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On the $(3, N)$ Maurer-Cartan equation

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Abstract

Deformations of the 3-differential of 3-differential graded algebras are controlled by the $(3, N)$ Maurer-Cartan equation. We find explicit formulae for the coefficients appearing in that equation, introduce new geometric examples of N-differential graded algebras, and use these results to study N Lie algebroids.

1 Introduction

In this work we study deformations of the N-differential of a N-differential graded algebra. According to Kapranov [18] and Mayer [24, 25] a N-complex over a field k is a $\mathbb{Z}$-graded k-vector space $V = \bigoplus_{n \in \mathbb{Z}} V_n$ together with a degree one linear map $d : V \longrightarrow V$ such that $d^N = 0$. Remarkably, there are at least two generalizations of the notion of differential graded algebras to the context of N-complexes. A choice, introduced first by Kerner in [20, 21] and further studied by Dubois-Violette [13, 14] and Kapranov [18], is to fix a primitive N-th root of unity q and define a q-differential graded algebra A to be a $\mathbb{Z}$-graded associative algebra together with a linear operator $d : A \longrightarrow A$ of degree one such that $d(ab) = d(a)b + q^{\bar{a}} a d(b)$ and $d^N = 0$. There are several interesting examples and constructions of q-differential graded algebras [1, 2, 6, 8, 9, 15, 16, 19, 21].

We work within the framework of N-differential graded algebras (N-dga) introduced in [4]. This notion does not depend on the choice of a N-th primitive root of unity, and thus it is better adapted for differential geometric applications. A N-differential graded algebra A consist of a $\mathbb{Z}$-graded associative algebra $A = \bigoplus_{n \in \mathbb{Z}} A_n$ together with a degree one linear map $d : A \longrightarrow A$ such that $d^N = 0$ and $d(ab) = d(a)b + (-1)^{\bar{a}} a d(b)$ for $a, b \in A$. The main question regarding this definition is whether there are interesting examples of N-differential graded algebras. Much work still needs to be done, but already a variety of examples has been constructed in [4, 5]. These examples may be classified as follows:

- Deformations of 2-dga into N-dga. This is the simplest and most direct way to construct N-differential graded algebras. Take a differential graded algebra A with differential d and consider the deformed derivation $d + e$ where $e : A \longrightarrow A$ is a degree one derivation. It is possible to write down explicitly the equations that determine under which conditions $d + e$ is a N-differential, and thus turns A into a N-differential graded algebra. In other words one can explicitly write down the condition $(d + e)^N = 0$.

- N flat connections. Let E be a vector bundle over a manifold M provided with a flat connection ∂_E. Differential forms on M with values in $End(E)$ form a differential graded algebra. An $End(E)$-valued one form T determines a deformation of this algebra into a N-differential graded algebra with differential of the form $\partial_E + [T, \ \ ]$ if and only if T is a N-flat connection, i.e., the curvature of T is N-nilpotent.

- Differential forms of depth $N \geq 3$. Attached to each affine manifold M there is a $(dim(M)(N - 1) + 1)$-differential graded algebra $\Omega_N(M)$, called the algebra of differential forms of depth N on M, constructed as the usual differential forms allowing higher order differentials, i.e., for affine coordinates x_i on M, there are higher order differentials $d^j x_i$ for $1 \leq j \leq N - 1$.

- Deformations of N-differential graded algebras into M-differential graded algebras. If we are given a N-differential graded algebra A with differential d, one can study under which condition a deformed derivation $d+e$, where e is a degree one derivation of A, turns A into a M-differential graded algebra, i.e., one can determine conditions ensuring that $(d + e)^M = 0$. In [4] we showed that e must satisfy a system of non-linear equations, which we called the (N, M) Maurer-Cartan equation.

- Algebras A_∞^N. These are not so much examples of N-differential graded algebras but rather a homotopy generalization of such notion. A_∞^N algebras are studied in [7].

This paper has three main goals. The first one is to introduce new geometric examples of N-differential graded algebras. We first review the constructions of N-differential graded algebras outlined above and then proceed to consider the new examples:

- Differential forms on finitely generated simplicial sets. We construct a contravariant functor $\Omega_N : \mathrm{set}^{\Delta^{op}} \longrightarrow N^{il}dga$ from the category of simplicial sets generated in finite dimensions to $N^{il}dga$, the category of nilpotent differential graded algebras, i.e., N-differential graded algebras for some $N \geq 1$. For a simplicial set s we let $\Omega_N(s)$ be the algebra of algebraic differential forms of depth N on the algebro-geometric realization of s. For each integer K we define functor $Sing_{\leq K} : Top \longrightarrow \mathrm{set}^{\Delta^{op}}$, thus we obtain contravariant functors $\Omega_N \circ Sing_{\leq K} : Top \longrightarrow N^{il}dga$ assigning to each topological space X a nil-differential graded algebra.

- Difference forms on finitely generated simplicial sets. We construct a contravariant functor D_N defined on $\mathrm{set}^{\Delta^{op}}$ with values in a category whose objects are graded algebras which are also N-complexes for some N, with the N-differential satisfying a twisted Leibnitz rule. For a simplicial set s we let $D_N(s)$ be the algebra of difference forms of depth N on the integral lattice in the algebro-geometric realization of s.

Again, for each integer $K \geq 0$ we obtain a functor $D_N \circ Sing_{\leq K}$ defined on Top assigning to each topological space X a twisted nil-differential graded algebra.

Our second goal is to study the construction of N-differential graded algebras as deformations of 3-differential graded algebras. Although in [4] a general theory solving this sort of problem was proposed, our aim here is to provided a solution as explicit as possible. We consider exact and infinitesimal deformations of 3-differentials in Section 3.

Our final goal in this work is to find applications of N-differential graded algebras to Lie algebroids. In Section 4 we review the concept of Lie algebroids introduced by Pradines [27], which generalizes both Lie algebras and tangent bundles of manifolds. A Lie algebroid E may be defined as a vector bundle together with a degree one differential d on $\Gamma(\bigwedge E^*)$. We generalize this notion to the world of N-complexes, that is we introduce the concept of N Lie algebroids and construct several examples of such objects.

2 Examples of N-differential graded algebras

In this section we give a brief summary of the known examples of N-dgas and introduce new examples of N-dgas of geometric nature.

Definition 1. Let $N \geq 1$ be an integer. A N-complex is a pair (A, d), where A is a $\mathbb{Z}$-graded vector space and $d : A \longrightarrow A$ is a degree one linear map such that $d^N = 0$.

Clearly a N-complex is also a M-complex for $M \geq N$. N-complexes are also referred to as N-differential graded vector spaces. A N-complex (A, d) such that $d^{N-1} \neq 0$ is said to be a proper N-complex. Let (A, d) be a N-complex and (B, d) be a M-complex, a morphism $f : (A, d) \longrightarrow (B, d)$ is a linear map $f : A \longrightarrow B$ such that $df = fd$. One of the most interesting features of N-complexes is that they carry cohomological information. Let (A, d) be a N-complex, $a \in A^i$ is p-closed if $d^p(a) = 0$, and is p-exact if there exists $b \in A^{i-N+p}$ such that $d^{N-p}(b) = a$, for $1 \leq p < N$. The cohomology groups of (A, d) are the spaces

$$_pH^i(A) = Ker\{d^p : A^i \longrightarrow A^{i+p}\}/Im\{d^{N-p} : A^{i-N+p} \longrightarrow A^i\},$$

for $i \in \mathbb{Z}$ and $p = 1, 2, ..., N-1$.

Definition 2. A N-differential graded algebra (N-dga) over a field k, is a triple (A, m, d) where $m : A \otimes A \longrightarrow A$ and $d : A \longrightarrow A$ are linear maps such that:

1. $d^N = 0$, i.e., (A, d) is a N-complex.
2. (A, m) is a graded associative algebra.
3. d satisfies the graded Leibnitz rule $d(ab) = d(a)b + (-1)^{\bar{a}} ad(b)$.

The simplest way to obtain N-differential graded algebras is deforming differential graded algebras. Let $Der(A)$ be the Lie algebra of derivations on a graded algebra A. Recall that a degree one derivation d on A, induces a degree one derivation, also denoted by d, on $End(A)$. Let A be a 2-dga and $e \in Der(A)$. It is shown in [4] that e defines a deformation of A into a N-differential graded algebra if and only if $(d + e)^N = 0$, or equivalently, if and only if the curvature $F_e = d(e) + e^2$ of e satisfies $(F_e)^{\frac{N}{2}} = 0$ if N is even, or $(F_e)^{\frac{N-1}{2}}(d+e) = 0$ if N is odd. For example, consider the trivial bundle $M \times \mathbb{R}^n$ over M. A connection on $M \times \mathbb{R}^n$ is a $gl(n)$-valued one form a on M, and its curvature is $F_a = da + \frac{1}{2}[a, a]$. Let $\Omega(M, gl(n))$ be the graded algebra of $gl(n)$-valued forms on M. Thus the pair $(\Omega(M, gl(n)), d + [a, \ \ ])$ defines a N-dga if and only if $(F_a)^{\frac{N}{2}} = 0$ for N even, or $(F_a)^{\frac{N-1}{2}}(d + a) = 0$ for N odd.

Differential forms of depth N on simplicial sets

Fix an integer $N \geq 3$. We are going to construct the $(n(N-1)+1)$-differential graded algebra $\Omega_N(\mathbb{R}^n)$ of algebraic differential forms of depth N on $\mathbb{R}^n$. Let $x_1, ..., x_n$ be coordinates on $\mathbb{R}^n$, and for $0 \leq i \leq n$ and $0 \leq j < N$, let $d^j x_i$ be a variable of degree j. We identify $d^0 x_i$ with x_i.

Definition 3. The $(n(N-1)+1)$-differential graded algebra $\Omega_N(\mathbb{R}^n)$ is given by

- $\Omega_N(\mathbb{R}^n) = \mathbb{R}[d^j x_i] / \langle d^j x_i d^k x_i \mid j, k \geq 1 \rangle$ as a graded algebras.
- The $(n(N-1)+1)$-differential $d : \Omega_N(\mathbb{R}^n) \longrightarrow \Omega_N(\mathbb{R}^n)$ is given by $d(d^j x_i) = d^{j+1} x_i$, for $0 \leq j \leq N-2$, and $d(d^{N-1} x_i) = 0$.

One can show that d is $(n(N-1)+1)$-differential as follows:

1. It is easy to check that $\Omega_N(\mathbb{R})$ is a N-dga.
2. If A is a N-dga and B is a P-dga, then $A \otimes B$ is a $(N+P-1)$-dga.
3. $\Omega_N(\mathbb{R}^n) = \Omega_N(\mathbb{R})^{\otimes n}$

We often write $\Omega_N(x_1, ..., x_n)$ instead of $\Omega_N(\mathbb{R}^n)$ to indicate that a choice of affine coordinates $(x_1, ..., x_n)$ on $\mathbb{R}^n$ has been made.

Let Δ be the category such that its objects are non-negative integers; morphisms in $\Delta(n, m)$ are order preserving maps $f : \{0, ..., n\} \longrightarrow \{0, ..., m\}$. The category of simplicial sets $\mathrm{Set}^{\Delta^{op}}$ is the category of contravariant functors $\Delta \longrightarrow \mathrm{Set}$. Explicitly, a simplicial set $s : \Delta^{op} \longrightarrow \mathrm{Set}$ is a functorial correspondence assigning:

- A set s_n for each integer $n \geq 0$. Elements of s_n are called simplices of dimension n.
- A map $s(f) : s_m \longrightarrow s_n$ for each $f \in \Delta(n, m)$.

Let Aff be the category of affine varieties, and let $A : \Delta \longrightarrow \mathrm{Aff}$ be the functor sending $n \geq 0$, into $A(x_0, ..., x_n) = \Delta_n = \{(x_0, ..., x_n) \in \mathbb{R}^n \mid x_0 + ... + x_n = 1\}$. A sends $f \in \Delta(n, m)$ into $A(f) : A(x_0, ..., x_n) \to A(x_0, ..., x_m)$ given by $A(f)^*(x_j) = \sum_{f(i)=j} x_i$, for $0 \leq j \leq m$. Forms of depth N on the cosimplicial affine variety A are defined by the functor $\Omega_N : \Delta^{op} \longrightarrow N^{il}\mathrm{dga}$ sending $n \geq 0$ into

$$\Omega_N(n) = \Omega_N(x_0, ..., x_n) / \langle x_0 + ... + x_n - 1, \quad dx_0 + ... + dx_n \rangle .$$

A map $f \in \Delta(n, m)$ induces a morphisms $\Omega_N(f) : \Omega_N(m) \longrightarrow \Omega_N(n)$ given for $0 \leq j \leq m$ by

$$\Omega_N(f)(x_j) = \sum_{f(i)=j} x_i \quad \text{and} \quad \Omega_N(f)(dx_j) = \sum_{f(i)=j} dx_i.$$

Let $\mathrm{set}^{\Delta^{op}}$ be the full subcategory of $\mathrm{Set}^{\Delta^{op}}$ whose objects are simplicial sets generated in finite dimensions, i.e., simplicial sets s for which there is an integer K such that for each $p \in s_i$, $i \geq K$, there exists $q \in s_j$, $j \leq K$, with $p = s(f)(q)$ for some $f \in \Delta(p,q)$. We are ready to define the contravariant functor $\Omega_N : \mathrm{set}^{\Delta^{op}} \longrightarrow N^{il}\mathrm{dga}$ announced in the introduction. The nil-differential graded algebra $\Omega_N(s) = \bigoplus_{i=0}^{\infty} \Omega_N^i(s)$ associated with s is given by

$$\Omega_N^i(s) = \{a \in \prod_{n=0}^{\infty} \prod_{p \in s_n} \Omega_N^i(n) \mid a_{s(f)(p)} = \Omega_N(f)(a_p) \text{ for } p \in s_m \text{ and } f \in \Delta(n,m)\}.$$

A natural transformation $l : s \longrightarrow t$ induces a map $\Omega_N(l) : \Omega_N(t) \longrightarrow \Omega_N(s)$ given by the rule $[\Omega_N(l)(a)]_p = a_{l(p)}$ for $a \in \Omega_N(t)$ and $p \in s_n$.

For each integer $K \geq 0$ there is functor $(\ \ )_{\leq K} : \mathrm{Set}^{\Delta^{op}} \longrightarrow \mathrm{set}^{\Delta^{op}}$ sending a simplicial set s, into the simplicial set $s_{\leq K}$ generated by simplices in s of dimension lesser or equal to K. The singular functor $Sing : Top \longrightarrow \mathrm{Set}^{\Delta^{op}}$ sends a topological space X into the simplicial set $Sing(X)$ such that

$$Sing_n(X) = \{f : \Delta_n \longrightarrow X \mid f \text{ is continous } \}.$$

Thus, for each pair of integers N and K we have constructed a functor

$$\Omega_N \circ (\ \ )_{\leq K} \circ Sing : Top \longrightarrow N^{il}\mathrm{dga}$$

sending a topological space X into the nil-differential graded algebra $\Omega_N(Sing_{\leq K}(X))$.

Difference forms of depth N on simplicial sets

Next we construct difference forms of higher depth on finitely generated simplicial sets. Difference forms on discrete affine space were introduced by Zeilberger in [28]. We proceed to construct a discrete analogue of the functors from topological spaces to nil-differential graded algebras introduced above. First, we construct $D_N(\mathbb{Z}^n)$ the algebra of difference forms of depth N on $\mathbb{Z}^n$. Let $F(\mathbb{Z}^n, \mathbb{R})$ be the algebra concentrated in degree zero of $\mathbb{R}$-valued functions on the lattice $\mathbb{Z}^n$. Introduce variables $\delta^j m_i$ of degree j for $1 \leq i \leq n$ and $1 \leq j < N$. The graded algebra of difference forms of depth N on $\mathbb{Z}^n$ is given by

$$D_N(\mathbb{Z}^n) = F(\mathbb{Z}^n, \mathbb{R}) \otimes \mathbb{R}[\delta^j m_i] / \left\langle \delta^j m_i \delta^k m_i \mid j,k \geq 1 \right\rangle .$$

A form $\omega \in D_N(\mathbb{Z}^n)$ can be written as $\omega = \sum_I \omega_I dm_I$ where $I : \{1,..,n\} \longrightarrow \mathbb{N}$ is any map and $dm_I = \prod_{i=1}^n d^{I(i)} m_i$. The degree of dm_I is $|I| = \sum_{i=1}^n I(i)$. The finite difference $\Delta_i(g)$ of $g \in F(\mathbb{Z}^n, \mathbb{R})$ along the i-direction is given by

$$\Delta_i(g)(m) = g(m + e_i) - g(m),$$

where the vectors e_i are the canonical generators of $\mathbb{Z}^n$ and $m \in \mathbb{Z}^n$. The difference operator δ is defined for $1 \leq j \leq N-2$ by the rules

$$\delta(g) = \sum_{i=1}^n \Delta_i(g) \delta m_i, \ \ \delta(\delta^j m_i) = \delta^{j+1} m_i \ \text{ and } \ \delta(\delta^{N-1} m_i) = 0.$$

It is not difficult to check that if $\omega = \sum_I \omega_I dm_I$, then $\delta\omega = \sum_J (\delta\omega)_J dm_J$ where

$$(\delta\omega)_J = \sum_{J(i)=1} (-1)^{|J_{<i}|}\Delta_i \omega_{J-e_i} + \sum_{J(i)\geq 2} (-1)^{|J_{<i}|}\omega_{J-e_i}.$$

From the later formula we see that $(\delta\omega)_J$ is a linear combination of (differences of) functions ω_K with $|K| < |J|$. This fact implies that δ is nilpotent, indeed, one can check that $\delta^{n(N-1)+1} = 0$. All together we have proved the following result.

Theorem 1. *$D_N(\mathbb{Z}^n)$ is a graded algebra and the difference operator δ gives $D_N(\mathbb{Z}^n)$ the structure of a $(n(N-1)+1)$-complex.*

One can check that δ satisfies a twisted Leibnitz rule, so $D_N(\mathbb{Z}^n)$ is actually pretty close of being a N-dga. Let $\mathbb{Z}^{n,1} \subseteq \mathbb{Z}^{n+1}$ consists of tuples $(m_0, ..., m_n)$ such that $m_0 + ... + m_n = 1$. Consider the functor D_N defined on Δ^{op} sending $n \geq 0$ into

$$D_N(n) = F(\mathbb{Z}^{n,1}, \mathbb{R}) \otimes \left\langle \delta^j m_i \delta^k m_i,\ \delta m_0 + ... + \delta m_n \right\rangle.$$

A map $f \in \Delta(n, k)$ induces a morphisms $D_N(f) : D_N(k) \longrightarrow D_N(n)$ given for $g \in F(\mathbb{Z}^{k,1}, \mathbb{R})$ and $0 \leq j \leq k$ by

$$D_N(f)(g)(m_0, ..., m_n) = g\left(\sum_{f(i)=0} m_i, ..., \sum_{f(i)=k} m_i\right) \text{ and } D_N(f)(\delta m_j) = \sum_{f(i)=j} \delta m_i.$$

We extend D_N to the functor defined on $\mathrm{set}^{\Delta^{op}}$ sending a finitely generated simplicial set s into $D_N(s) = \bigoplus_{i=0}^{\infty} D_N^i(s)$ where

$$D_N^i(s) = \left\{ a \in \prod_{n=0}^{\infty} \prod_{p \in s_n} D_N^i(n) \mid a_{s(f)(p)} = D_N(f)(a_p) \text{ for } p \in s_k \text{ and } f \in \Delta(n,k) \right\}.$$

A natural transformation $l : s \longrightarrow t$ induces a map $\Omega_N(l) : \Omega_N(t) \longrightarrow \Omega_N(s)$ by the rule $[D_N(l)(a)]_p = a_{l(p)}$ for $a \in D_N(t)$ and $p \in s_n$. Thus for given integers N and K we have constructed a functor $D_N \circ (\ )_{\leq K} \circ Sing$ on Top sending a topological space X into a sort of nil-differential graded algebra satisfying a twisted Leibnitz rule $D_N\left(Sing_{\leq K}(X)\right)$. It would be interesting to compute the cohomology groups of the algebra of difference forms of higher depth on known simplicial sets. Even in the case of forms of depth 2 these groups have seldom been studied.

3 On the (3,N) curvature

Recall that a discrete quantum mechanical system is given by the following data:

1. A directed graph with set of vertices V and set of directed edges E. The Hilbert space of the system is $\mathcal{H} = \mathbb{C}^V$.

2. A map $\omega : E \longrightarrow \mathbb{R}$ assigning a weight to each edge.

3. Operators $U_n : \mathcal{H} \longrightarrow \mathcal{H}$ for $n \in \mathbb{N}$ given by $(U_n f)(y) = \sum_{x\in V_\Gamma} \omega_n(y,x) f(x)$ where the discretized kernel $\omega_n(y,x)$ is given by

$$\omega_n(y,x) = \sum_{\gamma\in P_n(x,y)} \prod_{e\in\gamma} \omega(e).$$

$P_n(x,y)$ denotes the set of paths in Γ from x to y of length n, i.e., sequences $(e_1, \cdots, e_n)$ of edges such that $s(e_1) = x$, $t(e_i) = s(e_{i+1})$, for $i = 1, ..., n-1$ and $t(e_n) = y$.

Let us introduce some notation. For $s = (s_1, ..., s_n) \in \mathbb{N}^n$ we set $l(s) = n$ and $|s| = \sum_i s_i$. For $1 < i \leq n$ we set $s_{<i} = (s_1, ..., s_{i-1})$; also we set $s_{>n} = s_{<1} = \emptyset$. $\mathbb{N}^{(\infty)}$ is equal to $\bigsqcup_{n=0}^{(\infty)} \mathbb{N}^n$ where by convention $\mathbb{N}^{(0)} = \{\emptyset\}$. Let A be a 3-dga and e be a degree one derivation on A. For $s \in \mathbb{N}^n$ we let $e^{(s)} = e^{(s_1)} \cdots e^{(s_n)}$, where $e^{(l)} = d^l(e)$ if $l \geq 1$, $e^{(0)} = e$ and $e^{\emptyset} = 1$. For $N \in \mathbb{N}$, we set $E_N = \left\{ s \in \mathbb{N}^{(\infty)} \mid s \neq \emptyset \text{ and } |s| + l(s) \leq N \right\}$ and for $s \in E_N$ we let $N(s) \in \mathbb{Z}$ be given by $N(s) = N - |s| - l(s)$.

The following data defines a discrete quantum mechanical system:

1. The set of vertices is $\mathbb{N}^{(\infty)}$.

2. There is a unique directed edge from s to t if and only if $t \in \{(0,s), s, (s+e_i)\}$, where $e_i \in \mathbb{N}^{l(s)}$ are the canonical vectors.

3. Edges are weighted according to the table:

source	*target*	*weight*
s	$(0,s)$	1
s	s	$(-1)^{\lvert s\rvert + l(s)}$
s	$(s+e_i)$	$(-1)^{\lvert s_{<i}\rvert + i - 1}$

$P_N(\emptyset, s)$ consists of paths $\gamma = (e_1, ..., e_N)$, such that $s(e_1) = \emptyset$, $t(e_N) = s$ and $s(e_{l+1}) = t(e_l)$. The weight $\omega(\gamma)$ of a path $\gamma \in P_N(\emptyset, s)$ is given by $\omega(\gamma) = \prod_{i=1}^N \omega(e_i)$. The following result, proved in [4], tell us when $d + e$ defines a deformation of a 3-dga into a N-dga.

Theorem 2. *$d+e$ defines a deformation of the 3-dga A into a N-dga if and only if the $(3,N)$ Maurer-Cartan equation holds $c_o + c_1 d + c_2 d^2 = 0$, where for $0 \leq k \leq 2$ we set*

$$c_k = \sum_{\substack{s\in E_N \\ N(s)=k \\ s_i<3}} c(s,N) e^{(s)} \quad \textit{and} \quad c(s,N) = \sum_{\gamma\in P_N(\emptyset,s)} \omega(\gamma).$$

Exact deformations

Let us first consider the deformation of a 3-dga into a 3-dga. According to Theorem 2 the derivation $d+e$ defines a 3-dga if and only if $c_o + c_1 d + c_2 d^2 = 0$ where

$$c_k = \sum_{\substack{s\in E_3 \\ N(s)=k \\ s_i<3}} c(s,3) e^{(s)} \quad \text{and } c(s,3) = \sum_{\gamma\in P_3(\emptyset,s)} \omega(\gamma).$$

Let us compute the coefficients c_k. We have that

$$E_3 = \{\emptyset, (0), (1), (2), (0,0), (1,0), (0,1), (0,0,0)\} .$$

Let us first compute c_0. There are four vectors s in E_3 such that $N(s) = 0$, these are $(2), (1,0)$, $(0,1)$ and $(0,0,0)$. The only path from $\emptyset$ to (2) of length 3 is

$$\emptyset \longrightarrow (0) \longrightarrow (1) \longrightarrow (2)$$

of weight 1. Since $e^{(2)} = d^2(e)$, then we have that $c((2), 3) = d^2(e)$. The unique path from $\emptyset$ to $(1,0)$ of length 3 is

$$\emptyset \longrightarrow (0) \longrightarrow (0,0) \longrightarrow (1,0)$$

of weight 1. Since $e^{(1,0)} = d(e)e$ we have that $c((1,0), 3) = d(e)e$. There are two paths from $\emptyset$ to $(0,1)$ of length 3, namely

$$\emptyset \longrightarrow (0) \longrightarrow (0,0) \longrightarrow (0,1)$$

$$\emptyset \longrightarrow (0) \longrightarrow (1) \longrightarrow (0,1)$$

of weight -1 and 1, respectively. Thus $c((1,0), 3) = 0$ since the sum of the weights vanishes. The unique path from $\emptyset$ to $(0,0,0)$ of length 3 is

$$\emptyset \longrightarrow (0) \longrightarrow (0,0) \longrightarrow (0,0,0)$$

of weight 1. Since $e^{(0,0,0)} = e^3$, then $c((0,0,0), 3) = e^3$. Thus we have shown that

$$c_0 = d^2(e) + d(e)e + e^3.$$

We proceed to compute c_1. The vectors in E_3 such that $N(s) = 1$ are (1) and $(0,0)$. Paths from $\emptyset$ to (1) of length 3 are

$$\emptyset \longrightarrow \emptyset \longrightarrow (0) \longrightarrow (1)$$

$$\emptyset \longrightarrow (0) \longrightarrow (0) \longrightarrow (1)$$

$$\emptyset \longrightarrow (0) \longrightarrow (1) \longrightarrow (1)$$

of weight 1, -1 and 1, respectively. Since $e^{(1)} = d(e)$, then $c((1), 3) = d(e)$. Paths from $\emptyset$ to $(0,0)$ of length 3 are

$$\emptyset \longrightarrow (0) \longrightarrow (0) \longrightarrow (0,0)$$

$$\emptyset \longrightarrow \emptyset \longrightarrow (0) \longrightarrow (0,0)$$

$$\emptyset \longrightarrow (0) \longrightarrow (0,0) \longrightarrow (0,0).$$

The corresponding weights are, respectively, -1, 1 and 1. We have that $e^{(0,0)} = e^2$, thus $c((0,0), 3) = e^2$ and $c_1 = d(e) + e^2$.

Finally we compute c_2. (0) is the only vector in E_3 such that $N(s) = 2$. The paths from $\emptyset$ to (0) of length 3 are

$$\emptyset \longrightarrow \emptyset \longrightarrow \emptyset \longrightarrow (0)$$

$$\emptyset \longrightarrow \emptyset \longrightarrow (0) \longrightarrow (0)$$
$$\emptyset \longrightarrow (0) \longrightarrow (0) \longrightarrow (0).$$

The corresponding weights are, respectively, 1, -1 and 1. Since $e^{(0)} = e$, then we have that $c_2 = c((0), 3) = e$. Altogether we have proven the following result.

Theorem 3. *$d + e$ defines a deformation of the 3-dga A into a 3-dga if and only if*

$$(d^2(e) + d(e)e + e^3) + (d(e) + e^2)d + ed^2 = 0.$$

Consider now deformations of a 3-dga into a 4-dga. Again by Theorem 2 we must have $c_0 + c_1 d + c_2 d^2 = 0$. We proceed to compute the coefficients c_k. We have that

$$E_4 = \{\emptyset, (0), (1), (2), (0,0), (1,0), (0,1), (2,0), (0,2), (1,1), \\ (0,0,0), (1,0,0), (0,1,0), (0,0,1), (0,0,0,0)\}.$$

(0) is the only vector in E_4 such that $N(s) = 3$. Paths of length 4 from $\emptyset$ to (0) are of the form $\underbrace{\emptyset \to \cdots \to \emptyset}_{i} \to \underbrace{(0) \to \cdots \to (0)}_{j}$ with weight $(-1)^j$, where $i + j = 3$, thus we have that $c_3 = (1 - 1 + 1 - 1)e = 0$.

We compute c_2. Vectors in E_4 with $N(s) = 2$ are $(0,0)$ and (1). Paths from $\emptyset$ to $(0,0)$ of length 2 are of the form $\underbrace{\emptyset \to \cdots \to \emptyset}_{i} \to \underbrace{(0) \to \cdots \to (0)}_{j} \to \underbrace{(0,0) \to \cdots \to (0,0)}_{k}$ of weight $\sum_{i+j+k=2}(-1)^j = 2$, thus $c((0,0), 4)e^{(0,0)} = 2e^2$. Paths from $\emptyset$ to (1) of length 2 are of the form $\underbrace{\emptyset \to \cdots \to \emptyset}_{i} \to \underbrace{(0) \to \cdots \to (0)}_{j} \to \underbrace{(1) \to \cdots \to (1)}_{k}$ with weight $\sum_{i+j+k=2}(-1)^j = 2$, thus $c((1), 4)e^{(1)} = 2d(e)$ and $c_2 = 2(e^2 + d(e))$.

Let us now compute c_1. Vectors in E_4 with $N(s) = 1$ are $(0,0,0)$, $(1,0)$, $(0,1)$ and (2). Paths from $\emptyset$ to $(0,0,0)$ are of 5 types. Paths of the form

$$\underbrace{\emptyset \to \cdots \to \emptyset}_{i} \to \underbrace{(0) \to \cdots \to (0)}_{j} \to \underbrace{(0,0) \to \cdots \to (0,0)}_{k} \to \underbrace{(0,0,0) \to \cdots \to (0,0,0)}_{l}$$

with weight $\sum_{i+j+k+l=1}(-1)^j(-1)^l$, so that $c((0,0,0), 4)e^{(0,0,0)} = (1 - 1 + 1 - 1)e^3 = 0$. Paths of the form

$$\underbrace{\emptyset \to \cdots \to \emptyset}_{i} \to \underbrace{(0) \to \cdots \to (0)}_{j} \to \underbrace{(1) \to \cdots \to (1)}_{k} \to \underbrace{(0,1) \to \cdots \to (0,1)}_{l}$$

with weight

$$\sum_{i+j+k+l=1} (-1)^j(-1)^l = 1 - 1 + 1 - 1 = 0\,.$$

Path of the form

$$\underbrace{\emptyset \to \cdots \to \emptyset}_{i} \to \underbrace{(0) \to \cdots \to (0)}_{j} \to \underbrace{(0,0) \to \cdots \to (0,0)}_{k} \to \underbrace{(0,1) \to \cdots \to (0,1)}_{l}$$

of weight $\sum_{i+j+k+l=1}(-1)^j(-1)(-1)^l$ so that

$$c((0,1),4)e^{(0,1)} = ((1-1+1-1)+(1-1+1-1))ed(e) = 0.$$

Paths of the form

$$\underbrace{\emptyset \to \cdots \to \emptyset}_{i} \to \underbrace{(0) \to \cdots \to (0)}_{j} \to \underbrace{(0,0) \to \cdots \to (0,0)}_{k} \to \underbrace{(1,0) \to \overset{l}{\cdots} \to (1,0)}$$

of weight $\sum_{i+j+k+l=1}(-1)^j(-1)l$, thus $c((1,0),4)e^{(1,0)} = (1-1+1-1)d(e)e = 0$. There are also paths of the form

$$\underbrace{\emptyset \to \cdots \to \emptyset}_{i} \to \underbrace{(0) \to \cdots \to (0)}_{j} \to \underbrace{(1) \to \cdots \to (1)}_{k} \to \underbrace{(2) \to \cdots \to (2)}_{l}$$

of weight $\sum_{i+j+k+l=1}(-1)^j(-1)^l$, so we have $c((2),4)e^{(2)} = (1-1+1-1)d^2(e) = 0$. We have shown that

$$c_1 = c((0,0,0),4)e^{(0,0,0)} + c((0,1),4)e^{(0,1)} + c((1,0),4)e^{(1,0)} + c((2),4)e^{(2)} = 0.$$

Let us compute c_0. There are several types of paths in this case. Path

$$\emptyset \longrightarrow (0) \longrightarrow (0,0) \longrightarrow (0,0,0) \longrightarrow (0,0,0,0)$$

of weight 1, thus $c_q((0,0,0,0),4)a^{(0,0,0,0)} = a^4$. Paths

$$\emptyset \longrightarrow (0) \longrightarrow (0,0) \longrightarrow (0,0,0) \longrightarrow (0,0,1)$$

$$\emptyset \longrightarrow (0) \longrightarrow (1) \longrightarrow (0,1) \longrightarrow (0,0,1)$$

$$\emptyset \longrightarrow (0) \longrightarrow (0,0) \longrightarrow (0,1) \longrightarrow (0,0,1)$$

of weight 1, thus we have that $c((0,0,1),4)e^{(0,0,1)} = e^2d(e)$. Paths

$$\emptyset \longrightarrow (0) \longrightarrow (0,0) \longrightarrow (1,0) \longrightarrow (0,1,0)$$

$$\emptyset \longrightarrow (0) \longrightarrow (0,0) \longrightarrow (0,0,0) \longrightarrow (0,1,0)$$

of weight 0, thus $c((0,1,0),4)e^{(0,1,0)} = 0ad(a)a = 0$. Path

$$\emptyset \longrightarrow (0) \longrightarrow (0,0) \longrightarrow (0,0,0) \longrightarrow (1,0,0)$$

of weight 1, thus $c((1,0,0),4)e^{(1,0,0)} = d(e)e^2$. Path

$$\emptyset \longrightarrow (0) \longrightarrow (0,0) \longrightarrow (1,0) \longrightarrow (2,0)$$

of weight 1, so $c((2,0),4)e^{(2,0)} = d^2(e)e$. Paths

$$\emptyset \longrightarrow (0) \longrightarrow (0,0) \longrightarrow (0,1) \longrightarrow (0,2)$$

$$\emptyset \longrightarrow (0) \longrightarrow (1) \longrightarrow (0,1) \longrightarrow (0,2)$$

$$\emptyset \longrightarrow (0) \longrightarrow (1) \longrightarrow (2) \longrightarrow (0,2)$$

of weight 1, so that $c((0,2),4)e^{(0,2)} = ed^2(e)$. There are also paths

$$\emptyset \longrightarrow (0) \longrightarrow (0,0) \longrightarrow (1,0) \longrightarrow (1,1)$$

$$\emptyset \longrightarrow (0) \longrightarrow (1) \longrightarrow (0,1) \longrightarrow (1,1)$$

of weight 2, so that $c((1,1),4)e^{(1,1)} = (d(e))^2$. We see that

$$c_0 = e^4 + e^2d(e) + d(e)e^2 + d^2(e)e + ed^2(e) + (d(e))^2.$$

All together we have shown the following result.

Theorem 4. *$d + e$ defines a deformation of the 3-dga A into a 4-dga if and only if*

$$(e^4 + e^2d(e) + d(e)e^2 + d^2(e)e + ed^2(e) + (d(e))^2) + 2(e^2 + d(e))d^2 = 0.$$

Infinitesimal deformations

Let t be a formal parameter such that $t^2 = 0$.

Theorem 5. *Let (A, d) be a N-dga and e a degree one derivation on A, then we have*

$$(d + te)^N = t\sum_{k=0}^{N-1}\left(\sum_{p\in Par(k,N-k+1)}(-1)^{w(p)}\right)d^{N-k-1}(e)d^{N-k-1},$$

where

$$Par(k, N-k+1) = \{p = (p_1, \cdots, p_{N-k+1}) \mid \sum_{i=1}^{N-k+1} p_i = k\} \text{ and } w(p) = \sum_{i=1}^{N-k+1}(i-1)p_i.$$

Proof. From Theorem 2 we know that $D^N = \sum_{k=0}^{N-1} c_k d^k$. Since $t^2 = 0$, then

$$(te)^{(s)} = (te)^{(s_1)} \cdots (te)^{(s_{l(s)})} = t^{l(s)}e^{(s)} = 0$$

unless $l(s) \leq 1$. On the other hand we have that

$$E_N = \{(0), (1), \cdots, (N-1)\}.$$

Suppose that $l(s) = 1$ and $N(s) = N - |s| - l(s) = k$, thus $|s| = N - k - 1$. The unique vector s in E_N of length 1 such that $|s| = N - k - 1$ is $s = (N - k - 1)$. Therefore

$$c_k = \sum_{\substack{s\in E_N\\ N(s)=k\\ s_i<M}} c(s,N)e^{(s)} = c((N-k-1),N)e^{(s)} = c((N-k-1),N)d^{N-k-1}(e).$$

A path from $\emptyset$ to $(N - k - 1)$ of length N must be of the form

$$\emptyset\underbrace{\rightarrow\cdots\rightarrow}_{p_1}\emptyset \rightarrow (0)\underbrace{\rightarrow\cdots\rightarrow}_{p_2}(0) \rightarrow (1)\underbrace{\rightarrow\cdots\rightarrow}_{p_3}(1) \rightarrow \cdots \rightarrow (N-k-1)\underbrace{\rightarrow\cdots\rightarrow}_{p_{N-k+1}}(N-k-1)$$

with $(p_1+1)+(p_1+1)+\cdots+(p_{N-k}+1)+p_{N-k+1} = N$, i.e., $\sum_{i=1}^{N-k+1} p_i = k$. The weight of such path is

$$(-1)^{0p_1}(-1)^{(2-1)p_2}(-1)^{(3-1)p_2}...(-1)^{(N-k)p_{N-k+1}} = (-1)^{w(p)}, \quad \text{thus we get that}$$

$$c((N-k-1),N) = \sum_{\gamma\in P_N(\emptyset,s)} \omega(\gamma) = \sum_{p\in Par(k,N-k+1)} (-1)^{w(p)}.$$

■

Corollary 1. *e defines an infinitesimal deformation of the N-dga (A,d) into the N-dga $(A,d+e)$ if and only if*

$$\sum_{k=0}^{N-1}\left(\sum_{p\in Par(k,N-k+1)} (-1)^{w(p)}\right) d^{N-k-1}(e)d^{N-k-1} = 0. \tag{3.1}$$

4 N Lie algebroids

In this section we introduce the notion of N Lie algebroids and construct examples of such structures. We first review the notion of Lie algebroids, provide some examples, and write the definition of Lie algebroids in a convenient way for our purposes.

Lie algebroids

We review basic ideas around the notion of Lie algebroids; the interested reader will find much more information in [12, 23, 27]. The notion of Lie algebroids has gained much attention in the last few years because of its interplay with various branches of mathematics and theoretical physics, see [10, 11, 17]. We center our attention on the basic definitions and constructions of Lie algebroids and its relation with graded manifolds and differential graded algebras.

Definition 4. A Lie algebroid is a vector bundle $\pi : E \longrightarrow M$ together with:

- A Lie bracket $[\ ,\ ]$ on the space $\Gamma(E)$ of sections of E.
- A vector bundle map $\rho : E \longrightarrow TM$ over the identity, called the anchor, such that the induced map $\rho : \Gamma(E) \longrightarrow \Gamma(TM)$ is a Lie algebra morphism.
- The identity $[v, fw] = f[v,w] + (\rho(v)f)w$ must hold for sections v, w of E and f a smooth function on M.

Let $(x^1, ..., x^n)$ be coordinates on a local chart $U \subset M$, and let $\{e_\alpha \mid \alpha = 1,\dots,r\}$ be a basis of local sections of $\pi : E|_U \longrightarrow U$. Local coordinates on $E|_U$ are given by (x^i, y^α). Locally the Lie bracket and the anchor are given by $[e_\alpha, e_\beta]_E = C^\gamma_{\alpha\beta} e_\gamma$ and $\rho(e_\alpha) = \rho^i_\alpha \frac{\partial}{\partial x^i}$, respectively. The smooth functions $C^\gamma_{\alpha\beta}, \rho^i_\alpha$ are the structural functions of the Lie algebroid. The condition for ρ to be a Lie algebra homomorphism is written in local coordinates as

$$\rho^j_\alpha \frac{\partial \rho^i_\beta}{\partial x^j} - \rho^j_\beta \frac{\partial \rho^i_\alpha}{\partial x^j} = \rho^i_\gamma C^\gamma_{\alpha\beta}.$$

The other compatibility condition between ρ and $[\ ,\ ]$ is given by

$$\sum_{\mathrm{cycl}(\alpha,\beta,\gamma)} \left(\rho^i_\alpha \frac{\partial C^\mu_{\beta\gamma}}{\partial x^i} + C^\mu_{\alpha\nu} C^\nu_{\beta\gamma} \right) = 0,$$

where the sum is over indices α, β, γ such that the map $1, 2, 3 \longrightarrow \alpha, \beta, \gamma$ is a cyclic permutation. The simplest examples of Lie algebroids are described below; the reader will find further examples in the references listed at the beginning of this section.

Example. A finite dimensional Lie algebra $\mathfrak{g}$ may be regarded as a vector bundle over a single point. Sections are elements of $\mathfrak{g}$, the Lie bracket is that of $\mathfrak{g}$, and the anchor map is identically zero. The structural functions $C^\gamma_{\alpha\beta}$ are the structural constants $c^\gamma_{\alpha\beta}$ of $\mathfrak{g}$ and $\rho^i_\alpha = 0$.

Example. The tangent bundle $\pi : TM \longrightarrow M$ with anchor the identity map I_{TB} on TB and with the usual bracket on vector fields.

Exterior differential algebra of Lie algebroids

Sections $\Gamma(\bigwedge E)$ of a Lie algebroid E play the role of vector fields on a manifold and are called E vector fields. Sections of the dual bundle $\pi : E^* \longrightarrow M$ are called E 1-forms. Similarly sections $\Gamma(\bigwedge E^*)$ of $\bigwedge E^*$ are called E forms. The degree of a E form in $\Gamma(\bigwedge^k E^*)$ is k. Let us state and sketch the proof of a result of fundamental importance for the rest of this work.

Theorem 6. *Let E be a vector bundle. E is a Lie algebroid if and only if $\Gamma(\bigwedge E^*)$ is a differential graded algebra. A differential on $\bigwedge E^*$ is the same as a degree one vector field v on $E[-1]$ such that $v^2 = 0$.*

Above $E[-1]$ denotes the graded manifold whose underlying space is E with fibers placed in degree one. If E is a Lie algebroid one defines a differential

$$d : \Gamma(\wedge^k E^*) \longrightarrow \Gamma(\wedge^{k+1} E^*)$$

as follows:

$$\begin{aligned} d\theta(v_1, \ldots, v_{k+1}) &= \sum_i (-1)^{i+1} \rho(v_i) \theta(v_1, \ldots, \widehat{v}_i, \ldots, v_{k+1}) \\ &+ \sum_{i<j} (-1)^{i+j} \theta([v_i, v_j], v_1, \ldots, \widehat{v}_i, \ldots, \widehat{v}_j, \ldots v_{k+1}), \end{aligned}$$

for $v_1, \ldots, v_{k+1} \in \Gamma(E)$. The axioms for a Lie algebroid imply that:

1. $d^2 = 0$;
2. If $f \in C^\infty(M)$ and $v \in \Gamma(E)$, then $\langle df, v \rangle = \rho(v) f$;
3. d is a derivation of degree 1, i.e., $d(\theta \wedge \zeta) = d\theta \wedge \zeta + (-1)^{\overline{\theta}} \theta \wedge \zeta$.

Conversely, assume that d is a degree one derivation on $\Gamma(\bigwedge E^*)$ satisfying $d^2 = 0$. Then E is a Lie algebroid with the structural maps ρ and $[\ ,\ ]$ given by

$$\begin{aligned} \rho(v)f &= df(v) \\ \theta([v,w]) &= \rho(v)\theta(w) - \rho(w)\theta(v) - d\theta(v,w), \end{aligned}$$

for $v, w \in \Gamma(E)$, $f \in C^\infty(M)$ and $\theta \in \Gamma(\bigwedge^1 E)$. In local coordinates d is determined by

$$dx^i = \rho^i_\alpha\, e^\alpha \text{ and } de^\gamma = C^\gamma_{\alpha\beta}\, e^\alpha \wedge e^\beta,$$

where $\{e^\alpha \mid \alpha = 1, \dots, r\}$ is the dual basis of $\{e_\alpha \mid \alpha = 1, \dots, r\}$. It is not hard to see that the conditions $d^2x^i = 0$ and $d^2e^\alpha = 0$ are equivalent to the structural equations defining a Lie algebroid. Let us compute the exterior algebra of a few Lie algebroids.

Example. To the trivial Lie algebroid structure on a vector bundle E corresponds to the exterior algebra $\bigwedge E^*$ with vanishing differential.

Example. Chevalley-Eilenberg differential on $\bigwedge \mathfrak{g}^*$ arises from the Lie algebroid $\mathfrak{g} \longrightarrow \{\bullet\}$ of Example 4. The Chevalley-Eilenberg differential d takes the form

$$d\theta(v_1, \dots, v_{k+1}) = \sum_{i<j} (-1)^{i+j}\theta([v_i, v_j], v_1, \dots, \widehat{v_i}, \dots, \widehat{v_j}, \dots v_{k+1}),$$

for $v_i \in \mathfrak{g}$ and $\theta \in \bigwedge \mathfrak{g}^*$.

Example. The differential associated with the tangent bundle $TM \longrightarrow M$ Lie algebroid is de Rham differential.

N Lie algebroids

We are ready to introduce the main concept of this section. In the light of Theorem 6 it is rather natural to define a N Lie algebroid as a vector bundle E together with a degree one N-nilpotent vector field v on the graded manifold $E[-1]$. That definition, useful as it might be, rules out some significant examples that we would not like to exclude, thus, we prefer the more inclusive definition given below. Though not strictly necessary for our definition of N Lie algebroids, the study of nilpotent vector fields on graded manifolds is of independent interest, and we shall say a few words about them. Indeed our next result gives an explicit formula for the N-th power of a graded vector field.

Let $x^1, ..., x^m$ be local coordinates on a graded manifold and $\partial_1, ..., \partial_m$ be the corresponding vector fields. We recall that if x_i is a variable of degree $\overline{x}_i$, then ∂_i is of degree $-\overline{x}_i$, and dx_i is of degree $\overline{x}_i + 1$. Let $a^1, ..., a^m$ be functions of homogeneous degree depending on $x^1, ..., x^m$. For L a linearly ordered set and $f : L \longrightarrow [m]$ a map we define

$$\overline{f} = \sum_{i \in L} \overline{f(i)} \text{ and } \partial_f = \prod_{i \in L} \partial_{f(i)}.$$

Also we define the sign $s(f)$ by the rule

$$\partial_f = s(f)\partial_1^{|f^{-1}(1)|} \dots\ \partial_m^{|f^{-1}(m)|}.$$

Let $p : \mathbb{N} \longrightarrow \mathbb{Z}_2$ be the map such that $p(n)$ is 1 if n is even and -1 otherwise. Using induction on N one can show that:

Theorem 7.

$$(a^i\partial_i)^N = \sum s(f,\alpha)(\prod_{i=1}^{N} \partial_{f|_{\alpha^{-1}(i)}} a^{f(i)})\partial_{f|_{\alpha^{-1}(N+1)\sqcup N}},$$

where the sum runs over $f : [N] \longrightarrow [m]$ *and* $\alpha : [N-1] \longrightarrow [2, N+1]$ *such that* $\alpha(i) > i$. *The sign* $s(f,\alpha)$ *is given by*

$$s(f,\alpha) = p(\sum_{s=1}^{N-1} \sum_{s<j<\alpha(s)} \overline{x}_s\overline{a}^{f(j)} + \overline{x}_s\overline{f}|_{\alpha^{-1}(j)\cap[s+1,N-1]}).$$

Corollary 2.

$$(a^i\partial_i)^N = \sum_I c_I\partial_I,$$

where $I : [m] \longrightarrow \mathbb{N}$ *is such that* $1 \leq |I| := I(1) + ... + I(N) \leq N$, $\partial_I = \prod_{i=1}^m \partial_i^{I(i)}$, *and*

$$c_I = \sum \quad S(f,\alpha)\prod_{i=1}^{N}(\partial_{f(\alpha^{-1}(i))}a^{f(i)})$$

where the sum runs over maps $\alpha : [N-1] \longrightarrow [2, N+1]$ *with* $\alpha(i) > i$ *for* $i \in [N-1]$, *and* $f : [N] \longrightarrow [m]$ *such that* $|\{j \in \alpha^{-1}(N+1) \sqcup \{N\} \quad | \quad f(j) = i\}| = I(i)$, *for* $i \in [m]$. *The sign* $S(f,\alpha)$ *is given by*

$$S(f,\alpha) = s(f,\alpha)s(f|_{\alpha^{-1}(N+1)\sqcup\{N\}}).$$

Corollary 3. $(a^i\partial_i)^N = 0$ *if and only if* $c_I = 0$ *for* I *as above.*

For example for $N = 2$ one gets

$$(a^i\partial_i)^2 = \sum_{i,j} p(\overline{x}_i\overline{a}_j)a_ia_j\partial_i\partial_j + a_i\partial_i(a_j)\partial_j.$$

For $N = 3$ we get that

$$\begin{aligned}(a^i\partial_i)^3 \;&=\; \sum_{i,j,k} a_i\partial_i(a_j)\partial_j(a_k)\partial_k + p(\overline{x}_i\overline{a}_j)a_ia_j\partial_i\partial_j(a_k)\partial_k \\ &+\; p(\overline{x}_i\overline{a}_k)a_ia_j\partial_j(a_k)\partial_i\partial_k + p(\overline{x}_j\overline{a}_k)a_i\partial_i(a_j)a_k\partial_j\partial_k \\ &+\; \overline{x}_i\overline{a}_j)a_ia_j\partial_i(a_k)\partial_j\partial_k + p(\overline{x}_j\overline{a}_k + \overline{x}_i\overline{a}_j + \overline{x}_i\overline{a}_k)a_ia_ja_k\partial_i\partial_j\partial_k.\end{aligned}$$

For $N = 4$ the corresponding expression have 24 terms and we won't spell it out.

We return to the problem of defining N Lie algebroids. We need some general remarks on differential operators on associative algebras. Given an associative algebra A we let $DO(A)$ be the algebra of differential operators on A, i.e., the subalgebra of $End(A)$ generated by $A \subset End(A)$ and $Der(A) \subset End(A)$, the space of derivations of A. Thus $DO(A)$ is generated as a vector space by operators of the form $x_1 \circ x_2 \circ \cdots \circ x_n \in End(A)$ where x_i is in $A \sqcup Der(A)$. Notice that $DO(A)$ admits a natural filtration

$$\emptyset = DO_{\leq -1}(A) \subseteq DO_{\leq 0}(A) \subseteq DO_{\leq 1}(A) \subseteq \cdots \subseteq DO_{\leq k}(A) \subseteq \cdots \subseteq DO(A),$$

where $DO_{\leq k}(A) \subseteq DO(A)$ is the subspace generated by operators $x_1 \circ x_2 \circ \cdots \circ x_n$, where at most k operators among the x_i belong to $Der(A)$. Thus $DO(A)$ admits the following decomposition as graded vector space

$$DO(A) = \bigoplus_{k=0}^{\infty} DO_k(A) := \bigoplus_{k=0}^{\infty} DO_{\leq k}(A)/DO_{\leq k-1}(A).$$

Clearly $DO_0(A) = A$ and if A is either commutative or graded commutative, then

$$DO_1(A) = Der(A).$$

The projection map $\pi_1 : DO(A) \longrightarrow DO_1(A)$ induces a non-associative product

$$\diamond : DO_1(A) \otimes DO_1(A) \longrightarrow DO_1(A)$$

given by $s \diamond t = \pi_1(s \circ t)$ for $s, t \in DO_1(A)$. In particular if A is commutative or graded commutative we obtain a non-associative product

$$\diamond : Der(A) \otimes Der(A) \longrightarrow Der(A).$$

To avoid unnecessary use of parenthesis we assume that in the iterated applications of $\diamond$ we associate in the minimal form from right to left.

Definition 5. A N Lie algebroid is a vector bundle E together with a degree one derivation $d : \Gamma(\bigwedge E^*) \longrightarrow \Gamma(\bigwedge E^*)$, such that the result of N $\diamond$-compositions of d with itself vanishes, i.e., $d \diamond d \diamond \cdots \diamond d = 0$.

The notions of Lie algebroids and 2 Lie algebroids agree; indeed it is easy to check that $d \circ d = d \diamond d$ for any degree one derivation $d : \Gamma(\bigwedge E^*) \longrightarrow \Gamma(\bigwedge E^*)$. Let us now illustrate with an example the difference between the condition $d \circ d \circ \cdots \circ d = 0$ and the much weaker condition $d \diamond d \diamond \cdots \diamond d = 0$. Let $\mathbb{C}[x_1, ..., x_n]$ be the free graded algebra generated by graded variables x_i for $1 \leq i \leq n$. A derivation on $\mathbb{C}[x_1, ..., x_n]$ is a vector field $\partial = \sum a_i \partial_i$ where $a_i \in \mathbb{C}[x_1, ..., x_n]$. The condition $\partial^N = 0$ is rather strong and restrictive, it might be tackled with the methods provided above. In contrast, the condition $\partial \diamond \partial \diamond \cdots \diamond \partial = 0$ is much simpler and indeed it is equivalent to the condition $\partial^N(x_i) = 0$ for $1 \leq i \leq n$.

Definition 6. A N Lie algebra is a vector space $\mathfrak{g}$ together with a degree one derivation d on $\bigwedge \mathfrak{g}^*$ such that the N-th $\diamond$-composition of d with itself vanishes.

Our next result characterizes 3 Lie algebras in more familiar terms. For integers $k_1, k_2, ..., k_l$ such that $k_1 + k_2 + \cdots + k_l = n$, we let $Sh(k_1, k_2, \cdots, k_l)$ be the set of permutations

$$\sigma : \{1, \cdots, n\} \longrightarrow \{1, \cdots, n\}$$

such that σ is increasing on the intervals $[k_i + 1, k_{i+1}]$ for $0 \leq i \leq l$, $k_0 = 1$ and $k_{l+1} = n$. Assume we are given a map $[\ ,\ ] : \bigwedge^2 \mathfrak{g} \longrightarrow \mathfrak{g}$.

Theorem 8. *The pair $(\mathfrak{g}, [\ ,\ ])$ is a 3 Lie algebra if and only if for $v_1, v_2, v_3, v_4 \in \mathfrak{g}$ we have*

$$\sum_{\sigma \in Sh(2,1,1)} sgn(\sigma)[[[v_{\sigma(1)}, v_{\sigma(2)}], v_{\sigma(3)}]v_{\sigma(4)}] = \sum_{\sigma \in Sh(2,2)} sgn(\sigma)[[v_{\sigma(1)}, v_{\sigma(2)}], [v_{\sigma(3)}, v_{\sigma(4)}]],$$

Proof. One can show that a degree one differential on $\bigwedge \mathfrak{g}^*$ is necessarily the Chevalley-Eilenberg operator

$$d\theta(v_1,\ldots,v_{n+1}) = \sum_{i<j}(-1)^{i+j}\theta([v_i,v_j],v_1,\ldots,\widehat{v}_i,\ldots,\widehat{v}_j,\ldots v_{n+1}) \ ,$$

where $[\ \ ,\ \ ] : \bigwedge^2 \mathfrak{g} \longrightarrow \mathfrak{g}$ is an antisymmetric operator. We remark that we are not assuming, at this point, that the bracket $[\ \ ,\ \ ]$ satisfies any further identity. Jacobi identity arises when the square of d is set to be equal to zero, but do not do that since we want to investigate the weaker condition that the third $\diamond$-power of d be equal to zero. For $\theta \in \bigwedge^1 \mathfrak{g}^* = \mathfrak{g}^*$ the Chevalley-Eilenberg operator takes the simple form

$$d\theta(v_1,v_2) = -\theta([v_1,v_2]).$$

Moreover a further application of d to $d\theta$ yields

$$d^2\theta(v_1,v_2,v_3) = \sum_{\sigma\in Sh(2,1)} sgn(\sigma)\theta([[v_{\sigma(1)},v_{\sigma(2)}],v_{\sigma(3)}]).$$

From the last equation it is evident that Jacobi identity is equivalent to the condition $d^2 = 0$. We do not assume that Jacobi identity holds and proceed to compute the third $\diamond$-power of d. We obtain that

$$\begin{aligned} d\diamond d\diamond d\theta(v_1,v_2,v_3,v_4) &= \sum_{\sigma\in Sh(2,1,1)} sgn(\sigma)\theta([[[v_{\sigma(1)},v_{\sigma(2)}],v_{\sigma(3)}]v_{\sigma(4)}]) \\ &- \sum_{\sigma\in Sh(2,2)} sgn(\sigma)\theta([[v_{\sigma(1)},v_{\sigma(2)}],[v_{\sigma(3)},v_{\sigma(4)}]]). \end{aligned}$$

Thus $d\diamond d\diamond d = 0$ if and only if the condition from the statement of the Theorem holds. ■

Using local coordinates $\theta^1,...,\theta^m$ on the graded manifold $\mathfrak{g}[-1]$, it is not hard to show that a vector field of degree one on $\mathfrak{g}[-1]$ can be written as

$$\partial = \frac{1}{2}C^{\gamma}_{\alpha\beta}\theta^{\alpha}\theta^{\beta}\frac{\partial}{\partial\theta^{\gamma}}$$

where the constants $C^{\gamma}_{\alpha\beta}$ may be identified with the structural constants of $[\ \ ,\ \ ]$. The square of the vector field ∂ is given by

$$\begin{aligned} \partial^2 &= \left(\frac{1}{2}C^{\gamma}_{\alpha\beta}\theta^{\alpha}\theta^{\beta}\frac{\partial}{\partial\theta^{\gamma}}\right)\left(\frac{1}{2}C^{\sigma}_{\delta\varepsilon}\theta^{\delta}\theta^{\varepsilon}\frac{\partial}{\partial\theta^{\sigma}}\right) \\ &= \frac{1}{4}C^{\gamma}_{\alpha\beta}C^{\sigma}_{\gamma\varepsilon}\theta^{\alpha}\theta^{\beta}\theta^{\varepsilon}\frac{\partial}{\partial\theta^{\sigma}} - \frac{1}{4}C^{\gamma}_{\alpha\beta}C^{\sigma}_{\delta\gamma}\theta^{\alpha}\theta^{\beta}\theta^{\delta}\frac{\partial}{\partial\theta^{\sigma}} + \frac{1}{2}C^{\gamma}_{\alpha\beta}\theta^{\alpha}\theta^{\beta}C^{\sigma}_{\delta\varepsilon}\theta^{\delta}\theta^{\varepsilon}\frac{\partial}{\partial\theta^{\gamma}}\frac{\partial}{\partial\theta^{\sigma}}. \end{aligned}$$

Using the antisymmetry properties of $C^{\gamma}_{\alpha\beta}$ and the commutation rules for θ^{α} one can write together the first to terms. We find that

$$\partial\diamond\partial = \frac{1}{2}C^{\gamma}_{\alpha\beta}C^{\sigma}_{\gamma\varepsilon}\theta^{\alpha}\theta^{\beta}\theta^{\varepsilon}\frac{\partial}{\partial\theta^{\sigma}}\,.$$

The condition $\partial \diamond \partial = 0$ is equivalent to Jacobi identity. We assume that $\partial \diamond \partial \neq 0$ and proceed to compute consider the condition $\partial \diamond \partial \diamond \partial = 0$. We have that

$$\partial \circ (\partial \diamond \partial) = \left(\frac{1}{2}C^{\nu}_{\lambda\,\mu}\theta^{\lambda}\theta^{\mu}\frac{\partial}{\partial\theta^{\nu}}\right)\left(\frac{1}{2}C^{\gamma}_{\alpha\,\beta}C^{\sigma}_{\gamma\,\varepsilon}\theta^{\alpha}\theta^{\beta}\theta^{\varepsilon}\frac{\partial}{\partial\theta^{\sigma}}\right).$$

Using carefully the properties of $C^{\gamma}_{\alpha\,\beta}$ and θ^{α} we find that

$$\begin{aligned}\partial \circ (\partial \diamond \partial) &= \frac{1}{2}C^{\nu}_{\lambda\,\mu}C^{\gamma}_{\nu\,\beta}C^{\sigma}_{\gamma\,\varepsilon}\theta^{\lambda}\theta^{\mu}\theta^{\beta}\theta^{\varepsilon}\frac{\partial}{\partial\theta^{\sigma}}\\ &+ \frac{1}{4}C^{\nu}_{\lambda\,\mu}C^{\gamma}_{\alpha,\beta}C^{\sigma}_{\gamma\,\nu}\theta^{\lambda}\theta^{\mu}\theta^{\alpha}\theta^{\beta}\frac{\partial}{\partial\theta^{\sigma}}\\ &+ \frac{1}{4}C^{\nu}_{\lambda\,\mu}C^{\gamma}_{\alpha\,\beta}C^{\sigma}_{\gamma\,\varepsilon}\theta^{\lambda}\theta^{\mu}\theta^{\alpha}\theta^{\beta}\theta^{\varepsilon}\frac{\partial}{\partial\theta^{\nu}}\frac{\partial}{\partial\theta^{\sigma}}.\end{aligned}$$

Therefore we have shown that

$$\partial \diamond (\partial \diamond \partial) = \left(\frac{1}{2}C^{\nu}_{\lambda\,\mu}C^{\gamma}_{\nu\,\beta}C^{\sigma}_{\gamma\,\epsilon} - \frac{1}{4}C^{\gamma}_{\lambda\mu}C^{\sigma}_{\gamma\alpha}C^{\alpha}_{\beta\epsilon}\right)\theta^{\lambda}\theta^{\mu}\theta^{\beta}\theta^{\varepsilon}\frac{\partial}{\partial\theta^{\sigma}}.$$

Thus the condition $\partial \diamond (\partial \diamond \partial) = 0$ is equivalent to the following equations for fixed σ:

$$\sum_{\lambda,\mu,\beta,\varepsilon}\left(\frac{1}{2}C^{\nu}_{\lambda\,\mu}C^{\gamma}_{\nu\,\beta}C^{\sigma}_{\gamma\,\epsilon} - \frac{1}{4}C^{\gamma}_{\lambda\mu}C^{\sigma}_{\gamma\alpha}C^{\alpha}_{\beta\epsilon}\right)\theta^{\lambda}\theta^{\mu}\theta^{\beta}\theta^{\varepsilon} = 0\,.$$

Let us now go back to the case of Lie algebroids as opposed to Lie algebras. There is a natural degree one vector field on the graded manifold $T_{[-1]}\mathbb{R}^n$, namely, de Rham differential. We now investigate whether it is possible to deform, infinitesimally, de Rham differential into a 3-differential. In local coordinates $(x_1, \ldots, x_n, \theta_1, \ldots, \theta_n)$ on $T_{[-1]}\mathbb{R}^n$, with x_i of degree zero and θ_i of degree 1, de Rham operator takes the form

$$\partial = \delta^{i}_{\alpha}\theta^{\alpha}\frac{\partial}{\partial x^{i}}.$$

Let t be a formal infinitesimal parameter such that $t^2 = 0$. We are going to show that any set of functions a^i_α of degree zero on $T_{[-1]}\mathbb{R}^n$ determine a deformation of de Rham operator into a 3-$\diamond$ nilpotent operator given by

$$\partial_a = \left(\delta^{i}_{\alpha} + t a^{i}_{\alpha}\right)\theta^{\alpha}\frac{\partial}{\partial x^{i}}.$$

Theorem 9. $\partial_a \diamond \partial_a = t\,\dfrac{\partial a^{j}_{\beta}}{\partial x^{\alpha}}\,\theta^{\alpha}\,\theta^{\beta}\dfrac{\partial}{\partial x^{j}}$ *and* $\partial_a \diamond (\partial_a \diamond \partial_a) = 0$.

Proof.

$$\begin{aligned}\partial_a^2 &= \left(\delta^{i}_{\alpha} + t a^{i}_{\alpha}\right)\theta^{\alpha}\frac{\partial}{\partial x^{j}}\left(\delta^{j}_{\beta} + t a^{j}_{\beta}\right)\theta^{\beta}\frac{\partial}{\partial x^{j}}\\ &= t\frac{\partial a^{j}_{\beta}}{\partial x^{\alpha}}\,\theta^{\alpha}\,\theta^{\beta}\frac{\partial}{\partial x^{j}} + t\left(a^{i}_{\alpha}\delta^{j}_{\beta}\right)\theta^{\alpha}\,\theta^{\beta}\frac{\partial}{\partial x^{i}}\frac{\partial}{\partial x^{j}} + t^2\left(a^{i}_{\alpha}\frac{\partial a^{j}_{\beta}}{\partial x^{i}}\right)\theta^{\alpha}\,\theta^{\beta}\,\frac{\partial}{\partial x^{j}}.\end{aligned}$$

Since $t^2 = 0$ the third term on the right hand side of the expression above vanishes. The second term also vanishes because it is a contraction of even and odd indices. So we get that

$$\partial_a \diamond \partial_a = t \, \frac{\partial a^j_\beta}{\partial x^\alpha} \, \theta^\alpha \, \theta^\beta \frac{\partial}{\partial x^j} \, .$$

The third power of ∂_a is given by

$$\partial_a \diamond (\partial_a \diamond \partial_a) = t \frac{\partial^2 a^j_\beta}{\partial x^\gamma \partial x^\alpha} \theta^\gamma \, \theta^\alpha \, \theta^\beta \frac{\partial}{\partial x^j} = 0.$$

It also vanishes because it includes a contraction of even and odd indices. ■

The nilpotency condition for the operator $\partial_a \diamond \partial_a$ is $\dfrac{\partial a^j_\beta}{\partial x^\alpha} \, \theta^\alpha \, \theta^\beta = 0$ for $j = 1, \ldots, n$. It is not hard to find examples of matrices a^j_β such that $\partial_a \diamond \partial_a = 0$, for example

$$a = \begin{bmatrix} x^1 & \frac{(x^4)^2}{2} & x^1 & x^1 \\ x^2 & x^2 & x^3 & x^2 \\ x^3 & x^3 & x^2 & x^4 \\ x^4 & x^4 \, x^1 & x^4 & x^3 \end{bmatrix} .$$

More importantly there are also matrices a^j_β such that $\partial_a \diamond \partial_a \neq 0$, for example

$$a = \begin{bmatrix} x^1 \, x^4 & x^1 & x^1 & x^1 \\ x^2 & x^2 \, x^4 & x^2 & x^2 \\ x^3 & x^3 & x^3 \, x^4 & x^3 \\ x^4 & x^4 & x^4 & x^1 \, x^4 \end{bmatrix} .$$

We now consider full deformations as opposed to infinitesimal ones. Let

$$\partial_a = \left(\delta^i_\alpha + a^i_\alpha\right) \theta^\alpha \frac{\partial}{\partial x^i}$$

be a vector field. We think of ∂_a as a deformation of de Rham differential with deformation parameters a^i_α.

Theorem 10.

$$\partial_a \diamond (\partial_a \diamond \partial_a) = \left(\delta^l_\gamma + a^l_\gamma\right) \left\{ \frac{\partial a^i_\alpha}{\partial x^l} \frac{\partial a^j_\beta}{\partial x^i} + a^i_\alpha \frac{\partial^2 a^j_\beta}{\partial x^l \partial x^i} \right\} \theta^\gamma \theta^\alpha \theta^\beta \frac{\partial}{\partial x^j}.$$

Proof. Since

$$\begin{aligned} \partial_a^2 &= \left[\left(\delta^i_\alpha + a^i_\alpha\right) \theta^\alpha \frac{\partial}{\partial x^i} \right] \left(\delta^j_\beta + a^j_\beta\right) \theta^\beta \frac{\partial}{\partial x^j} \\ \partial_a \diamond \partial_a &= \left(\delta^i_\alpha + a^i_\alpha\right) \frac{\partial a^j_\beta}{\partial x^i} \theta^\alpha \theta^\beta \frac{\partial}{\partial x^j}, \end{aligned}$$

we get

$$\begin{aligned}
\partial_a \diamond (\partial_a \diamond \partial_a) &= \left(\delta^l_\gamma + a^l_\gamma\right)\theta^\gamma \frac{\partial}{\partial x^l} \diamond \left[\left(\delta^i_\alpha + a^i_\alpha\right)\frac{\partial a^j_\beta}{\partial x^i}\theta^\alpha\theta^\beta \frac{\partial}{\partial x^j}\right] \\
&= \left(\delta^l_\gamma + a^l_\gamma\right)\left\{\frac{\partial a^i_\alpha}{\partial x^l}\frac{\partial a^j_\beta}{\partial x^i} + a^i_\alpha \frac{\partial^2 a^j_\beta}{\partial x^l \partial x^i}\right\}\theta^\gamma\theta^\alpha\theta^\beta \frac{\partial}{\partial x^j}.
\end{aligned}$$

■

Corollary 4. $\partial_a \diamond (\partial_a \diamond \partial_a) = 0$ *if for fixed indices* $\alpha, \beta, \lambda, j$ *the following identity holds*

$$\left(\delta^l_\gamma + a^l_\gamma\right)\left\{\frac{\partial a^i_\alpha}{\partial x^l}\frac{\partial a^j_\beta}{\partial x^i} + a^i_\alpha \frac{\partial^2 a^j_\beta}{\partial x^l \partial x^i}\right\}\theta^\gamma\theta^\alpha\theta^\beta = 0.$$

Corollary 5. *Each matrix* $A = (A^j_\beta) \in M_n(\mathbb{R})$ *such that* $A^2 = 0$ *determines a 3 Lie algebroid structure on* $T\mathbb{R}^n$ *with differential given by* $(\delta^i_\alpha + A^i_\alpha x_\alpha)dx^\alpha \frac{\partial}{\partial x^i}$.

Our final result describes explicitly the conditions defining a 3 Lie algebroid. Let E be a vector bundle over M. A vector field on $E[-1]$ of degree one is given in local coordinates by

$$\partial = \rho^i_\alpha \theta^\alpha \frac{\partial}{\partial x^i} + \frac{1}{2}C^\gamma_{\alpha\beta}\theta^\alpha\theta^\beta \frac{\partial}{\partial \theta^\gamma}$$

where ρ^i_α and $C^\gamma_{\alpha\beta}$ are functions of the bosonic variables only.

Theorem 11. $\partial \diamond (\partial \diamond \partial) = 0$ *if and only if for fixed* γ *and* i *the following identity holds:*

$$\begin{aligned}
&\left[\frac{1}{2}\rho^j_\nu \frac{\partial}{\partial x^j}\left(\rho^i_\beta \frac{\partial C^\gamma_{\sigma\mu}}{\partial x^i}\right) + \frac{1}{2}\rho^j_\nu \frac{\partial (C^\gamma_{\alpha\beta}C^\alpha_{\sigma\mu})}{\partial x^j} + \frac{1}{2}\rho^i_\beta \frac{\partial C^\gamma_{\lambda\mu}}{\partial x^i}C^\lambda_{\mu\sigma} - \frac{1}{4}\rho^i_\beta C^\beta_{\nu\sigma}\frac{\partial C^\gamma_{\lambda\mu}}{\partial x^i} + \right. \\
&\qquad \left. + \left(\frac{1}{2}C^\gamma_{\alpha\beta}C^\alpha_{\lambda\mu}C^\lambda_{\nu\sigma} - \frac{1}{4}C^\alpha_{\beta\mu}C^\gamma_{\alpha\epsilon}C^\epsilon_{\nu\sigma}\right)\right]\theta^\nu\theta^\sigma\theta^\mu\theta^\beta = 0\,,
\end{aligned}$$

$$\begin{aligned}
&\left[\rho^l_\gamma \frac{\partial}{\partial x^l}\left(\rho^j_\nu \frac{\partial \rho^i_\gamma}{\partial x^j}\right) + \frac{1}{2}\left(\rho^l_\sigma \frac{\partial}{\partial x^l}\left(\rho^i_\alpha C^\alpha_{\nu\gamma}\right) + \rho^j_\epsilon \frac{\partial \rho^i_\gamma}{\partial x^j}C^\epsilon_{\sigma\nu} - \right.\right. \\
&\qquad \left.\left. - \rho^j_\gamma \frac{\partial \rho^i_\epsilon}{\partial x^j}C^\epsilon_{\sigma\nu} + \rho^i_\alpha C^\alpha_{\beta\gamma}C^\beta_{\sigma\nu}\right)\right]\theta^\sigma\theta^\nu\theta^\gamma = 0\,.
\end{aligned}$$

Proof. We sketch the rather long proof. For $\partial = \rho^i_\alpha \theta^\alpha \frac{\partial}{\partial x^i} + \frac{1}{2}C^\gamma_{\alpha\beta}\theta^\alpha\theta^\beta \frac{\partial}{\partial \theta^\gamma}$, we have

$$\partial \diamond \partial = \left(\rho^j_\beta \frac{\partial \rho^i_\gamma}{\partial x^j} + \frac{1}{2}\rho^i_\alpha C^\alpha_{\beta\gamma}\right)\theta^\beta\theta^\gamma \frac{\partial}{\partial x^i} + \left(\frac{1}{2}\rho^i_\beta \frac{\partial C^\gamma_{\lambda\mu}}{\partial x^i} + \frac{1}{2}C^\gamma_{\alpha\beta}C^\alpha_{\lambda\mu}\right)\theta^\lambda\theta^\mu\theta^\beta \frac{\partial}{\partial \theta^\gamma}.$$

As in the previous theorem one finds that the condition $\partial \diamond (\partial \diamond \partial) = 0$ is equivalent to the following identities

$$\begin{aligned}
&\left[\frac{1}{2}\rho^j_\nu \frac{\partial}{\partial x^j}\left(\rho^i_\beta \frac{\partial C^\gamma_{\sigma\mu}}{\partial x^i}\right) + \frac{1}{2}\rho^j_\nu \frac{\partial (C^\gamma_{\alpha\beta}C^\alpha_{\sigma\mu})}{\partial x^j} + \frac{1}{2}\rho^i_\beta \frac{\partial C^\gamma_{\lambda\mu}}{\partial x^i}C^\lambda_{\mu\sigma} - \frac{1}{4}\rho^i_\beta C^\beta_{\nu\sigma}\frac{\partial C^\gamma_{\lambda\mu}}{\partial x^i} + \right. \\
&\qquad \left. + \left(\frac{1}{2}C^\gamma_{\alpha\beta}C^\alpha_{\lambda\mu}C^\lambda_{\nu\sigma} - \frac{1}{4}C^\alpha_{\beta\mu}C^\gamma_{\alpha\epsilon}C^\epsilon_{\nu\sigma}\right)\right]\theta^\nu\theta^\sigma\theta^\mu\theta^\beta \frac{\partial}{\partial \theta^\gamma} = 0\,,
\end{aligned}$$

and

$$\left[\rho^l_\gamma \frac{\partial}{\partial x^l}\left(\rho^j_\nu \frac{\partial \rho^i_\gamma}{\partial x^j}\right) + \frac{1}{2}\left(\rho^l_\sigma \frac{\partial}{\partial x^l}\left(\rho^i_\alpha C^\alpha_{\nu\gamma}\right) + \rho^j_\epsilon \frac{\partial \rho^i_\gamma}{\partial x^j} C^\epsilon_{\sigma\nu} - \rho^j_\gamma \frac{\partial \rho^i_\epsilon}{\partial x^j} C^\epsilon_{\sigma\nu}\right) + \right.$$

$$\left. + \frac{1}{2}\,\rho^i_\alpha C^\alpha_{\beta\gamma} C^\beta_{\sigma\nu}\right] \theta^\sigma \theta^\nu \theta^\gamma \frac{\partial}{\partial x^i} = 0.$$

■

Needless to say further research is necessary in order to have a better grasp of the meaning and applications of the notion of N Lie algebroids. We expect that this approach will lead towards new forms of infinitesimal symmetries, and for that reason alone it should find applications in various problems in mathematical physics. In our forthcoming work [3] we are going to discuss some applications of N Lie algebroids in the context of Batalin-Vilkovisky algebras and the master equation.

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