$

The simple local systems that follow from (3.34) are presented below.

G.1. Adopting $c_i = 0, i > 3$ in (3.39) we obtain by differentiation the following hyperbolic system:

$$p_\tau = c_3 e^{-q}, \quad q_{\tau x} = 2c_1 e^{q+2u} - c_3 p_x e^{-q}, \quad u_{\tau x} = c_1 e^{q+2u} + c_2 e^{-4u}, \tag{3.40}$$

where $p = n - m, \; q = n + m$.

G.2. If $c_i = 0, i > 4$ and $c_4 \neq 0$ in (3.39), then one can remove c_1 and c_3 using the shifts along w_1 and w_3. This gives the following system:

$$u_\tau = c_2 w_2 + c_4 w_4, \quad m_\tau = 2c_4 w_1 w_3, \quad n_\tau = 2c_4 w_1 (w_3 + e^{-m-n}). \tag{3.41}$$

The substitution $z = w_3 e^m, \bar{z} = e^n$ (see point A.4.) reduces this system to the following form:

$$\begin{aligned} u_{\tau x} &= c_2 e^{-4u} + c_4(2z\bar{z} + 1)e^{2u}, \\ z_{\tau x} &= F^{-1}\bar{z} z_\tau z_x + 2c_4 z F e^{2u}, \\ \bar{z}_{\tau x} &= F^{-1} z \bar{z}_\tau \bar{z}_x + 2c_4 \bar{z} F e^{2u}, \end{aligned} \tag{3.42}$$

where $F = z\bar{z} + 1$. Exact integrability of this system has been proved in [11].

G.3. If $(c_1, c_2, c_3, c_8) \neq 0$ and the other $c_i = 0$ in (3.39), then introducing the new unknown functions $p = w_1, q = w_2, r = n - m$ we can reduce the system to the following form:

$$\begin{aligned} &r_\tau = c_3 p_x \sqrt{q_x}, \quad (p_x^{-1} p_{\tau x})_x = 4c_1 p_x + 2c_2 q_x + 4c_8 (pq)_x - \frac{c_3 r_x}{p_x \sqrt{q_x}}, \\ &(q_x^{-1} q_{\tau x})_x = -4c_1 p_x - 4c_2 q_x - 8c_8 p q_x - 4c_8 q p_x. \end{aligned} \tag{3.43}$$

We prove in the next section that zero curvature representations exist for all presented above nonlocal systems.

4 Zero curvature representations

Here we present matricies U and V that form the zero curvature representations (1.1) for nonlocal and selected local systems from the previous section. The matrices U from [9, 10, 11] are written in a slightly different form for completeness. A spectral parameter is denoted as λ everywhere.

A. Elements of the hierarchy of system "a" possess the zero curvature representations with the following common matrix U:

$$U = \begin{pmatrix} q_x & 0 & m_x \\ \lambda & -q_x - r_x & \lambda \\ 0 & 1 & r_x \end{pmatrix}. \tag{4.1}$$

where

$$q = \frac{1}{3}(2u + m + 2n), \quad r = \frac{1}{3}(2u - 2m - n).$$

Performing the gauge transformation $U_0 = S^{-1}(US - S_x)$, where
$S = \operatorname{diag}\{e^q, e^{-q-r}, e^r\}$, we obtain a matrix which is more appropriate for system (3.1):

$$U_0 = \begin{pmatrix} 0 & 0 & m_x e^{-m-n} \\ \lambda e^{n+2u} & 0 & \lambda e^{2u-m} \\ 0 & e^{m-2u} & 0 \end{pmatrix}.$$

Then setting $U = U_0, V = V(w_i)$ in (1.1), we solve that equation and find the components of V:

$$\begin{aligned} V_{11} = {} & \frac{1}{3}(c_6 - 2c_8)\lambda^{-1} - c_1 w_1 - c_3 w_3 + (w_2 w_3 - w_4)(c_6 w_1 - c_4) \\ & - c_5 w_1 w_3 + c_7(w_1 w_2 - w_5) + c_8(w_1 w_4 - w_3 w_5), \\ V_{22} = {} & \frac{1}{3}(c_8 - 2c_6)\lambda^{-1} + c_1 w_1 + c_2 w_2 - c_4 w_4 + c_5 w_6 \\ & + c_6(w_1 w_4 - w_2 w_6) - w_1(c_7 w_2 + c_8 w_4), \end{aligned}$$

$$V_{33} = \frac{1}{3}(c_8 + c_6)\lambda^{-1} - c_2 w_2 + w_3(c_3 + c_4 w_2) + c_5(-w_6 + w_3 w_1)$$
$$+ w_2 c_6(w_6 - w_1 w_3) + w_5(c_7 + c_8 w_3),$$
$$V_{12} = \lambda^{-1}\big(c_1 + c_5 w_3 + c_6(w_4 - w_2 w_3) - c_7 w_2 - c_8 w_4\big), \quad V_{13} = (c_7 + c_8 w_3)\lambda^{-1},$$
$$V_{21} = c_6 w_1 - c_4 - c_8 w_1, \quad V_{23} = c_4 w_3 - c_2 + c_6(w_6 - w_3 w_1) + w_1(c_7 + c_8 w_3),$$
$$V_{31} = c_6 w_1 w_2 - c_3 - c_4 w_2 - c_5 w_1 - c_8 w_5, \quad V_{32} = (c_5 - c_6 w_2)\lambda^{-1}.$$

So, system (3.1) is integrable and any of its corollaries are integrable too. In particular, system (3.3) possesses the zero curvature representation with matrix (4.1), where $m_x = q_x - 2p_x$, and the following matrix V:

$$V = \begin{pmatrix} 2ae^{r-q} & c_1\lambda^{-1}e^{2q+r} & 0 \\ 0 & -ae^{r-q} & -c_2 e^{-q-2r} \\ -3ae^{r-q} & 0 & -ae^{r-q} \end{pmatrix}.$$

B. Elements of the hierarchy of system "b" possess the zero curvature representations with the following common matrix U:

$$U = \begin{pmatrix} m_x - u_x & 1 & -m_x & -1 \\ 0 & u_x + n_x & 0 & -m_x \\ 0 & 1 & -u_x - n_x & -1 \\ \lambda & 0 & 0 & u_x - m_x \end{pmatrix}. \tag{4.2}$$

Using it we find the zero curvature representation for (3.11) with the following matrices

$$U_0 = \begin{pmatrix} 0 & e^{2u-m+n} & -m_x e^{-m-n} & -e^{2u-2m} \\ 0 & 0 & 0 & -m_x e^{-m-n} \\ 0 & e^{2u+2n} & 0 & -e^{2u-m+n} \\ \lambda e^{2m-2u} & 0 & 0 & 0 \end{pmatrix},$$

which is gauge equivalent to (4.2), and the following matrix $V = (V_{ij})$:

$$V_{11} = c_6(w_6 + w_2 w_3) - c_2 w_2 - c_3 w_3 + c_7 w_3 w_6 + c_8\big(1/(2\lambda) - w_2 w_8 + w_3 w_9\big),$$
$$V_{22} = c_6(w_2 w_3 - w_6) - c_1 w_1 - c_3 w_3 + c_7(w_3 w_6 - w_7) + c_8\big(1/(2\lambda) - w_1 w_7 + w_3 w_9\big),$$
$$V_{33} = -V_{22}, \quad V_{44} = -V_{11}, \quad V_{12} = -\lambda^{-1}(c_6 + c_7 w_3 + c_8 w_1 w_3),$$
$$V_{13} = \lambda^{-1} c_8 w_3 + c_5 n_x e^{n-3m+2u}, \quad V_{14} = \lambda^{-1}(c_2 - 2c_6 w_3 - c_7 w_3^2) + \lambda^{-1} c_8(w_8 - w_1 w_3^2),$$
$$V_{21} = -c_8 w_2 w_3 - \lambda c_5 e^{n-m}, \quad V_{23} = c_1 + \lambda c_5 e^{-2m} + c_8(w_7 - w_2 w_3^2),$$
$$V_{24} = V_{13}, \quad V_{31} = c_3 - c_6 w_2 - c_7 w_6 - c_8 w_9, \quad V_{32} = c_4 e^{2m} + \lambda^{-1}(c_7 + c_8 w_1),$$
$$V_{34} = \lambda^{-1}(c_6 + c_7 w_3 + c_8 w_1 w_3), \quad V_{41} = c_8 w_2 - \lambda c_5 e^{2n}, \quad V_{42} = V_{31}, \quad V_{43} = -V_{21}.$$

Hence system (3.11) is integrable and any of its corollaries are integrable too. In particular, system (3.13) possesses the zero curvature representation with matrix (4.2), where $u = r + q - p, m = r - p, n = p - q$, and the following matrix V:

$$V = \begin{pmatrix} 0 & 0 & 0 & c_2\lambda^{-1}e^{-2q} \\ 0 & c_3 e^{q-r} & c_1 e^{2r} & 0 \\ c_3 e^{q-r} & 0 & -c_3 e^{q-r} & 0 \\ 0 & c_3 e^{q-r} & 0 & 0 \end{pmatrix}.$$

C. Elements of the hierarchy of system "c" possess the zero curvature representations with the following common matrix U:

$$U = \begin{pmatrix} -q_x & 0 & 0 & \lambda & 0 \\ 0 & -r_x & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & -m_x & -1 & q_x & 0 \\ -m_x & 0 & 0 & 0 & r_x \end{pmatrix}. \tag{4.3}$$

where $q = m + u, r = n - u$. Using it we find the zero curvature representation for (3.18) with the matrix U_0

$$U_0 = \begin{pmatrix} 0 & 0 & 0 & \lambda e^{2(m+u)} & 0 \\ 0 & 0 & -e^{n-u} & 0 & 0 \\ e^{-m-u} & 0 & 0 & 0 & -e^{n-u} \\ 0 & -m_x e^{-m-n} & -e^{-m-u} & 0 & 0 \\ -m_x e^{-m-n} & 0 & 0 & 0 & 0 \end{pmatrix},$$

which is gauge equivalent to (4.3), and the following matrix V:

$$\begin{aligned}
&V_{12} = -c_4\lambda e^{n-m} - 2(c_8 + c_9 w_1 + c_{10} w_7), \quad V_{14} = c_4 \lambda e^{2n}, \\
&V_{13} = -c_5 - c_8(w_1 + 2w_2) - 2c_9 w_1 w_2 - 2c_{10}(w_1 w_5 + w_2 w_7 - w_8), \\
&V_{15} = -c_3 - c_5 w_2 - c_6 w_1 - c_7 w_7 - c_8 w_2(w_2 + w_1) - c_9 w_2^2 w_1 \\
&\quad - c_{10}(2w_1 w_2 w_5 - 2w_8 w_2 + w_7 w_2^2 - 2w_1 w_6 + w_9), \\
&V_{21} = \lambda^{-1}(c_6 + c_7(e^{-n-m} + w_3) + c_8 w_2 + c_9 w_2^2 + 2c_{10}(w_5 w_2 - w_6)), \quad V_{23} = 0, \\
&V_{24} = V_{15}, \quad V_{25} = -c_7\lambda^{-1}, \quad V_{31} = -\lambda^{-1}(c_8 + 2c_9 w_2 + 2c_{10} w_5), \\
&V_{32} = -(e^{-n-m} + w_3)(c_5 + c_8(w_1 + 2w_2) + 2c_9 w_1 w_2 + 2c_{10}(w_2 w_7 - w_8 + w_5 w_1)) \\
&\quad - c_2 + c_8(2w_5 + w_7) + 2c_9 w_8 + 2c_{10} w_7 w_5, \\
&V_{34} = c_5 + c_8(w_1 + 2w_2) + 2c_9 w_1 w_2 + 2c_{10}(w_2 w_7 - w_8 + w_5 w_1), \quad V_{35} = 0, \\
&V_{41} = \lambda^{-1}(e^{-n-m} + w_3)\big(2c_6 + c_7(w_3 + e^{-n-m}) + 2c_8 w_2 + 2c_9 w_2^2 + 4c_{10}(w_5 w_2 - w_6)\big) \\
&\quad - \lambda^{-1}(-c_1 + 2c_8 w_5 + 4c_9 w_6 + 2c_{10} w_5^2), \\
&V_{42} = -c_4 e^{n-3m-2u} n_x - 2c_9\lambda^{-1} - 2c_{10}\lambda^{-1}(e^{-n-m} + w_3), \\
&V_{43} = -\lambda^{-1}(c_8 + 2c_9 w_2 + 2c_{10} w_5), \quad V_{52} = -\lambda c_4 e^{-2m}, \\
&V_{45} = \lambda^{-1}(-c_6 - c_7(e^{-n-m} + w_3) - w_2 c_8 - c_9 w_2^2 - 2c_{10}(w_5 w_2 - w_6)), \\
&V_{51} = -c_4 e^{n-3m-2u} n_x + 2c_9\lambda^{-1} + 2c_{10}\lambda^{-1}(e^{-n-m} + w_3), \\
&V_{53} = -(e^{-n-m} + w_3)(c_5 + c_8(w_1 + 2w_2) + 2w_1 w_2 c_9 + 2c_{10}(w_2 w_7 - w_8 + w_5 w_1)) \\
&\quad - c_2 + c_8(2w_5 + w_7) + 2c_9 w_8 + 2c_{10} w_7 w_5, \\
&V_{54} = -2c_8 - 2c_{10} w_7 - 2w_1 c_9 + \lambda c_4 e^{n-m}, \quad V_{33} = -4c_{10}/(5\lambda), \\
&V_{11} = F + c_1 w_1 - c_5 w_5 + c_6 w_7 - c_8(-w_8 + 2w_6 + 2w_5 w_1) - c_9(-w_9 + 4w_1 w_6) \\
&\quad - 2/5 c_{10}(5w_7 w_6 - 5w_5 w_8 + 5w_5^2 w_1 - 3\lambda^{-1}),
\end{aligned}$$

$$
\begin{aligned}
V_{22} &= F + c_2 w_2 - c_6 w_7 - c_7 w_{10} - c_8(2w_5 w_2 + w_2 w_7 - 2w_6) - c_9(-w_9 + 2w_8 w_2) \\
&\quad - 2/5 c_{10}(-5w_7 w_6 + 5w_7 w_5 w_2 + 2\lambda^{-1}), \\
V_{44} &= -F - c_1 w_1 + c_5 w_5 - c_6 w_7 + c_8(-w_8 + 2w_6 + 2w_5 w_1) + c_9(-w_9 + 4w_1 w_6) \\
&\quad + 2/5 c_{10}(5w_7 w_6 - 5w_5 w_8 + 5w_5^2 w_1 + 3\lambda^{-1}), \\
V_{55} &= -F - c_2 w_2 + c_6 w_7 + c_7 w_{10} + c_8(2w_5 w_2 + w_2 w_7 - 2w_6) + c_9(-w_9 + 2w_8 w_2) \\
&\quad + 2/5 c_{10}(-5w_7 w_6 + 5w_7 w_5 w_2 - 2\lambda^{-1}), \\
F &= (e^{-n-m} + w_3)(c_3 + c_5 w_2 + c_6 w_1 + c_7 w_7 + c_8 w_2(w_2 + w_1) + c_9 w_1 w_2^2 \\
&\quad + c_{10}(-2w_1 w_6 - 2w_8 w_2 + 2w_1 w_2 w_5 + w_9 + w_7 w_2^2)).
\end{aligned}
$$

This means that any local system obtained from (3.18) is integrable. In particular, system (3.20) has the zero curvature representation with matrix (4.3) and the following matrix V:

$$
V = \begin{pmatrix}
0 & 0 & 0 & 0 & -c_3 e^{-q-r} \\
0 & c_3 e^{-q-r} & 0 & -c_3 e^{-q-r} & 0 \\
0 & -c_2 e^r & 0 & 0 & 0 \\
c_1 \lambda^{-1} e^{2q} & 0 & 0 & 0 & 0 \\
0 & 0 & -c_2 e^r & 0 & -c_3 e^{-q-r}
\end{pmatrix}.
$$

D. Elements of the hierarchy of system "d" possess the zero curvature representations with the following common matrix U:

$$
U = \begin{pmatrix}
-f & -\lambda & 0 & -1 \\
-1 & -g & 1 & 0 \\
-m_x & 0 & f & -1 \\
0 & 0 & \lambda & g
\end{pmatrix}, \tag{4.4}
$$

where $2f = m_x + n_x, 2g = n_x - m_x + 2u_x$. Using it we find the zero curvature representation for (3.24) with the matrix U_0

$$
U_0 = \begin{pmatrix}
0 & -\lambda e^{m-u} & 0 & -e^{n+u} \\
-e^{u-m} & 0 & e^{n+u} & 0 \\
-m_x e^{-m-n} & 0 & 0 & -e^{u-m} \\
0 & 0 & \lambda e^{m-u} & 0
\end{pmatrix},
$$

which is gauge equivalent to (4.4), and the following matrix V:

$$
\begin{aligned}
V_{12} &= \frac{1}{2}(w_1^2c_{10} - w_1c_8 - c_6) - c_7w_2 - c_9w_1^2w_2 - c_{11}w_1w_2,\\
V_{13} &= c_3 + c_4w_1 + c_5w_1^2 + c_6w_2 + c_7w_2^2 + c_8w_1w_2 + c_9w_1^2w_2^2 - c_{10}w_1^2w_2 + c_{11}w_1w_2^2,\\
V_{14} &= -\frac{1}{2}\lambda^{-1}(c_4 + 2c_5w_1 + c_8w_2 + 2c_9w_1w_2^2 - 2c_{10}w_1w_2 + c_{11}w_2^2),\\
V_{21} &= \frac{1}{2}c_1\lambda^{-1} - \frac{1}{2}c_4\lambda^{-1}w_3 + c_5\lambda^{-1}(w_4 - w_1w_3) + \frac{1}{2}c_8\lambda^{-1}(w_6 - w_2w_3)\\
&\quad + c_9(w_8 - w_1w_2^2w_3 + \lambda^{-1}w_2)\lambda^{-1} + \frac{1}{2}c_{10}(2w_1w_2w_3 - 2w_9 - \lambda^{-1})\lambda^{-1}\\
&\quad + \frac{1}{2}c_{11}\lambda^{-1}(2w_7 - w_2^2w_3),\\
V_{23} &= -\lambda^{-1}(c_5w_1 - c_{10}w_1w_2 + c_9w_1w_2^2) - \frac{1}{2}\lambda^{-1}(c_4 + c_8w_2 + c_{11}w_2^2),\\
V_{24} &= \lambda^{-2}(c_5 + c_9w_2^2 - c_{10}w_2), \quad V_{31} = c_9\lambda^{-2},\\
V_{32} &= \frac{1}{2}c_2 + \frac{1}{2}c_6w_3 + c_7(w_2w_3 - w_6) + c_9(w_1\lambda^{-1} - w_{11} + w_1^2w_2w_3)\\
&\quad + \frac{1}{2}c_8(w_1w_3 - w_4) + \frac{1}{2}c_{10}(2w_5 - w_1^2w_3) + \frac{1}{2}c_{11}(2w_1w_2w_3 - 2w_9 + \lambda^{-1}),\\
V_{34} &= -\frac{1}{2}c_1\lambda^{-1} + \frac{1}{2}\lambda^{-1}c_4w_3 + c_5\lambda^{-1}(w_1w_3 - w_4) + \frac{1}{2}c_8\lambda^{-1}(w_2w_3 - w_6)\\
&\quad + c_9\lambda^{-1}(w_1w_2^2w_3 - w_8 + w_2\lambda^{-1}) - \frac{1}{2}c_{10}\lambda^{-1}(2w_1w_2w_3 - 2w_9 + \lambda^{-1})\\
&\quad + \frac{1}{2}c_{11}\lambda^{-1}(w_2^2w_3 - 2w_7),\\
V_{41} &= -\frac{1}{2}c_2 - \frac{1}{2}c_6w_3 + c_7(w_6 - w_2w_3) + c_9(w_1\lambda^{-1} + w_{11} - w_1^2w_2w_3)\\
&\quad + \frac{1}{2}c_8(w_4 - w_1w_3) + \frac{1}{2}c_{10}(w_1^2w_3 - 2w_5) + \frac{1}{2}c_{11}(\lambda^{-1} - 2w_1w_2w_3 + 2w_9),\\
V_{42} &= c_7 + c_9w_1^2 + c_{11}w_1,\\
V_{43} &= -\frac{1}{2}c_6 - c_7w_2 - \frac{1}{2}c_8w_1 - c_9w_1^2w_2 + \frac{1}{2}c_{10}w_1^2 - c_{11}w_1w_2,\\
V_{11} &= -\frac{1}{2}c_1w_1 + \frac{1}{2}c_2w_2 + c_3w_3 + \frac{1}{2}c_4(2w_1w_3 - w_4) + c_5w_1(w_1w_3 - w_4)\\
&\quad + \frac{1}{2}c_6(2w_2w_3 - w_6) - \frac{1}{4}c_8(\lambda^{-1} - 4w_1w_2w_3 + 2w_1w_6 + 2w_2w_4)\\
&\quad + w_2c_7(w_2w_3 - w_6) - c_9(w_1w_8 - w_1^2w_2^2w_3 + w_2w_{11} - 2w_{10} + w_1w_2\lambda^{-1})\\
&\quad - \frac{1}{2}c_{10}(2w_1^2w_2w_3 - 2w_1w_9 + w_{11} - 2w_2w_5 - w_1\lambda^{-1})\\
&\quad + \frac{1}{2}c_{11}(2w_1w_2^2w_3 + w_8 - 2w_1w_7 - 2w_2w_9 - w_2\lambda^{-1}),\\
V_{22} &= \frac{1}{2}c_1w_1 + \frac{1}{2}c_2w_2 - \frac{1}{2}c_4w_4 + c_5(w_1w_4 - 2w_5) + \frac{1}{2}c_6w_6 + c_7(2w_7 - w_2w_6)\\
&\quad + \frac{1}{4}c_8(\lambda^{-1} - 2w_2w_4 + 2w_1w_6) + c_9(w_1w_8 - w_2w_{11} + w_1w_2\lambda^{-1})\\
&\quad + \frac{1}{2}c_{10}(2w_2w_5 - 2w_1w_9 + w_{11} - w_1\lambda^{-1}) + \frac{1}{2}c_{11}(w_2\lambda^{-1} - 2w_2w_9 + w_8 + 2w_1w_7),
\end{aligned}
$$

$$
\begin{aligned}
V_{33} &= \frac{1}{2}c_1w_1 - \frac{1}{2}c_2w_2 - c_3w_3 + \frac{1}{2}c_4(w_4 - 2w_1w_3) + c_5w_1(w_4 - w_1w_3)\\
&+ \frac{1}{2}c_6(w_6 - 2w_2w_3) + \frac{1}{4}c_8(2w_1w_6 + 2w_2w_4 - 4w_1w_2w_3 - \lambda^{-1})\\
&+ c_7w_2(w_6 - w_3w_2) + c_9(w_1w_8 - w_1^2w_2^2w_3 + w_2w_{11} - 2w_{10} - w_1w_2\lambda^{-1})\\
&+ \frac{1}{2}c_{10}(2w_1^2w_2w_3 - 2w_1w_9 + w_{11} - 2w_2w_5 + w_1\lambda^{-1})\\
&- \frac{1}{2}c_{11}(2w_1w_2^2w_3 + w_8 - 2w_1w_7 - 2w_2w_9 + w_2\lambda^{-1}),\\
V_{44} &= -\frac{1}{2}c_1w_1 - \frac{1}{2}c_2w_2 + \frac{1}{2}c_4w_4 + c_5(2w_5 - w_1w_4) - \frac{1}{2}c_6w_6 + c_7(w_2w_6 - 2w_7)\\
&+ \frac{1}{4}c_8(2w_2w_4 - 2w_1w_6 + \lambda^{-1}) + c_9(w_2w_{11} - w_1w_8 + w_1w_2\lambda^{-1})\\
&- \frac{1}{2}c_{10}(2w_2w_5 - 2w_1w_9 + w_{11} + w_1\lambda^{-1}) + \frac{1}{2}c_{11}(2w_2w_9 - 2w_1w_7 - w_8 + w_2\lambda^{-1}).
\end{aligned}
$$

This means that any local system obtained from (3.24) is integrable. In particular, system (3.25) has the zero curvature representation with matrix (4.4), where $u_x = q_x + r_x - p_x, m_x = r_x - p_x, n_x = p_x - q_x$, and the following matrix V:

$$
V = \begin{pmatrix}
-\frac{1}{2}c_3e^{q-r} & 0 & c_3e^{q-r} & 0\\
\frac{1}{2}c_1\lambda^{-1}e^{-q} & -\frac{1}{2}c_3e^{q-r} & 0 & 0\\
0 & \frac{1}{2}c_2e^r & \frac{1}{2}c_3e^{q-r} & -\frac{1}{2}c_1\lambda^{-1}e^{-q}\\
-\frac{1}{2}c_2e^r & 0 & 0 & \frac{1}{2}c_3e^{q-r}
\end{pmatrix}.
$$

E. Elements of the hierarchy of system "e" possess the zero curvature representations with the following common matrix U:

$$
U = \begin{pmatrix}
q_x & 0 & 1 & 1\\
-2\lambda m_x & r_x & 0 & \lambda\\
-\lambda & 1 & -r_x & 0\\
0 & -1 & 2m_x & -q_x
\end{pmatrix}, \tag{4.5}
$$

where $q = u - \frac{1}{2}(m+n), r = u + \frac{1}{2}(m+n)$. Using it we find the zero curvature representation for (3.30) with the matrix U_0

$$
U_0 = \begin{pmatrix}
0 & 0 & e^{-2u} & e^{m+n-2u}\\
-2\lambda m_x e^{-m-n} & 0 & 0 & \lambda e^{-2u}\\
-\lambda e^{2u} & e^{m+n+2u} & 0 & 0\\
0 & e^{2u} & 2m_x e^{-m-n} & 0
\end{pmatrix},
$$

which is gauge equivalent to (4.5), and the following matrix V:

$$
\begin{aligned}
V_{44} = -V_{11} &= 2c_2w_2 - c_3w_3 + c_6(w_6 - 2w_3w_1) + c_7(w_7 + 2w_3w_2) + c_8(2w_8 - w_3w_6)\\
&+ c_9w_3w_7 + c_{10}(w_{10} - 2w_2w_6 + 2w_1w_2w_3) - \frac{1}{4}c_{11}(\lambda^{-1} + 8w_2w_8 - 4w_3w_{10})\\
&- c_{12}(\lambda^{-1} - w_6w_7 + 8w_2w_8 - 2w_3w_{10} + 2w_1w_3w_7 - 2w_2w_3w_6),\\
V_{33} = -V_{22} &= 2c_1w_1 + c_3w_3 + c_6(w_6 + 2w_1w_3) + c_7(w_7 - 2w_3w_2) + c_8w_3w_6\\
&+ c_9(2w_9 - w_3w_7) - c_{10}(w_{10} - 2w_1w_7 + 2w_1w_2w_3)\\
&+ \frac{1}{4}c_{11}(\lambda^{-1} + 8w_1w_9 - 4w_3w_{10}) + c_{12}(w_6w_7 - 2w_3w_{10} + 2w_1w_3w_7 - 2w_2w_3w_6),
\end{aligned}
$$

$$
\begin{aligned}
V_{12} &= \frac{1}{2}\lambda^{-1}(c_3 + c_8 w_6 - c_9 w_7 - c_{11} w_{10}) + \lambda^{-1}(c_6 w_1 - c_7 w_2 - c_{10} w_1 w_2) \\
&\quad + c_{12}\lambda^{-1}(w_1 w_7 - w_2 w_6 - w_{10}), \\
V_{13} &= \lambda^{-1}\big(- 2c_5 \lambda e^{n-m-2u} - c_6 - c_8 w_3 + c_{10} w_2 + c_{11} w_2 w_3 + c_{12}(4w_2 w_3 - w_7)\big), \\
V_{14} &= -c_4 e^{-2u+2m} - c_5 e^{2n-2u} + \frac{1}{2}\lambda^{-1}(c_8 - c_{11} w_2 - 4c_{12} w_2), \\
V_{21} &= -4c_5 \lambda u_x e^{-2m} - c_{10} - c_{11} w_3 - 2c_{12} w_3, \\
V_{23} &= 2c_1 - 4c_5 \lambda e^{-2u-2m} + 4c_6 w_3 + 2c_8 w_3^2 + 2c_{10}(w_7 - 2w_2 w_3) + 2c_{11}(w_9 - w_3^2 w_2) \\
&\quad + 4c_{12} w_3 (w_7 - 2w_2 w_3), \\
V_{24} &= -2c_5 \lambda e^{n-m-2u} - c_6 - c_8 w_3 + c_{10} w_2 + c_{11} w_2 w_3 + c_{12}(4w_2 w_3 - w_7), \\
V_{31} &= -2c_5 \lambda e^{n+2u-m} - c_7 - c_9 w_3 - c_{10} w_1 - c_{11} w_1 w_3 - c_{12} w_6, \\
V_{32} &= c_4 e^{2m+2u} + c_5 e^{2n+2u} - \frac{1}{2}\lambda^{-1}(c_9 + c_{11} w_1), \\
V_{34} &= -\frac{1}{2}c_3 - c_6 w_1 + c_7 w_2 - \frac{1}{2}c_8 w_6 + \frac{1}{2}c_9 w_7 + c_{10} w_1 w_2 + \frac{1}{2}c_{11} w_{10} \\
&\quad + c_{12}(w_2 w_6 + w_{10} - w_1 w_7), \\
V_{41} &= -2c_2 + 4\lambda c_5 e^{2u-2m} - 4c_7 w_3 - 2c_9 w_3^2 + 2c_{10}(w_6 - 2w_1 w_3) \\
&\quad + 2c_{11}(w_8 - w_1 w_3^2) + 4c_{12}(2w_8 - w_3 w_6), \\
V_{42} &= -2c_5 e^{n+2u-m} - \lambda^{-1}(c_7 + c_9 w_3 + c_{10} w_1 + c_{11} w_1 w_3 + c_{12} w_6), \\
V_{43} &= 4c_5 u_x e^{-2m} + \lambda^{-1}(c_{10} + c_{11} w_3 + 2c_{12} w_3).
\end{aligned}
$$

This means that any local system obtained from (3.30) is integrable. In particular, system (3.32) has the zero curvature representation with matrix (4.5), where $m = (r - p - q)/2$, and the following matrix V:

$$
V = \begin{pmatrix}
-\frac{1}{2}c_3 e^{q-r} & \frac{1}{2}c_3 \lambda^{-1} e^{q-r} & 0 & 0 \\
0 & \frac{1}{2}c_3 e^{q-r} & 2c_1 e^{2r} & 0 \\
0 & 0 & -\frac{1}{2}c_3 e^{q-r} & -\frac{1}{2}c_3 e^{q-r} \\
-2c_2 e^{-2q} & 0 & 0 & \frac{1}{2}c_3 e^{q-r}
\end{pmatrix}.
$$

F. Elements of the hierarchy of system “f” possess the zero curvature representations with the following common matrix U:

$$
U = \begin{pmatrix}
-u_x - \frac{1}{2}q_x & -\lambda & -1 & 0 \\
1 & u_x - \frac{1}{2}q_x & 0 & 0 \\
0 & 2\lambda m_x & u_x + \frac{1}{2}q_x & \lambda \\
2m_x & 0 & 1 & \frac{1}{2}q_x - u_x
\end{pmatrix}, \tag{4.6}
$$

where $q = m + n$. Using it we find the zero curvature representation for (3.30) with the matrix

$$
U_0 = \begin{pmatrix}
0 & -\lambda e^{2u} & -e^{m+n+2u} & 0 \\
e^{-2u} & 0 & 0 & 0 \\
0 & 2\lambda m_x e^{-m-n} & 0 & \lambda e^{-2u} \\
2m_x e^{-m-n} & 0 & e^{2u} & 0
\end{pmatrix},
$$

which is gauge equivalent to (4.6), and the following matrix V:

$$
\begin{aligned}
V_{12} &= -c_1 - c_4 w_2 + 2c_6(w_4 - w_1 w_2) - c_8 w_7 + c_9(2w_8 - \lambda^{-1}) \\
&\quad - c_{10} w_5 + c_{11}(2w_{10} - \lambda^{-1} w_3), \quad V_{13} = c_{11}(2\lambda^2)^{-1}, \\
V_{14} &= -\frac{1}{2}c_3 - c_5 w_2 - \frac{1}{2}c_7 w_5 + \frac{1}{4}c_8(\lambda^{-1} - 2w_8) \\
&\quad + c_9(w_1^2 w_2 - 2w_6) + c_{10}(w_4 - w_1 w_2) + \frac{1}{2}c_{11}(w_9 - w_1\lambda^{-1}), \\
V_{21} &= -\lambda^{-1}(c_5 + c_7 w_3 + c_8 w_1 w_3 - c_9 w_1^2 + c_{10} w_1 - c_{11} w_1^2 w_3), \\
V_{23} &= -\frac{1}{2}\lambda^{-1}(c_3 + 2c_5 w_2 + c_7 w_5) - \frac{1}{4}c_8\lambda^{-2}(1 + 2\lambda w_8) \\
&\quad + c_9\lambda^{-1}(w_1^2 w_2 - 2w_6) + c_{10}\lambda^{-1}(w_4 - w_1 w_2) + \frac{1}{2}c_{11}\lambda^{-2}(w_1 + \lambda w_9), \\
V_{24} &= \frac{1}{2}\lambda^{-1}(c_7 + c_8 w_1 - c_{11} w_1^2), \quad V_{31} = -2c_2 - 2c_4 w_1 - 4c_5 w_3 \\
&\quad - 2w_3^2(c_7 + c_8 w_1 - c_{11} w_1^2) + 2w_1^2(2c_9 w_3 - c_6) - 4c_{10} w_1 w_3, \\
V_{32} &= c_4 + 2c_6 w_1 + w_3^2(c_8 - 2c_{11} w_1) - 4c_9 w_1 w_3 + 2c_{10} w_3, \\
V_{34} &= c_5 + c_7 w_3 + c_8 w_1 w_3 - w_1^2(c_9 + c_{11} w_3) + c_{10} w_1, \\
V_{41} &= -\lambda^{-1}\big(c_4 + 2c_6 w_1 + 2c_{10} w_3 + w_3^2(c_8 - 2c_{11} w_1) - 4c_9 w_1 w_3\big), \\
V_{42} &= 2\lambda^{-1}(c_6 - 2c_9 w_3 - c_{11} w_3^2), \\
V_{43} &= -\lambda^{-1}\big(c_1 + c_4 w_2 + 2c_6(w_1 w_2 - w_4) + c_8 w_7\big) + c_9\lambda^{-2}(1 + 2\lambda w_8) \\
&\quad - c_{10}\lambda^{-1} w_5 + c_{11}\lambda^{-2}(w_3 + 2\lambda w_{10}), \\
V_{11} &= c_1 w_1 + 2c_2 w_2 + c_3 w_3 + c_4(2w_1 w_2 - w_4) + c_5(2w_2 w_3 + w_5) \\
&\quad + 2c_6 w_1(w_1 w_2 - w_4) + c_7 w_3 w_5 + c_8(w_1 w_7 - (2\lambda)^{-1} w_3 + w_3 w_8 - w_{10}) \\
&\quad + c_9(\lambda^{-1} w_1 - 2w_1 w_8 + 4w_3 w_6 + w_9 - 2w_1^2 w_2 w_3) \\
&\quad + c_{10}(w_1 w_5 - 2w_3 w_4 - (2\lambda)^{-1} + 2w_1 w_2 w_3) \\
&\quad + c_{11}(\lambda^{-1} w_1 w_3 - 2w_1 w_{10} + 2w_{11} - w_3 w_9), \\
V_{22} &= -c_1 w_1 + c_3 w_3 - c_4 w_4 + c_5(2w_2 w_3 - w_5) + 2c_6(w_1 w_4 - 2w_6) \\
&\quad + c_7(w_3 w_5 - 2w_7) + c_8((2\lambda)^{-1} w_3 - w_1 w_7 + w_3 w_8 - w_{10}) \\
&\quad + c_9(2w_1 w_8 + 4w_3 w_6 - w_9 - 2w_1^2 w_2 w_3 - \lambda^{-1} w_1) \\
&\quad + c_{10}((2\lambda)^{-1} - w_1 w_5 - 2w_3 w_4 + 2w_1 w_2 w_3) \\
&\quad - c_{11}(\lambda^{-1} w_1 w_3 - 2w_1 w_{10} + w_3 w_9), \\
V_{33} &= -c_1 w_1 - 2c_2 w_2 - c_3 w_3 + c_4(w_4 - 2w_1 w_2) - c_5(2w_3 w_2 + w_5) \\
&\quad + 2c_6 w_1(w_4 - w_1 w_2) - c_8((2\lambda)^{-1} w_3 + w_1 w_7 + w_3 w_8 - w_{10}) \\
&\quad - c_7 w_3 w_5 + c_9(2w_1 w_8 - 4w_3 w_6 - w_9 + 2w_1^2 w_2 w_3 + \lambda^{-1} w_1) \\
&\quad - c_{10}((2\lambda)^{-1} + w_1 w_5 - 2w_3 w_4 + 2w_1 w_2 w_3) \\
&\quad + c_{11}(\lambda^{-1} w_1 w_3 + 2w_1 w_{10} - 2w_{11} + w_3 w_9),
\end{aligned}
$$

$$
\begin{aligned}
V_{44} &= c_1 w_1 - c_3 w_3 + c_4 w_4 + c_5(w_5 - 2w_2 w_3) + 2c_6(2w_6 - w_1 w_4) \\
&\quad + c_7(2w_7 - w_3 w_5) + c_8((2\lambda)^{-1} w_3 + w_1 w_7 - w_3 w_8 + w_{10}) \\
&\quad + c_9(w_9 - 2w_1 w_8 - 4w_3 w_6 + 2w_1^2 w_2 w_3 - \lambda^{-1} w_1) \\
&\quad + c_{10}(w_1 w_5 + 2w_3 w_4 + (2\lambda)^{-1} - 2w_1 w_2 w_3) \\
&\quad + c_{11}(w_3 w_9 - \lambda^{-1} w_1 w_3 - 2w_1 w_{10}).
\end{aligned}
$$

This means that any local system obtained from (3.34) is integrable. In particular, system (3.35) has the zero curvature representation with matrix (4.6), where $2m = q - p$, and the following matrix V:

$$V = \begin{pmatrix} -\frac{1}{2}c_3e^{-q} & -c_1e^{-2u} & 0 & -\frac{1}{2}c_3e^{-q} \\ 0 & -\frac{1}{2}c_3e^{-q} & -\frac{1}{2}c_3\lambda^{-1}e^{-q} & 0 \\ -2c_2e^{q+2u} & 0 & \frac{1}{2}c_3e^{-q} & 0 \\ 0 & 0 & -c_1\lambda^{-1}e^{-2u} & \frac{1}{2}c_3e^{-q} \end{pmatrix}.$$

G. Elements of the hierarchy of system “g” possess the zero curvature representations with the following common matrix U:

$$U = \begin{pmatrix} m_x + n_x & -n_x & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -n_x \\ 0 & 0 & 2u_x & \lambda & 0 \\ 2 & -1 & 0 & -2u_x & 0 \\ 0 & 0 & -2 & 0 & -m_x - n_x \end{pmatrix}. \tag{4.7}$$

Using it we find the zero curvature representation for (3.39) with the matrix U_0

$$U_0 = \begin{pmatrix} 0 & -n_xe^{-m-n} & 0 & 0 & 0 \\ 0 & 0 & -e^{2u} & 0 & -n_xe^{-m-n} \\ 0 & 0 & 0 & \lambda e^{-4u} & 0 \\ 2e^{m+n+2u} & -e^{2u} & 0 & 0 & 0 \\ 0 & 0 & -2e^{m+n+2u} & 0 & 0 \end{pmatrix}$$

which is gauge equivalent to (4.7), and the following matrix V:

$$\begin{aligned}
&V_{11} = -F_1 - 4c_{10}(5\lambda)^{-1}, \quad V_{55} = F_1 - 4c_{10}(5\lambda)^{-1},\\
&V_{22} = -4c_{10}(5\lambda)^{-1}, \quad V_{33} = 6c_{10}(5\lambda)^{-1} - F_2, \quad V_{44} = 6c_{10}(5\lambda)^{-1} + F_2,\\
&\ F_1 = 2\big(c_3 + 2c_4w_1 + 2c_5w_8 + c_6w_4 + c_7w_5 + 2c_{10}(w_1w_5 - w_4w_8)\big)(w_3 + e^{-m-n})\\
&\qquad + 2c_1w_1 - 2c_6w_6 - 2c_7w_7 + 2c_8w_8 + 2c_9(2w_9 - w_1w_8) + 4c_{10}(w_6w_8 - w_1w_7),\\
&\ F_2 = 2c_1w_1 + 2c_2w_2 + 2c_4w_4 + 2c_5(2w_2w_4 - w_5) + 2c_6w_6\\
&\qquad + 2c_7(2w_2w_6 - w_7) + 2c_8(2w_1w_2 - w_8) + 2w_1c_9(w_1w_2 - w_8)\\
&\qquad - 2c_{10}(2w_6w_8 + 2w_1w_7 - w_4w_5 + w_2w_4^2 - 4w_1w_2w_6),\\
&V_{12} = 0, \quad V_{15} = -\frac{1}{2}c_9\lambda^{-1},\\
&V_{13} = \lambda^{-1}\Big(2(c_{10}w_4 - c_5)(w_3 + e^{-m-n}) - (c_7 + 2c_{10}w_1)(w_3 + e^{-m-n})^2\\
&\qquad\qquad - c_8 - c_9w_1 - 2c_{10}w_6\Big),\\
&V_{14} = \big(c_6 + c_7w_2 + 2c_{10}(w_1w_2 - w_8)\big)(w_3 + e^{-m-n})^2\\
&\qquad + 2\big(c_4 + c_5w_2 + c_{10}(w_5 - w_2w_4)\big)(w_3 + e^{-m-n})\\
&\qquad + c_1 + c_8w_2 + c_9(w_1w_2 - w_8) + 2c_{10}(w_2w_6 - w_7),\\
&V_{21} = -2c_3 - 4c_4w_1 - 4c_5w_8 - 2c_6w_4 - 2c_7w_5 - 4c_{10}(w_1w_5 - w_4w_8),\\
&V_{23} = 2\lambda^{-1}\big(c_{10}w_4 - c_5 - (c_7 + 2w_1c_{10})(w_3 + e^{-m-n})\big),
\end{aligned}$$

$$V_{24} = 2\big(c_6 + c_7 w_2 + 2c_{10}(w_1 w_2 - w_8)\big)(w_3 + e^{-m-n}) + 2c_4 + 2c_5 w_2 + 2c_{10}(w_5 - w_2 w_4),$$
$$V_{25} = 0, \quad V_{31} = -2c_6 - 2w_2 c_7 - 4c_{10}(w_1 w_2 - w_8), \quad V_{32} = V_{24},$$
$$V_{34} = 0, \quad V_{35} = -V_{14}, \quad V_{41} = -2\lambda^{-1}(c_7 + 2c_{10} w_1),$$
$$V_{42} = 2\lambda^{-1}\Big((c_7 + 2c_{10} w_1)(w_3 + e^{-m-n}) + c_5 - c_{10} w_4\Big),$$
$$V_{43} = -2\lambda^{-1}\big(c_2 + 2c_5 w_4 + 2c_7 w_6 + 2c_8 w_1 + c_9 w_1^2 + c_{10}(4w_1 w_6 - w_4^2)\big),$$
$$V_{51} = 0, \quad V_{53} = -2\lambda^{-1}(c_7 + 2c_{10} w_1),$$
$$V_{52} = -2c_3 - 4c_4 w_1 - 4c_5 w_8 - 2c_6 w_4 - 2c_7 w_5 - 4c_{10}(w_1 w_5 - w_4 w_8),$$
$$V_{54} = 2c_6 + 2c_7 w_2 + 4c_{10}(w_1 w_2 - w_8).$$

This means that any local system obtained from (3.39) is integrable. In particular, system (3.40) has the zero curvature representation with matrix (4.7), where $2m = q - p, 2n = p + q$, and the following matrix V:

$$V = \begin{pmatrix} -p_\tau & 0 & 0 & c_1 e^{q+2u} & 0 \\ -2p_\tau & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -c_1 e^{q+2u} \\ 0 & 0 & -2\lambda^{-1} c_2 e^{-4u} & 0 & 0 \\ 0 & -2p_\tau & 0 & 0 & p_\tau \end{pmatrix}.$$

Conclusion

It has been proved in [9, 10, 11] that the zero curvature representations found there are nontrivial. As all zero curvature representations presented here contain the same U-matrices by modulo of a gauge transformation, then all zero curvature representations constructed here are nontrivial too.

The method that is used here for constructing several exactly integrable hyperbolic systems can be applied to other integrable evolution systems. This method of obtaining integrable hyperbolic systems is much easier than direct classification.

If some of the hyperbolic systems presented in this paper are interesting from the application viewpoint, their zero curvature representations can be easily found as shown.

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