\( q \to \infty \) limit of the quasitriangular WZW model

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Abstract
We study the \( q \to \infty \) limit of the \( q \)-deformation of the WZW model on a compact simple and simply connected target Lie group. We show that the commutation relations of the \( q \to \infty \) current algebra are underlied by certain affine Poisson structure on the group of holomorphic maps from the disc into the complexification of the target group. The Lie algebroid corresponding to this affine Poisson structure can be integrated to a global symplectic groupoid which turns out to be nothing but the phase space of the \( q \to \infty \) limit of the \( q \)-WZW model. We also show that this symplectic groupoid admits a chiral decomposition compatible with its (anomalous) Poisson-Lie symmetries. Finally, we dualize the chiral theory in a remarkable way and we evaluate the exchange relations for the \( q \to \infty \) chiral WZW fields in both the original and the dual pictures.

1 Introduction
The goal of the present paper is to study the \( q \to \infty \) limit of the quasitriangular WZW model [9], which is the \( q \)-deformation of the standard WZW model [21] (the standard WZW model corresponds to the limit \( q \to 1 \). It will turn out that in the \( q \to \infty \) limit the quasitriangular WZW model simplifies considerably, while enjoying chiral decomposability and other agreeable features of its finite \( q \) analogues. In particular, we shall find that the elliptic \( r \)-matrices, which characterize the chiral exchange relations, become trigonometric in the \( q \to \infty \) limit. On the other hand, we shall show, rather remarkably, that despite of this simplification, the symmetry pattern of the \( q \to \infty \) model is richer and more intricate than in the case of finite \( q \).

An important tool for analyzing the rich structure of the quasitriangular WZW model is the theory of affine Poisson groups and the associated concept of anomalous Poisson-Lie symmetry of dynamical systems. Affine Poisson groups have been introduced by Dazord and Sondaz [3] as generalizations of Poisson-Lie groups. To every affine Poisson bivector \( \Pi^* \) on a Lie group \( G^* \) there are associated two Poisson-Lie bivectors \( \Pi^*_L, \Pi^*_R \) on \( G^* \). We shall denote by \( G_L \) and \( G_R \) the dual Poisson-Lie groups of the Poisson-Lie groups \( (G^*, \Pi^*_L) \) and \( (G^*, \Pi^*_R) \), respectively. In [13, 14], Lu has pointed out the following fact: if \( \mu \) is a smooth Poisson map from a symplectic manifold \( P \) into \( (G^*, \Pi^*) \) then there exists a pair

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of (infinitesimal) Poisson-Lie symmetries of $P$ with the symmetry groups $G_L$ and $G_R$. If $\Pi^*$ is itself a Poisson-Lie bivector, then $\Pi^* = \Pi^*_L = \Pi^*_R$ and $G_L = G_R \equiv G$. The Poisson map $\mu : P \to (G^*, \Pi^*)$ is then said to be an \textit{equivariant} moment map of the pair of the $G$ Poisson-Lie symmetries of $P$. Note that we still speak about the \textit{pair} of symmetries because even in the equivariant case the action of the symmetry group $G_L = G$ on $P$ need not coincide with the action of $G_R = G$ on $P$. If $\Pi^*$ is Poisson but not a Poisson-Lie structure, then we say that the Poisson map $\mu : P \to (G^*, \Pi^*)$ is an \textit{anomalous} moment map generating the pair of anomalous $G_L$ and $G_R$ Poisson-Lie symmetries of $P$.

We remark that even in the anomalous case, $G_L$ and $G_R$ may be isomorphic as the Lie groups. This happens for the quasitriangular WZW model if $q$ is finite. (In the limit $q \to 1$, the symmetry structure of the model simplifies even more: not only $G_L$ is isomorphic to $G_R$ but also the $G_L = G$ action on the phase space $P$ of the model coincides with the $G_R = G$ action.) We shall show in this article that, at the opposite end of the range of the deformation parameter $q$, the full structural richness of the concept of the anomalous Poisson-Lie symmetry is realized. We mean by this that in the limit $q \to \infty$ the group $G_L$ is no longer isomorphic to the group $G_R$. This fact leads to an interesting duality in the $q \to \infty$ WZW theory which interchanges the roles of the symmetry groups $G_L$ and $G_R$ in the description of the dynamics of the model.

The plan of the paper is as follows. In Sections 2 and 3, we review the elements of the theory of the affine Poisson groups and of the anomalous Poisson-Lie symmetries. In particular, we shall identify the moment maps of the anomalous Poisson-Lie symmetries of the symplectic groupoids integrating the affine Poisson groups. In Section 4, we turn our attention to the Poisson-Lie anomalies of loop group symmetries. The phase space of the $q \to \infty$ limit of the quasitriangular WZW model [9] will be shown to have the structure of the symplectic groupoid integrating certain affine Poisson structure on the group $G^*$ whose elements are holomorphic maps from the disc into the complexification of the target group of the WZW model. We shall also establish the chiral decomposition of the $q \to \infty$ WZW theory and show that the chiral model is also Poisson-Lie symmetric. Moreover, we shall work out in detail the exchange (braiding) relations and $q \to \infty$ current algebra relations in the chiral sector. On the top of it, we shall describe also the remarkable duality of the chiral $q \to \infty$ WZW model which permits to express the symplectic structure of the model in terms of the two dual groups $G_L$ and $G_R$ naturally associated to the affine Poisson structure on $G^*$.

## 2 Affine Poisson groups

The affine Poisson groups were introduced by Dazord and Sondaz [3] and the basic facts about them can be found in [3, 13, 22]. We give here a short summary of some of the results contained in those papers. Note that all manifolds, maps or sections of bundles will be understood to be smooth, moreover, slightly abusing terminology, we shall often speak about e.g. vectors and forms instead of vector fields and form fields, respectively.

A manifold $P$ equipped with a bivector $\Pi$ is called Poisson if the following bracket defines
a Lie algebra commutator on the space of functions on $P$

$$\{x, y\} = \Pi(dx, dy).$$

Here $x, y$ are any functions on $P$. Note that a map $\mu$ from another Poisson manifold $P'$ into $P$ is called Poisson if it intertwines the corresponding Poisson brackets, i.e.

$$\mu^*\{x, y\} = \{\mu^*x, \mu^*y\}'.$$

It is well-known [4, 16, 12, 3] that the Poisson bivector $\Pi$ induces a Lie algebra structure also on the space consisting of 1-form fields on $P$. It is defined by the following formula [3]:

$$\{\alpha_1, \alpha_2\} = L_{\Pi(\alpha_1, \cdot)}\alpha_2 - L_{\Pi(\alpha_2, \cdot)}\alpha_1 - d(\Pi(\alpha_1, \alpha_2)), \quad (2.1)$$

where $L_v\alpha$ denotes the Lie derivative of the 1-form $\alpha$ with respect to the vector field $v$.

**Definition 1.** Let $R(G^*)$ be the space of right-invariant 1-forms on a group manifold $G^*$. An affine Poisson structure on $G^*$ is a Poisson bivector $\Pi^*$ such that the bracket (2.1) defines the Lie algebra structure on $R(G^*)$. When the affine Poisson bivector vanishes at the unit element $e^* \in G^*$, the group $(G^*, \Pi^*)$ is called the Poisson-Lie group.

**Lemma 1.** A Poisson bivector $\Pi^*$ on $G^*$ defines an affine Poisson structure if and only if the bracket (2.1) defines a Lie algebra structure on the space $L(G^*)$ of left-invariant 1-forms on $G^*$.

We remark that both spaces $R(G^*)$ and $L(G^*)$ can be naturally identified with the dual vector space $\mathfrak{g}$ of the Lie algebra $G^*$ of $G^*$. Thus the affine Poisson structure on $G^*$ defines two (not necessarily) isomorphic Lie algebra structures on $\mathfrak{g}$, we denote them $\mathcal{G}_L = (\mathfrak{g}, [], L)$ and $\mathcal{G}_R = (\mathfrak{g}, [], R)$.

**Lemma 2.** Let $(G^*, \Pi^*)$ be an affine Poisson group and denote by $M$ the value of the bivector $\Pi^*$ at the group unit $e^*$. Then bivectors

$$\Pi^*_L \equiv \Pi^* - L_*M, \quad \Pi^*_R \equiv \Pi^* - R_*M$$

are both Poisson-Lie bivectors on $G^*$ and a bivector

$$\Pi_{op}^* \equiv \Pi^* - R_*M - L_*M$$

is an affine Poisson bivector on $G^*$.

The symbols $L_*$ and $R_*$ stand for the right and left translations from $e^*$ onto the whole group manifold $G^*$. The bivectors $\Pi^*_L$ and $\Pi^*_R$ are referred to as the left and right Poisson-Lie structures associated to the affine Poisson structure $\Pi^*$; $\Pi_{op}^*$ is called the opposed affine Poisson structure with respect to $\Pi^*$. It turns out that $(\Pi_{op}^*)_\text{op} = \Pi^*$ and $\Pi_{op}^*$ (resp. $\Pi^*_L$) is the right (resp. left) associated Poisson-Lie structures to the affine Poisson structure $\Pi_{op}^*$. 
Lemma 3. To every affine Poisson group \((G^*, \Pi^*)\) it can be associated a Lie algebra \(D\) such that:

1) There exist three injective Lie algebra homomorphisms \(\varsigma : G^* \to D\), \(\varsigma_L : G_L \to D\) and \(\varsigma_R : G_R \to D\) such that \(D = \varsigma(G^*) + \varsigma_L(G_L)\) and also \(D = \varsigma(G^*) + \varsigma_R(G_R)\).

2) There is an \(Ad\)-invariant non-degenerate symmetric bilinear form \((.,.)_D\) on \(D\) which vanishes when restricted to each of the subalgebras \(\varsigma(G^*), \varsigma_L(G_L)\) and \(\varsigma_R(G_R)\).

Note that the symbol \(+\) means the direct sum of vector spaces but not necessarily the direct sum of Lie algebras (the latter denoted usually by \(\oplus\)).

The affine Poisson structure on a simply connected group \(G^*\) can be completely reconstructed from the Lie algebra \(D\) by considering a simply connected group \(\hat{D}\) whose Lie algebra is \(D\). The explicite formula for \(\Pi^*\) is as follows

\[
R_{g^*} - \Pi^*(g^*)(\xi_1, \xi_2) = -(Ad_{g^*} - 1)\varsigma_L(\xi_1), p_R Ad_{g^*} - 1 \varsigma_L(\xi_2))_D, \quad g^* \in G^*, \xi_1, \xi_2 \in G.
\]

or, equivalently

\[
L_{g^*} - \Pi^*(g^*)(\xi_1, \xi_2) = -(Ad_{g^*} - 1)\varsigma_R(\xi_2), p_L Ad_{g^*} - 1 \varsigma_R(\xi_1))_D, \quad g^* \in G^*, \quad \xi_1, \xi_2 \in G.
\]

Here \(\varsigma : G^* \to D\) is the Lie group homomorphism integrating the inclusion map \(G^* \hookrightarrow D\) and \(p_R, p_L : D \to D\) are projectors with the kernel \(\varsigma(G^*)\) and the respective images \(\varsigma_R(G_R)\) and \(\varsigma_L(G_L)\). The group \(D\) is called the double of the affine Poisson group \((G^*, \Pi^*)\).

Note that the projectors \(p_L, p_R\) and their respective adjoints \(p_L^*, p_R^*\) with respect to the bilinear form \((.,.)_D\) can be all viewed as elements of \(D \otimes D^*\). Since the dual \(D^*\) can be identified with \(D\) via \((.,)_D\), we can view them also as elements of \(D \otimes D\). In the latter case we denote them as \(P_L, P_R, P_L^*\) and \(P_R^*\), respectively. Obviously, \((P_L - P_L^*)\) and \((P_R - P_R^*)\) are in \(D \wedge D\).

Lemma 4. The following bivector on the group manifold \(D\) is Poisson:

\[
\Pi_D = \frac{1}{2} L_*(P_L - P_L^*) + \frac{1}{2} R_*(P_R - P_R^*).
\]

Moreover, the bivector \(\Pi_D\) is invertible on an open subset \(S\) of elements of \(D\) which can be simultaneously decomposed as products \(\varsigma(u)\varsigma_L(v_L)\) and \(\varsigma_R(v_R)\varsigma(\tilde{u})\) for some \(u, \tilde{u} \in G^*, v_L \in G_L\) and \(v_R \in G_R\).

Remark: The symplectic manifold \((S, \Pi_D)\) is nothing but the so-called global symplectic groupoid integrating the so-called Lie algebroid that corresponds to the affine Poisson structure \((P, \Pi^*)\) (see [13, 15] for more details). We do not describe here the groupoid structure of \(S\) since we shall not need it in our study of the \(q \to \infty\) limit of the quasitriangular WZW model. However, we shall continue to use the term symplectic groupoid in order not to give to the well-known structure a new name.

3 Poisson-Lie symmetry

In this paper, we shall use somewhat abbreviated terminology, by calling the Poisson-Lie symmetry of a Poisson manifold \(P\) what is usually referred to in the literature as the infinitesimal Poisson-Lie symmetry with moment map (cf. [13, 11, 5]). Moreover, we use the results of [13, 14] to rewrite the definition of this concept in the following form:
Definition 2. A Poisson manifold $(P, \Pi)$ is Poisson-Lie symmetric with respect to an affine Poisson group $(G^*, \Pi^*)$ if it exists a Poisson map $\mu : (P, \Pi) \to (G^*, \Pi^*)$.

If $\Pi^*$ in $e^* \vDash \Pi^*$ vanishes (does not vanish), the Poisson-Lie symmetry is called equivariant (anomalous). The Poisson map $\mu$ is referred to as the moment map and, as it is established in the following lemma, it permits to express infinitesimal symmetry transformations in terms of Poisson brackets on the manifold $P$.

Lemma 5. Let $\lambda (\rho)$ be the left (right) invariant Maurer-Cartan form on the group manifold $G^*$. If $\mu : (P, \Pi) \to (G^*, \Pi^*)$ is a Poisson map, then the section $\Pi(\lambda, \mu^* \lambda) \in G^* \otimes TP$ realizes a left $G_L$ action on $P$ and the section $\Pi(\mu^* \rho, \cdot) \in G^* \otimes TP$ a right $G_R$ action on $P$.

We observe that the Poisson-Lie symmetric manifolds always admit the simultaneous actions of two symmetry Lie algebras. However, in the equivariant case, the Lie algebras $G_L$ and $G_R$ are necessarily isomorphic and, moreover, if the affine Poisson group $G^*$ is Abelian, even the $G_L$ and $G_R$ actions on $P$ coincide.

Definition 3. We say that the double $D$ of an affine Poisson group $(G^*, \Pi^*)$ is proper if the images of the group homomorphisms $\varsigma_L, \varsigma_R, \varsigma_P$ are all simply connected and if the unit $e_D$ of $D$ is the unique element of the intersection $\varsigma_L(G_L) \cap \varsigma(G^*)$ and also of the intersection $\varsigma_R(G_R) \cap \varsigma(G^*)$.

The properness of a double $D$ means that every element $K \in S \subset D$ can be unambiguously decomposed in two ways: as $K = \varsigma_R(v_R)\varsigma(u)$, $v_R \in G_R$, $u \in G^*$ and as $K = \varsigma(\tilde{u})\varsigma_L(v_L)$, $v_L \in G_L$, $\tilde{u} \in G^*$. These decompositions obviously define four maps $\Lambda_L : S \to G^*$, $\Lambda_R : S \to G^*$, $\Xi_R : S \to G_R$, $\Xi_L : S \to G_L$ as follows:

$$
\Lambda_L(K) \equiv \tilde{u}, \quad \Lambda_R(K) \equiv u^{-1}, \quad \Xi_L(K) = v_R, \quad \Xi_R(K) = v_L^{-1}.
$$

Theorem 1. The maps $\Lambda_L : (S, \Pi_D) \to (G^*, \Pi_{op}^*)$ and $\Lambda_R : (S, \Pi_D) \to (G^*, \Pi^*)$ are both Poisson.

Proof. We have to show that

$$
\Lambda_L, \Pi_D(K) = \Pi_{op}^*(\Lambda_L(K)).
$$

First we note that $p_L^* + p_L^* = p_R^* + p_R^* \in D^* \otimes D^*$ is nothing but the identity map from $D$ to $D$. It then follows that $P_L + P_L^* = P_R^* + P_R^* \in D^* \otimes D$ is $Ad$-invariant since it is the dual of the bilinear form $(\cdot, \cdot)_D$. Thus from the equality $L_*(P_L + P_L^*) = R_*(P_R + P_R^*)$ we deduce

$$
\Pi_D = L_*(P_L - R_*P_R^*).
$$

From the very definition of the map $\Lambda_L$ it follows that

$$
\Lambda_L R_{L_s} L_{\varsigma_L(v_L)} w = \Lambda_L w, \quad L_*(\Xi_L)^{-1}(u) w = L_*(\Lambda_L w)
$$

for a whatever vector $w \in T_K S$ and whatever elements $v_L \in G_L$ and $u \in G^*$. (Here e.g. $L_K_s$ means the left transport by the element $K \in D$.) By using the relations (3.2), we easily arrive at

$$
\Lambda_L R_{K^*} P_L = L_{\Lambda_L(K)} S \Lambda_L Ad_{L_s} L(\Xi_L^{-1}(K)) P_L, \quad \Lambda_R R_{K^*} P_R^* = L_{\Lambda_L(K)} \Lambda_L Ad_{L_s}(\Lambda_L^{-1}(K) P_R^*.
$$

(3.3)
Denote by $\eta < \mathcal{W}$.

We thus arrive to the announced conclusion

Taking into account Eqs. (3.3), we infer

$$< \Lambda_L \Pi_D(K), L_{\Lambda_L(K)}^* (\xi_1 \otimes \xi_2) >= < \Lambda_L \left( Ad_{\xi_1 (\mathcal{W}_R^{-1} (K))} P_L - Ad_{\xi_2 (\Lambda_L^{-1} (K))} P_R^* \right), \xi_1 \otimes \xi_2 >=$$

$$= 0 - (Ad_{\xi_1 (\Lambda_L(K))} \varsigma_L (\xi_2), p_R Ad_{\xi_2 (\Lambda_L(K))} \varsigma_L (\xi_1)) = < \Pi^*_R (\Lambda_L(K)), L_{\Lambda_L(K)}^* (\xi_1 \otimes \xi_2) > .$$

In a similar manner, we show that

$$\Lambda_R : \Pi_D(K) = \Pi^* (\Lambda_R(K)).$$

$\blacksquare$

**Remark:** We note that our study of the structure of the symplectic groupoids of affine Poisson groups is similar in spirit to the study of the symplectic groupoids of Poisson-Lie groups in [1].

Because the moment map $\Lambda_R$ is Poisson with respect to the affine Poisson structure $\Pi^*$, it simultaneously realizes the right $\mathcal{G}_R$ Poisson-Lie symmetry and the left $\mathcal{G}_L$ Poisson-Lie symmetries. However, because the moment map $\Lambda_L$ is Poisson with respect to the *opposed* affine Poisson structure $\Pi^*_R$, it simultaneously realizes the left $\mathcal{G}_R$ Poisson-Lie symmetry and the right $\mathcal{G}_L$ Poisson-Lie symmetry. In the applications of our general theory, presented in the next chapter, we shall need the explicit formulae for the right Poisson-Lie symmetries of $S$ induced by the moment maps $\Lambda_L$ and $\Lambda_R$.

**Theorem 2.** The section $\Pi (\Lambda_R^* \rho, .)$ generates the infinitesimal version of the natural right $G_R$ action on $D$: $(g_R K) \rightarrow \varsigma_R (g_R^{-1} K)$ for $g_R \in G_R$ and $K \in D$. Similarly, the section $\Pi (\Lambda_R^* \rho, .)$ generates the infinitesimal version of the natural right $G_L$ action on $D$: $(g_R K) \rightarrow K \varsigma_L (g_L)$ for $g_L \in G_L$ and $K \in D$.

**Proof.** Consider a point $K \in S$ and elements $\tilde{\eta} \in D^*, \xi \in \mathcal{G}$. We set $\rho \xi \equiv \langle \rho, \xi \rangle = R^* \xi$ and write

$$< \Pi_D, \Lambda_L^* \rho \xi \otimes R_{K^{-1}}^* \tilde{\eta} >= < L_K, P_L - R_K^* P_{R^*}, \Lambda_L^* \rho \xi \otimes R_{K^{-1}}^* \tilde{\eta} >=$$

$$= < P_L, (R_{\Lambda_L(K)^{-1}} \Lambda_L L_K)^* \xi \otimes (L_K R_{K^{-1}})^* \tilde{\eta} > - < P_{R^*}, (R_{\Lambda_L(K)^{-1}} \Lambda_L L_K)^* \xi \otimes \tilde{\eta} > .$$

Denote by $\eta$ the element of $\mathcal{D}$ which corresponds to $\tilde{\eta} \in D^*$ upon the identification by the bilinear form $\langle ., . \rangle_D$. Then

$$< \Pi_D, \Lambda_L^* \rho \xi \otimes R_{K^{-1}}^* \tilde{\eta} >= < R_{\Lambda_L(K)^{-1}} \Lambda_L L_K)^* \xi, p_L Ad_K \eta > - < (R_{\Lambda_L(K)^{-1}} \Lambda_L L_K)^* \xi, p_{R^*} \eta > .$$

Taking into account Eqs. (3.3), we infer

$$< \Pi_D, \Lambda_L^* \rho \xi \otimes R_{K^{-1}}^* \tilde{\eta} >=$$

$$= (\varsigma_R (\xi), Ad_{\Lambda_L(K)} p_L^* \left( Ad_{\varsigma_L (\mathcal{W}_R^{-1} (K))} p_L Ad_K \eta - Ad_{\varsigma_L (\Lambda_L^{-1} (K))} p_R \eta \right))_D =$$

$$= - (\varsigma_R (\xi), p_{R^*} \eta)_D = - < \tilde{\eta}, \varsigma_R (\xi) >= - < R_K, \varsigma_R (\xi), R_{K^{-1}}^* \tilde{\eta} > .$$

We thus arrive to the announced conclusion

$$\Pi_D (\Lambda_L^* \rho \xi, .) = - R \varsigma_R (\xi).$$ (3.4)
In a similar manner, we show that
\[ \Pi_D(\Lambda^*_R \rho_{k_{*}},,) = L_{*}S_L(\xi). \] (3.5)

In Section 4, we shall need an explicit formula for the symplectic form \( \omega_D \) corresponding to the Poisson bivector \( \Pi_D \) on a proper double \( D \).

**Theorem 3.** Let \( (D, \Pi_D) \) be a proper double and \( S \subset D \) be the corresponding symplectic groupoid. Denote by \( \rho, \rho_L \) and \( \rho_R \) the right-invariant Maurer-Cartan forms on the respective groups \( G^*, G_L \) and \( G_R \). The symplectic form \( \omega_S \) on \( S \) is then given by the following formula
\[ \omega_S = \frac{1}{2}(\Lambda^*_L \rho \wedge \Xi L \rho R) D + \frac{1}{2}(\Lambda^*_R \rho \wedge \Xi R \rho L) D. \] (3.6)

**Proof.** Choose a basis \( t_i \) of \( G^* \) and the basis \( T^i_L \) of \( G_L \) and \( T^i_R \) of \( G_R \) such that
\[ (t_i, T^i_L) = \delta^i_j, \quad (t_i, T^j_R) = \delta^i_j. \] (3.7)

The form \( \omega_S \) can be then rewritten as
\[ \omega_S = \frac{1}{2}(\Lambda^*_L \rho, T^i_R) D \wedge (\Xi L \rho R, t_i) D + \frac{1}{2}(\Lambda^*_R \rho, T^i_L) D \wedge (\Xi R \rho L, t_i) D. \]

We are going to show that the 2-form \( \omega_S \) is the inverse of the Poisson bivector \( \Pi_D \) restricted to \( S \).

Consider a point \( K \in S \) and four linear subspaces of the tangent space \( T_K S \) defined as \( S_L = L_K S_L, S_R = R_K S_R, S_L = L_K S^* \) and \( S_R = R_K S^* \). At every \( K \in S \) (but not necessarily at every \( K \in D \)!) the tangent space \( T_K S \) can be decomposed as \( T_K S = S_L + S_R \) and \( T_K P = S_L + S_R \), respectively. We introduce a projector \( \Pi_{LR} \) on \( S_R \) with a kernel \( S_L \), a projector \( \Pi_{RR} \) on \( S_R \) with a kernel \( S_L \), a projector \( \Pi_{RL} \) on \( S_R \) with a kernel \( S_L \) and a projector \( \Pi_{LL} \) on \( S_L \) with a kernel \( S_L \). Note that the first subscript stands for the kernel and the second for the image. Then we have
\[
< (\Lambda^*_L \rho, T^i_R) D, t >= (R_K T^i_L, \Pi_{LR} t) D, \]
\[
< (\Xi L \rho R, t_i) D, t >= (R_K T^i_L, \Pi_{LR} t) D, \]
\[
< (\Lambda^*_R \rho, T^i_L) D, t >= -(L_K T^i_R, \Pi_{RL} t) D, \]
\[
< (\Xi R \rho L, t_i) D, t >= -(L_K T^i_R, \Pi_{RL} t) D, \]

where \( t \) is a vector at a point \( K \in S \).

Let us show how to demonstrate (3.8-11) on an example (3.8). For \( K \in S \), the vectors \( L_K T^i_L, R_K t_i \) form the basis of the tangent space \( T_K S \). Thus it is sufficient to prove (3.8) for \( t \) being one of the elements of the basis of \( T_K S \). For \( t = L_K T^i_L \), it is obvious that the r.h.s. of (3.8) vanishes. On the other hand, knowing that \( \Lambda_L(D s T^t_L) = \Lambda_L(K) \), we can evaluate the l.h.s.:
\[
< (\Lambda^*_L \rho, T^i_R) D, L_K T^i_L >= < (\rho, T^i_R) D, \Lambda_L(D s T^t_L) >= 0.
\]
For \( t = R_{K^s} t_j \), the r.h.s. of (3.8) gives
\[
(R_{K^s} T_{R^j}^t, \Pi_{L^R} R_{K^s} t_j)_D = (R_{K^s} T_{R^j}^t, R_{K^s} t_j)_D = \delta^j_i.
\]
On the other hand, knowing that \( \Lambda_L(e^{t_j} K) = e^{t_j} \Lambda_L(K) \), we can evaluate the l.h.s.:
\[
< (\Lambda_L^\dagger \rho, T_{R^j}^t)_D, R_{K^s} t_j > = < (\rho, T_{R^j}^t)_D, \Lambda_L, R_{K^s} t_j > = < (\rho, T_{R^j}^t)_D, R_{\Lambda_L(K)} t_j > = (R_{\Lambda_L^{-1}(K)} R_{\Lambda_L(K)} t_j, T_{R^j}^t)_D = (t_j, T_{R^j}^t)_D = \delta^j_i.
\]
By using the relations (3.8-11), we can evaluate the form \( \omega_S \) on any two vectors \( t, u \in T_K P \) in terms of the projectors:
\[
2 \omega_S(t, u) = (R_{K^s} T_{R^j}^t, \Pi_{L^R} u)_D (R_{K^s} t_i, \Pi_{L^R} u)_D + (L_{K^s} T_{R^j}^t, \Pi_{R^L} t_i)_D (L_{K^s} t_i, \Pi_{R^L} u)_D - (t \leftrightarrow u) = (\Pi_{L^R} t_i, \Pi_{L^R} u)_D + (\Pi_{R^L} t_i, \Pi_{R^L} u)_D - (t \leftrightarrow u).
\]
Here \((., .)_D\) is the bi-invariant metric at the point \( K \). By realizing that it holds
\[
(t, \Pi_{L^R} u)_D = (\Pi_{R^L} t_i, \Pi_{L^R} u)_D = (\Pi_{R^L} t, u)_D,
\]
we finally arrive at
\[
\omega_S(t, u) = (t, (\Pi_{L^R} - \Pi_{L^R}) u)_D.
\]
Now we can easily show that \( \omega_S \) is the symplectic form corresponding to the Poisson structure \( \Pi_D \) restricted to \( S \). First of all, we remark that \( \Pi_D \) can be written also as
\[
\Pi_D = L_* (T_{L^j}^t \otimes t_i) - R_* (t_i \otimes T_{R^j}^t).
\]
Then we conclude
\[
\Pi_D(., \omega_S(., u)) =
\]
\[
= L_{K^s} T_{L^j}^t (L_{K^s} t_i, (\Pi_{L^R} - \Pi_{L^R}) u)_D - R_{K^s} t_i (R_{K^s} T_{R^j}^t, (\Pi_{L^R} - \Pi_{L^R}) u)_D =
\]
\[
= (\Pi_{L^L} - \Pi_{R^R}) (\Pi_{L^R} - \Pi_{L^R}) u = (\Pi_{R^R} \Pi_{L^R} - \Pi_{L^L} \Pi_{L^R} + \Pi_{L^L} \Pi_{L^R}) u =
\]
\[
= (\Pi_{L^R} - \Pi_{L^L} \Pi_{L^R} + \Pi_{L^L} \Pi_{L^R}) u = (\Pi_{L^R} + \Pi_{L^L}) u + (\Pi_{L^L} - \Pi_{L^R} + \Pi_{L^L} \Pi_{L^R}) u =
\]
\[
= (\Pi_{L^R} + \Pi_{L^L}) u = u. \quad (3.12)
\]
From the equation (3.12), we learn that the form \( \omega_S \) is invertible and its inverse is nothing but the Poisson bivector \( \Pi_D \) restricted to \( S \). From this it also follows, by the way, that \( \omega_S \) is closed hence symplectic.
4 $q \to \infty$ WZW model on a compact group

4.1 $q \to \infty$ limit of a twisted Heisenberg double

We remind that the quasitriangular WZW model (or the $q$-WZW model for short) is the $q$-deformation of the standard WZW model. It exhibits the Poisson-Lie symmetries with respect to two different (chiral) actions of the polynomial loop group $L_{pol}K$ on the phase space of the model and those symmetries become Hamiltonian in the limit $q \to 1$. We believe that this paper would become too long if we attempted to review here the full structure of the $q$-WZW model, and, in particular, the details of the limit $q \to 1$. Thus, for the sake of economy, we shall point out only few facts concerning the finite $q$ that are indispensable for the good understanding of the $q \to \infty$ limit. The reader, who will feel a need to learn more about the situation for the finite $q$, can consult [9, 10].

The symplectic structure of the quasitriangular WZW model for finite $q$ is that of the twisted Heisenberg double of the complexified polynomial loop group $L_{pol}K$ [10]. The concept of the twisted Heisenberg double $(D, \kappa)$ is due to Semenov-Tian-Shansky [19] and it can be defined for every automorphism $\kappa$ of the double $D$ of a Poisson-Lie group $G^*$ such that $\kappa$ preserves the canonical invariant bilinear form $(.,.)_D$ on $Lie(D)$. Thus $(D, \kappa)$ is a Poisson manifold with the Poisson bivector $\Pi^\kappa_D$ defined as follows

$$\Pi^\kappa_D = \frac{1}{2}L_*(P-P^*) + \frac{1}{2}R_*\kappa_*(P-P^*).$$

(4.1)

In this formula, the elements $P, P^* \in D \wedge D$ are defined as in Lemma 4, but we abandon the subscripts $L, R$ in $P_L, P_R$ since for the Poisson-Lie group $P_L = P_R \equiv P$.

**Remark.** The notions of the symplectic grupoid and of the twisted Heisenberg double are not quite equivalent, inspite of certain similarity between their respective Poisson structures (2.4) and (4.1). However, in some circumstances they become equivalent; e.g. if the automorphism $\kappa$ preserves the subgroup $\varsigma(G^*)$. In this special case, the twisted Heisenberg double $(D, \kappa)$ of a Poisson-Lie group $G^*$ can be interpreted as the symplectic grupoid of certain $\kappa$-dependent affine Poisson structure on $G^*$ for which the original Poisson-Lie structure on $G^*$ is the left-associated Poisson-Lie structure. This particular situation takes place for the finite $q$ quasitriangular WZW model, however, as we shall see, in the limit $q \to \infty$ this is no more the case and the usage of the symplectic grupoids instead of the twisted Heisenberg doubles becomes essential.

Let us view a connected simply connected simple compact Lie group $K$ as a subgroup of the group of unitary matrices of order $n$. Following [18], the group of polynomial loops $L_{pol}K$ consists of matrix valued functions $\gamma(\sigma) \in K$, defined on the circle parametrized by an angular variable $\sigma$, for which there exist non-negative integers $p_+, p_-$ and $n \times n$-matrices $\gamma_k$ such that

$$\gamma(\sigma) = \sum_{k=p_+}^{k=p_-} \gamma_k e^{i k \sigma}.$$  

(4.2)

Denote by $(.,.)_K$ the negative-definite $Ad$-invariant Killing form on the Lie algebra $K \equiv Lie(K)$ and define a non-degenerate $Ad$-invariant bilinear form $(.,.)$ on $Lie(L_{pol}K)$ by the
following formula
\[
(a|b) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\sigma (a(\sigma), b(\sigma))_K.
\]
(4.3)

Denote by \(D\) the complexification \(L_{pol}K^C\) of \(L_{pol}K\) and view it as the real group. We note that the elements of \(D\) are also of the form (4.2), however, for each \(\sigma\), the result of the summation is in \(K^C\) and not just in \(K\). Let \(G^*\) be a subgroup of \(D\), consisting of the elements of \(L_{pol}K^C\) of the form (4.2), for which \(p_-=0\) and \(\gamma_0 \in AN\). Here \(AN\) is the subgroup of \(K^C\) defined by the Iwasawa decomposition \(K^C = KAN\). We remark that the elements of \(G^*\) can be viewed as the boundary values of the holomorphic maps \(\gamma : \{z \in C : |z| \leq 1\} \rightarrow K^C\). The following factorization lemma was proved in [18] and it has a crucial importance for the present paper:

**Lemma 6.** Any element \(l \in D\) can be factorized uniquely as \(l = uv\), \(u \in G^*\) and \(v \in L_{pol}K\) and the product map \(G^* \times L_{pol}K \rightarrow D\) is a diffeomorphism.

**Remark.** In what follows, we shall suppress the symbols \(s_L, s_R\), standing for the group homomorphisms introduced in Sec 2. We do it in order to avoid cumbersome formulae and we hope that the reader will easily reconstitute them from the context.

Let us introduce a nondegenerate bilinear form \((.,.)_D\) on \(D \equiv \text{Lie}(D)\) defined as
\[
(x,y)_D = \frac{1}{\epsilon} \text{Im}(x|y), \quad x,y \in D.
\]
(4.4)

Here \((.|.)\) is just the bilinear form (4.3) naturally extended to \(\text{Lie}(L_{pol}K^C)\), \(\text{Im}\) stands for the imaginary part and \(\epsilon \equiv \ln q\) is the deformation parameter. It is not difficult to establish that the restrictions of the bilinear form \((.,.)_D\) on \(\text{Lie}(L_{pol}K)\) and on \(\text{Lie}(G^*)\) both vanish. For \(\text{Lie}(L_{pol}K)\), it follows from the fact that the bilinear form \((.|.)\) is real when restricted to \(\text{Lie}(L_{pol}K)\) and for \(\text{Lie}(G^*)\), it follows from the fact that the integral of the product of two functions containing only non-negative Fourier modes vanishes unless both functions contain a zero mode. However, the zero mode part of \(\text{Lie}(G^*)\) is \(\text{Lie}(AN)\) and the restriction of \((.,.)_D\) on \(\text{Lie}(AN)\) vanishes (cf. Sec 4.4.1 of [9]).

The data \(D = L_{pol}K^C\), \(G^*, \ L_{pol}K\) and \((.,.)_D\), that we have just introduced, define a Poisson-Lie structure on \(G^*\). This follows from Lemma 3 and Eqs. (2.2), (2.3), for the special case when \(\mathcal{G}_L = \mathcal{G}_R = \text{Lie}(L_{pol}K)\). Let \(\kappa\) denote the following automorphism \(\kappa\) of \(D\), preserving the bilinear form \((.,.)_D\):
\[
\kappa(l)(\sigma) = l(\sigma + i\epsilon), \quad l \in L_{pol}K^C.
\]
(4.5)

The double \((D,\kappa)\) is then nothing but the twisted Heisenberg double of the Poisson-Lie group \(G^*\) and the Poisson bivector (4.1) determines the symplectic structure of the quasitriangular WZW model for finite \(q\). Note also that the integer parameter \(k\), appearing in (4.5), becomes the level of the standard WZW model in the limit \(q \rightarrow 1\).

In what follows, it will be useful to work with an appropriate choice of the basis of \(D = \text{Lie}(L_{pol}K^C)\). We normalize the extension of the Killing-Cartan form \((.,.)_K\) on \(K^C\) in such a way that the square of the length of the longest root is equal to two. We pick an
In what follows, we shall often denote a generic affine step generator as 
\[ \hat{\alpha}^{>}(4.1) \] of the quasitriangular WZW model for a finite 
\( q \) with the help of the respective basis of \( \text{Lie} \). Note that for the roots of the type \( \hat{\alpha}^{>} \), Here by \( \hat{\alpha}^{>} \) stands for the corresponding labels \((\alpha,n)\) and \( \hat{\alpha} \) is such that \( \alpha, \mu \) are arbitrary and \( n > 0 \), or \( \alpha > 0 \) and \( n = 0 \), we say that \( \hat{\alpha} > 0 \).

A basis of the Lie subalgebra \( \text{Lie}(L_{pol} K) \subset \text{Lie}(L_{pol} K^C) \) can be then chosen as \((T_{\hat{\mu}}^{\hat{\alpha}}, B_{\hat{\alpha}}^{\hat{\alpha}}, C_{\hat{\alpha}}^{\hat{\alpha}})\), \( \hat{\alpha} > 0 \) where

\[ T_{\hat{\mu}}^{\hat{\alpha}} = i H^{\hat{\mu}}, \quad B_{\hat{\alpha}}^{\hat{\alpha}} = \frac{i}{\sqrt{2}}(E^{\hat{\alpha}} + E^{-\hat{\alpha}}), \quad C_{\hat{\alpha}}^{\hat{\alpha}} = \frac{1}{\sqrt{2}}(E^{\hat{\alpha}} - E^{-\hat{\alpha}}). \]  

Here by \( -\hat{\alpha} \) we mean \((-\alpha,-n)\) for \( \hat{\alpha} = (\alpha,n) \) and \((\mu,-n)\) for \( \hat{\alpha} = (\mu,n) \). The meaning of the subscript \( L \) will become clear soon.

In a similar manner, a basis of \( \text{Lie}(G^*) \) will be denoted as \((t_\mu,b_{\hat{\alpha}},c_{\hat{\alpha}})\), \( \hat{\alpha} > 0 \) and it reads:

\[ t_\mu = H^{\mu}, \quad b_{\hat{\alpha}} = \frac{|\hat{\alpha}|^2}{\sqrt{2}} E^{\hat{\alpha}}, \quad c_{\hat{\alpha}} = -i \frac{|\hat{\alpha}|^2}{\sqrt{2}} E^{\hat{\alpha}}. \]

Note that for the roots of the type \( \hat{\alpha} = (\mu,n) \), we set \( |\hat{\alpha}|^2 = 2 \).

With the help of the respective basis of \( \text{Lie}(L_{pol} K) \) and of \( \text{Lie}(G^*) \), the Poisson structure (4.1) of the quasitriangular WZW model for a finite \( q \) can be rewritten as

\[ \mathcal{H}_D = L_s \left( T_{\hat{\mu}}^{\hat{\alpha}} \otimes t_\mu + B_{\hat{\alpha}}^{\hat{\alpha}} \otimes b_{\hat{\alpha}} + C_{\hat{\alpha}}^{\hat{\alpha}} \otimes c_{\hat{\alpha}} \right) - R_s \kappa_s \left( t_\mu \otimes T_{\hat{\mu}}^{\hat{\alpha}} + b_{\hat{\alpha}} \otimes B_{\hat{\alpha}}^{\hat{\alpha}} + c_{\hat{\alpha}} \otimes C_{\hat{\alpha}}^{\hat{\alpha}} \right). \]
In this expression, the limit \( q \to \infty \) can be directly performed with the help of Eqs. (4.6) and (4.7). The result is as follows

\[
\Pi^\infty_D = L_s \left( T^\mu_L \otimes t^\mu + B^\alpha_R \otimes b_\alpha + C^\alpha_L \otimes c_\alpha \right) - R_s \left( t^\mu \otimes T^\mu_R + b_\alpha \otimes B^\alpha_R + c_\alpha \otimes C^\alpha_R \right),
\]

where for \( n > 0 \) we define

\[
B^\alpha_R = \frac{i}{\sqrt{2}} E^{-\hat{\alpha}}, \quad C^\alpha_R = -\frac{1}{\sqrt{2}} E^{-\hat{\alpha}}, \quad \hat{\alpha} > 0
\]

and for \( n = 0 \)

\[
T^\mu_R = iH^\mu, \quad B^{(\alpha,0)}_R = \frac{i}{\sqrt{2}} (E^\alpha_0 + E^{-\alpha}_0), \quad C^{(\alpha,0)}_R = \frac{1}{\sqrt{2}} (E^\alpha_0 - E^{-\alpha}_0), \quad \alpha > 0.
\]

With the help of the theory of the affine Poisson groups, it is easy to understand the structure of the Poisson bivector \( \Pi^\infty_D \). In fact, it turns out that it defines a symplectic structure of the symplectic groupoid \( S_\infty \) corresponding to certain affine Poisson structure on the group \( G^* \). To see that, first we introduce a subgroup \( G_R \) of \( D \), consisting of the elements of \( L_{pol}K^C \) of the form (4.2), for which \( p_+ = 0 \) and \( \gamma_0 \in K \). We remark that the elements of \( G_R \) can be viewed as the boundary values of the holomorphic maps \( \tilde{\gamma} : \{ z \in \mathbb{C} \cup \infty : |z| \geq 1 \} \to K^C \). It is not difficult to see that the elements \( (T^\mu_R, B^\alpha_R, C^\alpha_R) \), \( \hat{\alpha} > 0 \) form the basis of the Lie algebra \( \text{Lie}(G_R) \). The data \( D = L_{pol}K^C, G^*, G_L \equiv L_{pol}K \) and \( G_R \) define the affine Poisson structure on \( G^* \), given by Eqs. (2.2),(2.3). Finally, by working out the corresponding bivector (2.4) in the chosen basis on \( \mathcal{G}_L \equiv \text{Lie}(L_{pol}K) \), \( \mathcal{G}_R \equiv \text{Lie}(G_R) \) and \( \mathcal{G}^* \equiv \text{Lie}(G^*) \), we obtain the formula (4.8).

4.2 \( q \to \infty \) current algebras

So far we have worked out the \( q \to \infty \) limit of the Poisson structure of the \( q \)-WZW model and we have established that it coincides with the Poisson structure of the symplectic groupoid of certain affine Poisson structure on the group \( G^* \). In order to establish the Poisson-Lie symmetries of this symplectic groupoid, we wish to show that the double \( D \) is proper in the sense of Definition 3. This is indeed the case because, first of all, the subgroups \( G_L, G_R \) and \( G^* \) are all simply connected (cf. [18]), secondly, because \( G_L \) intersects \( G^* \) only at the unit element \( e_D \in D \) (cf. Lemma 6) and, finally, because any common element of \( G_R \) and \( G^* \) defines a global holomorphic map from the Riemann sphere into \( K^C \) and, therefore, it must be \( \sigma \)-independent element of \( D \) belonging at the same time to \( K \) and to \( AN \). Obviously, only such element is again \( e_D \). We thus see that the hypothesis of Theorem 1 is satisfied and the \( q \to \infty \) limit of the \( q \)-WZW model enjoys a rich structure of the Poisson-Lie symmetries defined with the help of the moment maps \( \Lambda_L, \Lambda_R \) defined on the phase space \( S_\infty \equiv G_R G^* \cap G_L G^* \subset D \). The most interesting feature of this symmetry structure is the fact that the right and the left symmetry Lie algebras \( \mathcal{G}_R \) and \( \mathcal{G}_L \) are not isomorphic. This is a genuinely new phenomenon arising in the \( q \to \infty \) WZW model which has no analogue for any finite \( q \) and, as we shall see, it is in the origin of an interesting duality of the chiral version of the \( q \to \infty \) model.

Before studying the issue of the \( \mathcal{G}_L \to \mathcal{G}_R \) duality, we have to work out several general formulae in our loop group context, in particular we shall need the affine Poisson structures
\( \Pi^* \) and \( \Pi_{op}^* \) on the group \( G^* \). The bivector \( \Pi^* \), with respect to which the map \( \Lambda_R \) is Poisson, is given by the general formula (2.2), while its opposed bivector \( \Pi_{op}^* \), with respect to which the map \( \Lambda_L \) is Poisson, is given by the same formula with the subscripts \( L \) and \( R \) interchanged. As it is costumary in the studies of WZW models, we shall characterize the Poisson bivectors \( \Pi^* \) and \( \Pi_{op}^* \) by Poisson brackets of a set of particular coordinate functions on the group \( G^* \) called the ”Kac-Moody functions”. Actually, these Kac-Moody functions, pull-backed by the moment maps \( \Lambda_L \) and \( \Lambda_R \) to the phase space \( S_\infty = G_RG^{**} \cap G^*G_L = G_RG^* \), are the observables of the WZW model that generate the symmetries via the Poisson brackets on \( S_\infty \), and they are commonly referred to as the ”Kac-Moody” currents. Actually, for \( q = 1 \), the Poisson brackets of the Kac-Moody currents give the standard current algebra commutators; for a finite \( q \), they give the \( q \)-current algebra relations described in [9] and for \( q \to \infty \) they give \( \infty \)-current algebra relations that we are going to work out.

The Kac-Moody functions on the group \( G^* \), the Poisson brackets of which determine the bivectors \( \Pi^* \) and \( \Pi_{op}^* \), are obviously the same as for the finite \( q \). To describe them, we pick an irreducible unitary representation \( \Upsilon \) of the compact group \( \text{current algebra relations described in [9]} \) and for \( \sigma \), we wish to calculate Poisson brackets \( \{ \ldots \} \) of just holomorphic functions cannot fully descri be the Poisson structures \( \Pi^{*\dagger} \) and \( \Pi_{op}^{*\dagger} \). Thus, if \( \Pi^{*\dagger} \) of the matrix valued functions \( \Upsilon_{ij} \), \( (\Upsilon_{ij}^{*\dagger})^{-1} \) and, finally, matrix elements \( ij \) of the element \( l(\sigma) \) in the representation \( \Upsilon \).

The functions \( \Upsilon_{ij} \) are holomorphic. Since we regard \( G^* \) as the real group, the Poisson brackets of just holomorphic functions cannot fully describe the Poisson structures \( \Pi^* \) and \( \Pi_{op}^* \). In fact, we must also consider the antiholomorhic functions \( (\Upsilon_{ij}^{*\dagger})^{-1} \) and calculate their brackets with the holomorphic ones. The calculation is considerably simplified if one uses the notation of the matrix valued Poisson brackets [6, 9]. Thus, if \( V \) is a vector space and \( E \) and \( F \) two \( \text{End}(V) \)-valued functions on \( G^* \), we introduce a matrix Poisson bracket \( \{ E \otimes F \} \) as the \( (\text{End}(V) \otimes \text{End}(V)) \)-valued function on \( G^* \) defined as

\[
\{ E \otimes F \}^{ik,jl} = \{ E^{ij}, F^{kl} \}.
\]

We wish to calculate Poisson brackets \( \{ \ldots \}^* \) or \( \{ \ldots \}^{*\dagger} \) of the matrix valued functions \( \Upsilon_{ij} \), \( (\Upsilon_{ij}^{*\dagger})^{-1} \) restricted on \( G^* \). The calculation is not difficult, it is based on the following obvious relations

\[
\begin{align*}
\langle L_{\sigma} t, d\Upsilon_{ij} \rangle &= (T_{\sigma})^{ik} T(t)^{kj} \\
\langle R_{\sigma} t, d\Upsilon_{ij} \rangle &= T(t)^{ik} T_{\sigma}^{kj}
\end{align*}
\]

for every \( t \in \mathcal{D} \). The result reads:

\[
\{ \Upsilon_{ij} \otimes \Upsilon_{ij'} \}^{*\dagger} = r(\sigma - \sigma') \left( \Upsilon_{ij} \otimes \Upsilon_{ij'} \right) - \left( \Upsilon_{ij} \otimes \Upsilon_{ij'} \right) r(\sigma - \sigma').
\]

\[
\{ \Upsilon_{ij}^{-1} \otimes \Upsilon_{ij'}^{-1} \}^{*\dagger} = r(\sigma - \sigma') \left( \Upsilon_{ij}^{-1} \otimes \Upsilon_{ij'}^{-1} \right) - \left( \Upsilon_{ij}^{-1} \otimes \Upsilon_{ij'}^{-1} \right) r(\sigma - \sigma').
\]

\[
\{ \Upsilon_{ij}^{-1} \otimes \Upsilon_{ij'} \}^{*\dagger} = r(\sigma - \sigma') \left( \Upsilon_{ij}^{-1} \otimes \Upsilon_{ij'} \right) - \left( \Upsilon_{ij}^{-1} \otimes \Upsilon_{ij'} \right) (r + iC).
\]
moreover, we set

\[
\{ \mathcal{Y}_\sigma \triangleleft \mathcal{Y}_{\sigma'} \}^*_L = \{ \mathcal{Y}_\sigma \triangleleft \mathcal{Y}_{\sigma'} \}^*_R = \{ \mathcal{Y}_\sigma \triangleleft \mathcal{Y}_{\sigma'} \}^*; \\
\{ \mathcal{Y}_\sigma^{-1} \oplus \mathcal{Y}_{\sigma'}^{-1} \}^*_L = \{ \mathcal{Y}_\sigma^{-1} \oplus \mathcal{Y}_{\sigma'}^{-1} \}^*_R = \{ \mathcal{Y}_\sigma^{-1} \oplus \mathcal{Y}_{\sigma'}^{-1} \}^*; \\
\{ \mathcal{Y}_\sigma^{-1} \oplus \mathcal{Y}_{\sigma'}^{-1} \}^*_L = (r + iC) \left( \mathcal{Y}_\sigma^{-1} \otimes \mathcal{Y}_\sigma' \right) - \left( \mathcal{Y}_\sigma^{-1} \otimes \mathcal{Y}_\sigma' \right) (r + iC).
\]

(4.12)

Here the canonical \( r \)-matrix and the Casimir element \( C \) are given by the standard expressions

\[
r = \sum_{\alpha > 0} \frac{i|\alpha|^2}{2} (E^{-\alpha} \otimes E^{\alpha} - E^{\alpha} \otimes E^{-\alpha}),
\]

\[
C = \sum_{\mu} H^\mu \otimes H^\mu + \sum_{\alpha > 0} \frac{|\alpha|^2}{2} (E^{-\alpha} \otimes E^{\alpha} + E^{\alpha} \otimes E^{-\alpha}),
\]

moreover, we set

\[
r(\sigma - \sigma') = i \sum_{\mu} (H^\mu \otimes H^\mu)(1 + 2 \sum_{n > 0} e^{in(\sigma - \sigma')}) + \\
+ i \sum_{\alpha > 0} |\alpha|^2 (E^{-\alpha} \otimes E^{\alpha}) \sum_{n > 0} e^{in(\sigma - \sigma')} = r + C \cotg \frac{\sigma - \sigma'}{2}.
\]

(4.13)

For completeness, we evaluate explicitly also the left and the right Poisson-Lie brackets \( \{..\}^*_L \) and \( \{..\}^*_R \) associated to the affine Poisson structure \( \Pi^* \), by using the defining formulae (2.2) and (2.3) with all subscripts \( L, R \) set to \( L \) only, for \( \{..\}^*_L \), and to \( R \) only, for \( \{..\}^*_R \):

\[
\{ \mathcal{Y}_\sigma \triangleleft \mathcal{Y}_{\sigma'} \}^*_L = \{ \mathcal{Y}_\sigma \triangleleft \mathcal{Y}_{\sigma'} \}^*_R = \{ \mathcal{Y}_\sigma \triangleleft \mathcal{Y}_{\sigma'} \}^*; \\
\{ \mathcal{Y}_\sigma^{-1} \oplus \mathcal{Y}_{\sigma'}^{-1} \}^*_L = \{ \mathcal{Y}_\sigma^{-1} \oplus \mathcal{Y}_{\sigma'}^{-1} \}^*_R = \{ \mathcal{Y}_\sigma^{-1} \oplus \mathcal{Y}_{\sigma'}^{-1} \}^*; \\
\{ \mathcal{Y}_\sigma^{-1} \oplus \mathcal{Y}_{\sigma'}^{-1} \}^*_L = (r + iC) \left( \mathcal{Y}_\sigma^{-1} \otimes \mathcal{Y}_\sigma' \right) - \left( \mathcal{Y}_\sigma^{-1} \otimes \mathcal{Y}_\sigma' \right) (r + iC).
\]

Let us define the following Hermitian matrix valued observables on the phase space \( S_\infty = G_R G^* \cap G^* G_L \) of the \( q \to \infty \) WZW model:

\[
L(\sigma) = \Lambda_L^* (\mathcal{Y}_\sigma \mathcal{T}_\sigma'), \\
R(\sigma) = \Lambda_R^* (\mathcal{T}_\sigma' \mathcal{Y}_\sigma).
\]

By using the Poisson properties of the maps \( \Lambda_L : (S_\infty, \Pi_D^\infty) \to (G^*, \Pi_{op}) \) and \( \Lambda_R : (S_\infty, \Pi_D^\infty) \to (G^*, \Pi^*) \), we find out the basic commutation relations of the left and the right \( \infty \)-current algebras:

\[
\{ L(\sigma) \triangleleft L(\sigma') \}^\infty_D = \\
= \left( L(\sigma) \otimes L(\sigma') \right) \left( r + C \cotg \frac{\sigma - \sigma'}{2} \right) + \left( r + C \cotg \frac{\sigma - \sigma'}{2} \right) \left( L(\sigma) \otimes L(\sigma') \right) \\
- \left( L(\sigma) \otimes 1 \right) \left( 1 \otimes L(\sigma') \right) - \left( 1 \otimes L(\sigma') \right) \left( r - iC \right) \left( L(\sigma) \otimes 1 \right).
\]

(4.14)
\begin{align*}
\{ R(\sigma) \otimes R(\sigma') \}_D^\infty &= \left( -\left( R(\sigma) \otimes R(\sigma') \right) \left( r + C \cotg \frac{\sigma - \sigma'}{2} \right) - \left( r + C \cotg \frac{\sigma - \sigma'}{2} \right) \left( R(\sigma) \otimes R(\sigma') \right) \right) \\
&+ \left( R(\sigma) \otimes 1 \right) \left( r - iC \right) \left( 1 \otimes R(\sigma') \right) + \left( 1 \otimes R(\sigma') \right) \left( r + iC \right) \left( R(\sigma) \otimes 1 \right).
\end{align*}

(4.15)

The Poisson brackets (4.14) and (4.15) are the \( q \to \infty \) analogues of the ordinary \( q \to 1 \) left and right Kac-Moody relations (2.66) and (2.67) of Ref. [9]. Actually, for a generic finite \( q \), we have concentrated ourselves in [9] mainly on the left chiral version of the \( q \)-WZW model therefore we have detailed in (1.37) of [9] only the left \( q \)-Kac-Moody brackets. It is in fact the bracket (1.37) of [9] which is the finite \( q \) analogue of our left \( \infty \)-bracket (4.33).

### 4.3 Chiral decomposition

Both the standard WZW model [21, 8, 2] and its \( q \)-deformation [9] admit the so called chiral decomposition. This means, roughly speaking, that the phase space \( S_q \) of the model for each finite \( q \) can be represented as a "square" of a simpler symplectic manifold \( M_q \) which itself enjoys only one half of the full symmetry of \( S_q \). More precisely, by the "square" of a symplectic manifold \( M_q \) we mean the symplectic manifold \( S_q = M_q \times M_q // T \) where the notation \( //T \) means the symplectic reduction by an appropriate action of the Cartan torus \( T \subset K \) on \( M_q \times M_q \). The aim of the section is to show that the chiral decomposition takes place also in the \( q \to \infty \) case. However, there is a novelty: a remarkable duality in the description of the chiral symplectic manifold \( M_\infty \) related to the fact that the groups \( G_R \) and \( G_L \) are not isomorphic. We shall describe this duality in Sec 4.6 and, for the moment, we restrict ourselves to the explicite description of the chiral decomposition of the phase space \( S_\infty = G_R G^* \cap G^* G_L = G_R G^* \). Let us start by formulating and proving an auxiliary theorem on \( \infty \)-Cartan decomposition. We remind that, in order to avoid cumbersome notations, we shall not write explicitly the injection homomorphisms \( \varsigma \), hoping that the reader will restore them easily from the context.

**Theorem 4.** For every element \( s \in S_\infty \), there exist two elements \( k_l, k_r \in G_R \) and an element \( a \in A_+ \) such that

\[ s = k_l a \Xi_R(k_r a). \]

(4.16)

The ambiguity of this decomposition is given by the simultaneous right multiplication \( (k_l, k_r) \mapsto (k_l t, k_r t) \) by any element \( t \) of the Cartan torus \( T \subset K \).

Reciprocally, for every elements \( k_l, k_r \in G_R \) and every element \( a \in A_+ \) the product \( k_l a \Xi_R(k_r a) \) is in \( S_\infty \).

**Proof.** First of all we remind the notation. Thus \( A \) is the (real) subgroup of \( K^C \) given by the Iwasawa decomposition \( K^C = KAN \). Its Lie algebra \( \mathfrak{a} \) consists of the Hermitian elements of the Cartan subalgebra \( \mathfrak{h}^C \) and it is spanned by the elements \( \mathfrak{h}^\nu \) (cf. Sec 4.1). By \( A_+ \) we mean \( \exp A_+ \) where \( A_+ \) is the positive Weyl chamber in \( \mathfrak{a} \). For instance, for the groups \( K^C = SL(n, \mathbb{C}) \), \( A_+ \) consists of diagonal matrices with real positive entries ordered from the biggest one to the smallest one. We remind also that the Lie algebra
$T$ of the Cartan torus $T$ is spanned by the elements $T^\mu = i H^\mu$. Finally, we recall that the decomposition $D = G^* G_L$ is global hence the domain of definition of the map $\Xi_R$ (introduced in Eq. (3.1)), is the whole double $D$.

The (hermitian conjugated version of the) theorem 8.1.1. of [18] says that every element $u \in L_{pol} K^C$ can be decomposed as $u = u_- u_0$ where the Fourier expansion (4.2) of $u_-$ contains only the non-positive modes and $u_0$ is in the subgroup of $L_{pol} K$ consisting of the loops passing through the unit element of $K$ at $\sigma = 0$. Obviously, $u$ can be decomposed also as $u = u_N u' u_0$. Here $u_0$ as is before, $u'$ is in $K^C$ (in this context $K^C$ is viewed as the subgroup of $L K^C$ formed by the constant loops) and the Fourier expansion $u_N$ contains as before only the non-positive modes with the zero mode being equal to the unit element of $K^C$. By the classical theorem about the Cartan decomposition of a simple complex connected and simply connected group $K^C$ (cf. [23], p. 117), we may write $u' = u a u_r^{-1}$, where $u_1, u_r$ are the elements of $K$ and $a$ is in $A_+$. The ambiguity of this classical Cartan decomposition is given by the simultaneous right multiplication $(u_1, u_r) \rightarrow (u t, u_r t)$ by any element $t$ of the Cartan torus $T$. Thus we may decompose $u$ also as

$$u = u_N u a u_r^{-1} u_0.$$  

We note that $u_N u_1$ is an element of the subgroup $G_R$ of $L_{pol} K^C$ defined at the end of Sec 4.1 and $u_r^{-1} u_0$ is the element of the subgroup $G_L = L_{pol} K$. For the moment, we conclude that any $u$ in $L_{pol} K^C$ can be decomposed as

$$u = k_l a g_r^{-1},$$

where $k_l \in G_R$, $a \in A_+$ and $g_r \in G_L$. The ambiguity of this decomposition is given by the simultaneous right multiplication $(k_l, g_r) \rightarrow (k_l t, g_r t)$ by any element $t$ of the Cartan torus $T$.

Suppose now, that the element $u$ is in $S_\infty = G_R G^*$. This means that the element $a g_r^{-1}$ is in the domain of definition of the maps $\Xi_L$ and $\Lambda_R$ (cf. Eq. (3.1)). Set

$$k_r = \left( \Xi_L(a g_r^{-1}) \right)^{-1} \equiv \Xi_L^{-1}(a g_r^{-1}).$$

We see that $k_r$ is an element of $G_R$. Since the domain of definition of the map $\Xi_R$ is the whole double $D$, we can evaluate $\Xi_R(k_r a) \in G_L$:

$$\Xi_R(k_r a) = \Xi_R(\Xi_L^{-1}(a g_r^{-1}) a) = \Xi_R(\Lambda_R(a g_r^{-1}) \Xi_L^{-1}(a g_r^{-1}) a) = \Xi_R(g_r a^{-1} a) = g_r^{-1}.$$

We conclude that for every $u \in S_\infty$ there exist $k_l, k_r \in G_R$ and $a \in A_+$ such that

$$u = k_l a \Xi_R(k_r a).$$

It remains to deal with the ambiguity of this decomposition. When $g_r$ is replaced by $g_r t$, $t \in T$ then $k_r$ is replaced by $\Xi_L^{-1}(t^{-1} a g_r^{-1}) = \Xi_L^{-1}(a g_r^{-1}) t = k_r t$. This proves the first part of the theorem.

Reciprocally, let $k_l, k_r$ be in $G_R$ and $a \in A_+$. In order to show that $k_l a \Xi_R(k_r a)$ is in $S_\infty = G_R G^*$, it is obviously sufficient to show that $a \Xi_R(k_r a)$ is in $S_\infty$. We use the fact that the maps $\Xi_R$ and $\Lambda_L$ are defined everywhere on $D = L_{pol} K^C$ and we write

$$a \Xi_R(k_r a) = a \Xi_R(k_r a) \Lambda_L^{-1}(k_r a) \Lambda_L(k_r a) = a(k_r a)^{-1} \Lambda_L(k_r a) = k_r^{-1} \Lambda_L(k_r a).$$
The element \( k_r^{-1} \Lambda_L(k_r a) \) is evidently in \( G_R G^* \).

We have just established that the phase space \( S_\infty \) can be identified with the manifold \( (G_R \times A_+ \times G_R)/T \). The core of this section is the following theorem expressing the chiral decomposability of the \( q \to \infty \) WZW model:

**Theorem 5.** Parametrize by \( k_l \in G_R \), \( a_l \in A_+ \) the direct product \( M_\infty \equiv G_R \times A_+ \) and define the following 2-form \( \Omega_\infty \) on \( M_\infty \):

\[
\Omega_\infty(k_l, a_l) = \frac{1}{2}(d a_l a_l^{-1} \wedge k_l^{-1} d k_l)_{\mathcal{D}} + \frac{1}{2}(d \Xi_R(k_l a_l) \Xi_R^{-1}(k_l a_l) \wedge a_l^{-1} d a_l + a_l^{-1}(k_l^{-1} d k_l) a_l)_{\mathcal{D}}.
\]

(4.17)

Denote by \( \phi : G_R \times A_+ \times G_R \to S_\infty \) the map induced by the \( \infty \)-Cartan decomposition, i.e.

\[
\phi(k_l, a, k_r) = k_l a \Xi_R(k_r a).
\]

(4.18)

Then the pull-back \( \phi^* \omega_{S_\infty} \) of the grupoid symplectic form (3.6) can be written as

\[
\phi^* \omega_{S_\infty} = \Omega_\infty(k_l, a_l = a) - \Omega_\infty(k_r, a_r = a).
\]

**Proof.** The theorem says, in other words, that the restriction of the form \( \Omega_\infty - \Omega_\infty^l \) on the submanifold of \( M_\infty \times M_\infty \) defined by \( a_l = a_r \) is the same thing as the \( \phi^* \)-pull-back of the anomalous Semenov-Tian-Shansky form from \( S_\infty \) into \( G_R \times A_+ \times G_R \). In order to prove it let \( s \) denote an element of \( S_\infty \). The grupoid symplectic form \( \omega_{S_\infty} \) is given by Eq.(3.6)

\[
\omega_p = \frac{1}{2}(d \Lambda_L(s) \Lambda_L^{-1}(s) \wedge d \Xi_L(s) \Xi_L^{-1}(s))_{\mathcal{D}} + \frac{1}{2}(d \Lambda_R(s) \Lambda_R^{-1}(s) \wedge d \Xi_R(s) \Xi_R^{-1}(s))_{\mathcal{D}}.
\]

(4.19)

By using the \( \infty \)-Cartan parametrization (4.16) and the relations (3.1), we easily infer:

\[
\Lambda_L(s) = \Lambda_L(k_l a) = k_l a \Xi_R(k_l a),
\]

(4.20)

\[
\Lambda_R(s) = \Lambda_R(a \Xi_R(k_r a)) = \Xi_R^{-1}(k_r a) a^{-1} k_r^{-1},
\]

(4.21)

\[
\Xi_L(s) = k_l \Xi_L(a \Xi_R(k_r a)) = k_l k_r^{-1},
\]

(4.22)

\[
\Xi_R(s) = \Xi_R^{-1}(k_r a) \Xi_R(k_l a).
\]

(4.23)

By inserting the expressions (4.20-23) into (4.19), we find immediately

\[
\phi^* \omega_{S_\infty} = \frac{1}{2}(d a a^{-1} \wedge k_l^{-1} d k_l)_{\mathcal{D}} + \frac{1}{2}(d \Xi_R(k_l a) \Xi_R^{-1}(k_l a) \wedge a_l^{-1} d a_l + a_l^{-1}(k_l^{-1} d k_l) a_l)_{\mathcal{D}}
\]

\[
- \frac{1}{2}(d a a^{-1} \wedge k_r^{-1} d k_r)_{\mathcal{D}} - \frac{1}{2}(d \Xi_R(k_r a) \Xi_R^{-1}(k_r a) \wedge a_r^{-1} d a_r + a_r^{-1}(k_r^{-1} d k_r) a_r)_{\mathcal{D}}
\]

\[
= \Omega_\infty(k_l, a_l = a) - \Omega_\infty(k_r, a_r = a).
\]
Corollary. The chiral form $\Omega_{\infty}$ is symplectic. The manifold $S_{\infty}$ can be obtained by the symplectic reduction $S_{\infty} = M_{\infty} \times M_{\infty} // T$, where $T$ is the Cartan torus acting as $(k_1, k_r) \rightarrow (k_1 t, k_r t), t \in T$.

Proof. The closedness of $\Omega_{\infty}$ can be seen from the fact that $\Omega_{\infty}(k_1, a_1)$ is the pull-back of the closed form $\omega_{S_{\infty}}$ under the map $\chi : G_{\infty} \times A_{\infty} \rightarrow S_{\infty}$ given by $\chi(k_1, a_1) = k_1 a_1$. It is slightly more involved to show the non-degeneracy of $\Omega_{\infty}$. First of all we compute the contraction $\iota_v \Omega_{\infty}$ where $v = L_{k_1} T^\mu$. The result is

$$\Omega_{\infty}(., L_{k_1} T^\mu) = (T^\mu, da_l a_{-1})_D = dv^\mu,$$

where we have parametrized $a_l$ as $a_l = e^{\psi_l H^\mu}$. We observe that, whatever is the point $(k_1, a_1)$ in $M_{\infty}$, the vector $L_{k_1} T^\mu$ does not constitute a degeneracy direction of the form $\Omega_{\infty}$. If the form $\Omega_{\infty}$ had at some point a degeneracy vector, then the restriction of the form $\Omega_{\infty}(k_1, a_1) - \Omega_{\infty}(k_r, a_r)$ to the submanifold $a_l = a_r$ would have degeneracy vectors other than $L_{k_1} T^\mu + L_{k_r} T^\mu$. However, this is impossible since $\Omega_{\infty}(k_1, a_1) - \Omega_{\infty}(k_r, a_r)$ is the pull-back of the non-degenerate form $\omega_{P_{\infty}}$.

Having established that $(M_{\infty}, \Omega_{\infty})$ is a symplectic manifold (actually it is going to play the role of the phase space of the chiral $q \rightarrow \infty$ WZW model), it is easy to prove that $S_{\infty} = M_{\infty} \times M_{\infty} // T$. Indeed, due to relative minus sign of the left and the right chiral symplectic forms on each copy of $M_{\infty}$, we see from (4.24) that the Hamiltonian function $\psi^\mu - \psi_{\mu}$ generates the action of the vector $L_{k_1} T^\mu + L_{k_r} T^\mu$ on $M_{\infty} \times M_{\infty}$. The setting $a_l = a_r$ is nothing but saying that $\psi^\mu - \psi_{\mu} = 0$ and we conclude that the anomalous Semenov-Tian-Shansky form $\omega_{S_{\infty}}$ comes indeed from the symplectic reduction $M_{\infty} \times M_{\infty} // T$.

Remark. The proof of the corollary is not rigorous by strictly mathematical standards and shifts the paper to the level of rigour common in mathematical physics. Indeed, the propositions in Sections 2 and 3 were proved only for finite dimensional Lie groups, in particular the result about the non-degeneracy of the grupoid symplectic form (3.6). In our study of the infinite-dimensional loop groups, we still preserve the full mathematical rigour for certain propositions, e.g. Theorems 4 and 5, but in some cases we adopt the usual cavalier approach of mathematical physicists in treating the infinitesimal symplectic geometry of field theories. Thus, in the proof of the corollary, we assumed that the theorems of Sections 2 and 3 remains valid also in the infinite dimensional context.

4.4 Symmetry of the chiral model

There is a natural right action of the group $G_R$ on the chiral phase space $M_{\infty}$ given by $h \triangleright (k_1, a_1) = (h^{-1} k_1, a_1), h \in G_R, (k_1, a_1) \in M_{\infty}$. It can be interpreted as the restriction to $M_{\infty} \subset S_{\infty}$ of the Poisson-Lie symmetric right action $G_R \times S_{\infty} \rightarrow S_{\infty}$ given by the group multiplication on the anomalous double: $h^{-1} \triangleright s = h s, h \in G_R, s \in S_{\infty}$. Our next goal is to prove that also the restricted (or chiral) $G_R$-action is in fact a Poisson-Lie symmetry.

Theorem 6. Denote $\Pi^{\infty}$ the bivector inverse to the chiral symplectic form $\Omega_{\infty}$ and consider the map $\chi : G_R \times A_{\infty} \rightarrow S_{\infty}$ given by $\chi(k_1, a_1) = k_1 a_1$. Then it holds that the composition map $\Lambda_L \circ \chi : (M_{\infty}, \Pi^{\infty}) \rightarrow (G^*, \Pi_{\text{top}})$ is Poisson.
Proof. Consider a pair of functions \( x, y \) defined on \( G^* \), their pull-backs \( \Lambda^*_L x, \Lambda^*_L y \) defined on \( S_\infty \) and \( \chi^* \Lambda^*_L x, \chi^* \Lambda^*_L y \) defined on \( M_\infty \), and also the functions \( \chi^* \Lambda^*_L x \otimes 1, \chi^* \Lambda^*_L y \otimes 1 \) defined on \( M_\infty \times M_\infty \). Consider the submanifold \( O \) of \( M_\infty \times M_\infty \) defined by setting \( a_l = a_r \) and also the map \( \phi : O \to S_\infty \) defined by Eq. (4.18). Then the \( \infty \)-Cartan parametrization (4.16) and the definition (3.1) of the map \( \Lambda_L \) imply that

\[
(\chi^* \Lambda^*_L x \otimes 1)|_O = \phi^* \Lambda^*_L x.
\]

(4.25)

Stated in words, the restriction of \( \chi^* \Lambda^*_L x \otimes 1 \) to \( O \) is the same thing as the \( \phi^* \)-pull-back of the function \( \Lambda^*_L x \) defined on \( S_\infty \). Moreover, we have

\[
\Lambda_L(k_l a_l) = \Lambda_L(k_l a_l) = \Lambda_L(k_l a_l), \quad k_l \in G_R, a_l \in A_+, t \in T.
\]

This relation implies that the function \( \chi^* \Lambda^*_L x \otimes 1 \) is \( T \)-invariant with respect to the Cartan torus action defined in the Corollary of Theorem 5. Since we know that \( S_\infty = (M_\infty \times M_\infty)/T \), we infer

\[
\{\chi^* \Lambda^*_L x \otimes 1, \chi^* \Lambda^*_L y \otimes 1\}_{M_\infty \times M_\infty}|_O = \phi^* \{\Lambda^*_L x, \Lambda^*_L y\}_D \cap \infty,
\]

or, equivalently,

\[
(\{\chi^* \Lambda^*_L x, \chi^* \Lambda^*_L y\}_{M_\infty \times M_\infty} \otimes 1)|_O = \phi^* (\Lambda^*_L x, \Lambda^*_L y)_D \cap \infty.
\]

(4.26)

If we write the identity (4.25) for the function \( \{x, y\}_{op}^* \), Theorem 1 combined with Eq. (4.26) gives

\[
(\{\chi^* \Lambda^*_L x, \chi^* \Lambda^*_L y\}_{M_\infty \times M_\infty} \otimes 1)|_O = (\chi^* \Lambda^*_L \{x, y\}_{op}^* \otimes 1)|_O.
\]

Thus we conclude

\[
\{\chi^* \Lambda^*_L x, \chi^* \Lambda^*_L y\}_{M_\infty} = \chi^* \Lambda^*_L \{x, y\}_{op}^*.
\]

Corollary. The right action of the group \( G_R \) on the \( \infty \)-WZW chiral phase space \( M_\infty \),
given by \( h \triangleright (k, a) = (h^{-1}k, a) \), \( h \in G_R \), is the right Poisson-Lie symmetry corresponding to the moment map \( \Lambda_L \circ \chi \).

Proof. Consider a point \((k, a)\) in \( M_\infty \). The multiplication of \((k, a)\) on the left by an infinitesimal generator \( S \in G_R \) gives the vector \( v_S = (Sk, a) \) and its \( \chi^* \)-push-forward vector \( \chi_* v_S = Ska = R_{(ka)*} S \in T_{(ka)} S_\infty \). Denote also by \( v_S \) and \( \chi_* v_S \) the respective vector fields on \( M_\infty \) and \( S_\infty \) obtained by varying the point \((k, a)\). In the sense of Lemma 5, Theorem 1 and Eq. (3.4) of Sec 4.3, we know that

\[
-\chi_* v_S = \Pi_D^\infty (\Lambda^*_L (\rho, S) D, \ldots).
\]

This relation can be rewritten equivalently as

\[
\omega_{S_\infty} (\ldots, \chi_* v_S) = \Lambda_L (\rho, S) D
\]

(4.27)
\( q \rightarrow \infty \) WZW model

(note that the bivector inverse to the groupoid symplectic form \( \omega_{S_\infty} \) on \( S_\infty \) has been denoted by \( \Pi_\infty^\circ \)). The \( \chi^* \)-pull-back of Eq. (4.27) gives

\[
\Omega_\infty(.,v_S) = \chi^* A_L^* (\rho,S)_D, \tag{4.28}
\]
or, equivalently,

\[
v_S = -\Pi^\infty((\Lambda_L \circ \chi)^*(\rho,S)_D,.). \tag{4.29}
\]

4.5 Exchange relations

Our next goal is to invert the chiral symplectic form \( \Omega_\infty \) on \( M_\infty = G_R \times A_+ \). The strategy, which we shall use, will be that of Sec 5.1.2 of [9]. It consists in exploiting particular properties of the Lie derivatives of Poisson-Lie symmetric bivectors. Let \( u \) be a differential 1-form on \( M_\infty \) and \( v = R_s S, S \in G_R \) a right-invariant vector field on \( G_R \) viewed as the vector field on \( M_\infty \). We calculate the Lie derivate \( \mathcal{L}_v \) of the both sides of the following identity

\[
\Omega_\infty(.,\Pi_\infty(.,u)) = u
\]
and we obtain

\[
(\mathcal{L}_v \Omega_\infty)(.,\Pi_\infty(.,u)) + \Omega_\infty(.,(\mathcal{L}_v \Pi_\infty)(.,u)) = 0.
\]

From Eq. (4.28), we infer

\[
\mathcal{L}_v \Omega_\infty = d(\iota_v \Omega_\infty) = -d(\chi^* A_L^* (\rho,S)_D)) = -\chi^* A_L^* ((\rho,S'_D \wedge (\rho,S''_D), \tag{4.30}
\]
where the map \( S \rightarrow S' \wedge S'' \in \mathcal{G}_R \wedge \mathcal{G}_R \) is the dual to the Lie algebra commutator \([.,.]^* : \mathcal{G}^* \wedge \mathcal{G}^* \rightarrow \mathcal{G}^* \). The last equality follows from the well-known Maurer-Cartan identity

\[
d(R^* S) = R^* S' \wedge R^* S'', \quad S \in (\mathcal{G}^*), \tag{4.31}
\]
because \( \mathcal{G}_R \) can be identified with \( (\mathcal{G}^*)^* \). Finally, by inserting (4.30) into (4.31), we arrive at the following formula

\[
\mathcal{L}_v \Pi_\infty = -v' \wedge v'',
\]
where \( v' \equiv R_s S' \) and \( v'' \equiv R_s S'' \).

With the notations introduced after Lemma 3, introduce the following Poisson-Lie bivector \( \Pi_R \) on the group manifold \( G_R \):

\[
R_{g^{-1}} \Pi_R(g)(t_1,t_2) = -(Ad_{\varsigma_R(g^{-1})}(t_1),p_R Ad_{\varsigma_R(g^{-1})}(t_2))_D, \quad g \in G_R, t_1,t_2 \in \mathcal{G}^* \tag{4.32}
\]
and consider it as the bivector on \( M_\infty \). Then a straightforward calculation gives

\[
\mathcal{L}_v(\Pi_\infty + \Pi_R) = 0,
\]
hence $\Pi^\infty + \Pi_R$ must have the following shape

$$
\Pi^\infty + \Pi_R = \Sigma_{ij}(a)L_*(T_R \wedge T_R^\dagger) + \sigma^\mu(i) \sigma^\nu L_*(T_R \wedge \frac{\partial}{\partial \phi^\mu} + s^\mu \sigma^\nu \frac{\partial}{\partial \phi^\nu}),
$$

where $a = \exp(\sum \phi^\mu H^\mu)$ and $T_R^\dagger$ is a basis in $G_R$. The crucial observation is as follows: since at the group unit $e_R \in G_R$ the Poisson-Lie bivector $\Pi_R$ vanishes, in order to fully determine the unknown functions $\Sigma_{ij}(a), \sigma^\mu(i)$ and $s^\mu$, it is sufficient to invert the almost Darboux-like expression (4.33) and to write (everywhere in $M^\infty$): Here we use the notation

$$a^{2\alpha} \equiv e^{2\alpha(\phi^\mu H^\mu)}.$$

It is easy to invert this almost Darboux-like expression (4.33) and to write (everywhere in $M^\infty$):

$$
\Pi^\infty = -\Pi_{G_R} + L_*(T_R \wedge \frac{\partial}{\partial \phi^\mu} - \sum_{\alpha>0, \nu>0} a^{-2\alpha}|\hat{\alpha}|^2 L_*(B_R^\alpha \wedge C_R^\alpha) - \sum_{\alpha>0, \nu=0} \frac{|\hat{\alpha}|^2}{a^{2\alpha} - 1} L_*(B_R^\alpha \wedge C_R^\alpha).
$$

We wish to calculate the brackets of the principal variables $a \in A_+$ and $k \in G_R$ like $\{a \otimes k(\sigma)a\}_\infty$, $\{k(\sigma) \otimes k(\sigma')a\}_\infty$, $\{k(\sigma) \otimes k(\sigma')^{-1}\}_\infty$, etc. or more precisely, the brackets

$$\{a \otimes \Upsilon_\sigma\}_\infty, \{\Upsilon_\sigma \otimes \Upsilon_\sigma'\}_\infty, \{\Upsilon_\sigma \otimes \Upsilon_\sigma'^{-1}\}_\infty$$

where the matrix valued functions $\Upsilon_\sigma$ (introduced in Sec 4.2) are restricted to the subgroup $G_R \subset L_{pol}K^C$. However, we prefer to make explicit slightly modified Poisson brackets like $\{a \otimes k(\sigma)a\}_\infty$, $\{k(\sigma)a \otimes k(\sigma')a\}_\infty$, $\{k(\sigma) \otimes (k(\sigma')^{-1})\}_\infty$, since they contain equivalent information and are less cumbersome. With the help of Eqs.(4.32),(4.34), (4.9) and (4.10), we find

$$
\{a \otimes k(\sigma)a\}_\infty = -i(a \otimes k(\sigma)a)(H^\mu \otimes H^\mu),
$$

$$
\{k(\sigma)a \otimes k(\sigma')a\}_\infty = (r + C\cotg \frac{\sigma - \sigma'}{2}) \times (k(\sigma)a \otimes k(\sigma')a) - (k(\sigma)a \otimes k(\sigma')a)(r(a) + C\cotg \frac{\sigma - \sigma'}{2}),
$$
\[
\{ (k(\sigma)a)^{1-1} \otimes (k(\sigma')a)^{1-1} \}_\infty = (r + C\cotg \frac{\sigma - \sigma'}{2}) \times \\
- \left( (k(\sigma)a)^{1-1} \otimes (k(\sigma')a)^{1-1} \right) (r(a)) + C\cotg \frac{\sigma - \sigma'}{2},
\]
(4.37)

\[
\{ k(\sigma)a \otimes (k(\sigma')a)^{1-1} \}_\infty = \\
= (r - iC) \left( k(\sigma)a \otimes (k(\sigma')a)^{1-1} \right) - \left( k(\sigma)a \otimes (k(\sigma')a)^{1-1} \right) (r(a)) + C\cotg \frac{\sigma - \sigma'}{2},
\]
(4.38)

where

\[
r(a) = \sum_\alpha \frac{i|\alpha|^2 a^\alpha + a^{-\alpha}}{2} \frac{a^\alpha - a^{-\alpha}}{E^{-\alpha} \otimes E^\alpha}.
\]
(4.39)

Note that the summation in (4.39) runs over all roots \( \alpha \) and not only over the positive roots. It is not difficult to recognize in the expression (4.39) the canonical dynamical \( r \)-matrix associated to a simple Lie algebra (cf. Eq.(5.60) of [9]).

The Poisson brackets (4.35-38) encode full information about the symplectic structure of the chiral \( \infty \)-WZW model and they may be called the exchange relations. However, it is not obvious how to compare them with the exchange relations of the finite \( q \) WZW model (cf. Eq.(5.159) of Ref.[9]). The point is that the \( \infty \)-exchange relations involve the Poisson brackets of certain matrix-valued functions on the group \( G_R \) while the exchange relations for finite \( q \) involve the Poisson brackets of the matrix-valued functions on the group \( G_L = L_{pol}K \). It turns out, however, that the chiral \( \infty \)-WZW model enjoys a remarkable duality which enables to describe its dynamics also in terms of the functions on the group \( G_L \). Thus, in particular, we shall see in the next section that the exchange relations (4.35-38) can be equivalently rewritten in terms of dual exchange relations employing the matrix-valued functions on \( G_L \). It will turn out, rather satisfactorily, that the dual \( \infty \)-exchange relation can be obtained by the direct \( q \to \infty \) limit of the finite \( q \) exchange relations.

### 4.6 Duality

We have learned in Sec 3, that the anomalous Poisson-Lie moment maps realize at the same time the right and the left Poisson-Lie symmetries. In Corollary of Theorem 6, we have worked out the action of the right infinitesimal \( G_R \) symmetry on the phase space \( M_\infty \) of the chiral \( \infty \)-WZW model and we found that it can be lifted to the natural global right action of the group \( G_R \) on itself. More precisely it is given by \( h \triangleright (k,a) = (h^{-1}k,a) \), \( h \in G_R \), \( (k,a) \in G_R \times A_+ = M_\infty \). Consider now the infinitesimal action of the left \( G_L \) Poisson-Lie symmetry generated by the vector fields \( \Pi_\infty(l_1) \) for \( T \in G_L \) and we may ask the following question: Can we parametrize the points of the phase space \( M_\infty \) by using the group \( G_L \) in such a way that the action of the left \( G_L \) symmetry becomes...
just (the infinitesimal version of) the natural action of the group $G_L$ on itself? It turns out that the answer to this question is affirmative, i.e. we shall succeed to represent $M_\infty$ as a submanifold of the direct product $G_L \times A_-$ on which the Lie algebra $\mathcal{G}_L$ (but not the group $G_L$) acts by the right-invariant vector fields on $G_L$. The quantitative basis of this result is the following theorem:

**Theorem 7.** Parametrize by $\tilde{k} \in G_L$, $\tilde{a} \in A_-$ the direct product $G_L \times A_-$, where $A_- = \exp \mathcal{A}_-$ and $\mathcal{A}_- = -\mathcal{A}_+$ is the negative Weyl chambre. Define a 2-form $\tilde{\Omega}_\infty(\tilde{k}, \tilde{a})$ on the submanifold $\tilde{M}_\infty$ of $G_L \times A_-$ for which $\tilde{a}^{-1}\tilde{k}^{-1}$ is in the domain of definition of the maps $\tilde{\Xi}_L$ and $\Lambda_R$ (i.e. $\tilde{a}^{-1}\tilde{k}^{-1}$ is in $S_\infty$):

$$
\tilde{\Omega}_\infty(\tilde{k}, \tilde{a}) = \frac{1}{2} (d\tilde{a}^{-1} \wedge \tilde{k}^{-1} dk)_D + \frac{1}{2} (d\tilde{\Xi}_L(\tilde{a}^{-1}\tilde{k}^{-1})\tilde{\Xi}_L^{-1}(\tilde{a}^{-1}\tilde{k}^{-1}) \wedge \tilde{a}^{-1}d\tilde{a} + \tilde{a}^{-1}(\tilde{k}^{-1} dk)\tilde{a})_D.
$$

(4.40)

Then the manifold $(\tilde{M}_\infty, \tilde{\Omega}_\infty)$ is symplectic and the infinitesimal version of the right $G_L$ action $\tilde{h} \triangleright (\tilde{k}, \tilde{a}) = (\tilde{h}^{-1}\tilde{k}, \tilde{a})$, $\tilde{h} \in G_L$ is a right Poisson-Lie symmetry of $(\tilde{M}_\infty, \tilde{\Omega}_\infty)$.

**Proof.** Consider a map $U : M_\infty \rightarrow \tilde{M}_\infty$ defined everywhere in $M_\infty$ as follows

$$(\tilde{k}, \tilde{a}) = U(k, a) = (\tilde{\Xi}_L^{-1}(ka), a^{-1}), \quad (k, a) \in M_\infty.
$$

(4.41)

Similarly, consider a map $V : \tilde{M}_\infty \rightarrow M_\infty$ defined everywhere in $\tilde{M}_\infty$ as follows

$$(k, a) = V(\tilde{k}, \tilde{a}) = (\tilde{\Xi}_L^{-1}(a^{-1}\tilde{k}^{-1}), a^{-1}), \quad (\tilde{k}, \tilde{a}) \in \tilde{M}_\infty.
$$

We easily verify that $U \circ V$ is the identity map on $M_\infty$ and $V \circ U$ is the identity map on $\tilde{M}_\infty$. It follows that both $U, V$ are injective and surjective, hence they are diffeomorphisms inverse to each other.

By an easy direct calculation, we can relate the form (4.40) on $\tilde{M}_\infty$ to the symplectic form (4.17) on $M_\infty$ by pull-backs

$$
U^*\tilde{\Omega}_\infty = -\Omega_\infty, \quad V^*\Omega_\infty = -\tilde{\Omega}_\infty.
$$

(4.42)

This implies that $(\tilde{M}_\infty, \tilde{\Omega}_\infty)$ is a symplectic manifold.

We pick $T \in \mathcal{G}_L$ and we pull-forward by $U$ the vector field $\Pi(., \chi^*\Lambda^*_L < \lambda, T >) \in Vect(M_\infty)$, realizing the left $\mathcal{G}_L$ Poisson-Lie symmetry of $M_\infty$. The result is a vector field $\tilde{w}_T \in Vect(\tilde{M}_\infty)$ given by

$$
\tilde{w}_T \equiv U_* (\Pi(., \chi^*\Lambda^*_L < \lambda, T >)) = -\tilde{\Pi}_\infty(., V^*\chi^*\Lambda^*_L < \lambda, T >) = \tilde{\Pi}_\infty(., V^*\chi^*\Lambda^*_L J^* < \rho, T >),
$$

where $\tilde{\Pi}_\infty$ stands for the Poisson bivector inverse to the symplectic form $\tilde{\Omega}_\infty$, the map $\chi : M_\infty \rightarrow S_\infty$ was defined in Theorem 6 and the map $J : G^* \rightarrow G^*$ is just the inversion map $J(b) = b^{-1}$, $b \in G^*$. Note that $J^* \rho = -\lambda$.

Now consider a map $\tilde{\chi} : \tilde{M}_\infty \rightarrow S_\infty$ given by $\tilde{\chi}(\tilde{k}, \tilde{a}) = \tilde{a}^{-1}\tilde{k}^{-1}$. Then it is easy to see that

$$
\Lambda_R \circ \tilde{\chi} = J \circ \Lambda_L \circ \chi \circ V,
$$

(4.43)
which permits to rewrite
\[ \tilde{w}_T = \tilde{\Pi}^\infty(\cdot, \tilde{\chi}^* \Lambda^*_R < \rho, T >). \]

Obviously, we have
\[ \tilde{\Omega}_\infty(\cdot, \tilde{w}_T) = \tilde{\chi}^* \Lambda^*_R(\rho, T)_D. \]

Consider a point \((\tilde{k}, \tilde{a}) \in \tilde{M}_\infty\). The multiplication of \((\tilde{k}, \tilde{a})\) on the left by an infinitesimal generator \(T \in \mathcal{G}_L\) gives the vector \(\tilde{v}_T = (T \tilde{k}, \tilde{a})\) and its \(\tilde{\chi}_s\)-push-forward vector \(\tilde{\chi}_s \tilde{v}_T = -\tilde{a}^{-1} \tilde{k}^{-1} T = -L_{(\tilde{a}^{-1} \tilde{k}^{-1})^s} T \in T_{(\tilde{a}^{-1} \tilde{k}^{-1})^s} S_\infty\). Denote also by \(\tilde{v}_T\) and \(\tilde{\chi}_s \tilde{v}_T\) the respective vector fields on \(\tilde{M}_\infty\) and \(S_\infty\) obtained by varying the point \((\tilde{k}, \tilde{a})\). Then we learn from Theorem 2 (cf. Eq. (3.5)):
\[ \tilde{\chi}_s \tilde{v}_T = \Pi^\infty_D(\cdot, \Lambda^*_R(\rho, T)_D), \]
or, equivalently,
\[ \omega_{S_\infty}(\cdot, \tilde{\chi}_s \tilde{v}_T) = \Lambda^*_R(\rho, T)_D. \]

The relation (4.42) and Theorem 5 imply easily
\[ \tilde{\Omega}_\infty = \tilde{\chi}^* \omega_{S_\infty}, \]
where \(\omega_{S_\infty}\) is the grupoid symplectic form (3.6). This fact permits to infer from (4.45):
\[ \tilde{\Omega}_\infty(\cdot, \tilde{v}_T) = \tilde{\chi}^* \Lambda^*_R(\rho, T)_D. \]

By comparing Eqs. (4.44) and (4.46), we obtain
\[ \tilde{w}_T = \tilde{v}_T. \]

Thus we have established, that the left \(\mathcal{G}_L\) Poisson-Lie symmetry of the chiral phase space \(\tilde{M}_\infty\) is most conveniently described by using the (anti)symplectomorphism \(U : M_\infty \to \tilde{M}_\infty\). Indeed, as Eq. (4.47) shows, in the parametrization \(\tilde{M}_\infty = \tilde{G}_L \times A_-\) the \(\mathcal{G}_L\) symmetry action is nothing but the infinitesimal version of the natural left \(G_L\) action \(\tilde{h} \triangleright (\tilde{k}, \tilde{a}) = (\tilde{h} \tilde{k}, \tilde{a}), \tilde{h} \in G_L\).

Now the proof of the present theorem is almost finished. Obviously, the right action of \(G_L\) on \(\tilde{M}_\infty\), given by \(\tilde{h} \triangleright (\tilde{k}, \tilde{a}) = (\tilde{h}^{-1} \tilde{k}, \tilde{a}), \tilde{h} \in G_L\), is infinitesimaly generated by the vector fields \(-\tilde{v}_T, T \in \mathcal{G}_L\). We infer from (4.44) and (4.47) that
\[ -\tilde{v}_T = \tilde{\Pi}^\infty(\tilde{\chi}^* \Lambda^*_R < \rho, T >, ..) = \tilde{\Pi}^\infty((\Lambda^*_R \circ \tilde{\chi})^* < \rho, T >, ..) \]

It remains to show that the map \((\Lambda^*_R \circ \tilde{\chi}) : (M_\infty, \bar{P}_\infty) \to (G^*, \Pi^*)\) is Poisson, thus realizing the Poisson-Lie symmetry of \(M_\infty\). The Poisson property of the map \(\Lambda^*_R \circ \tilde{\chi}\) follows from Eq. (4.43). Indeed, we know from (4.42) that \(V : (M_\infty, \bar{P}_\infty) \to (M_\infty, -\Pi^\infty)\) is the Poisson map, Theorem 6 states that \(\Lambda_L \circ \chi : (M_\infty, \Pi^\infty) \to (G^*, \Pi^*_\text{op})\) is the Poisson map and, by using Eqs. (2.2),(2.3), it is not difficult to work out that \(J : (G^*, \Pi^*_\text{op}) \to (G^*, -\Pi^*)\) is the Poisson map. Thus the composition map \(J \circ \Lambda_L \circ \chi \circ V = \Lambda^*_R \circ \tilde{\chi}\) is also Poisson.
Interpretation in terms of duality. It is not so surprising that there is an (anti)-symplectomorphism $U$ transforming the infinitesimal left $G_L$ Poisson-Lie symmetry of $M_\infty$ into a more nicely looking right $G_L$ Poisson-Lie symmetry of $\hat{M}_\infty$. After all, to find this anti-symplectomorphism, it was sufficient to integrate the vector fields $\Pi^\infty(\cdot,(\Lambda_L \circ \chi)^*(\lambda, T))^D$ to $G_L$-orbits and use the $G_L$ orbits for the parametrization of the chiral phase space $M_\infty$ (it is in this way that we have constructed the diffeomorphism $U$). What is surprising and remarkable is that the $U$-transformed expression (4.40) has the same structure than the original one (4.17). Indeed, we have

$$\Omega_\infty(k, a) = \frac{1}{2}(daa^{-1} \wedge k^{-1} dk)_D + \frac{1}{2}((\Xi_R \circ \chi)^*\rho_R \triangleright a^{-1} da + a^{-1}(k^{-1} dk) a)_D,$$

$$\tilde{\Omega}_\infty(\tilde{k}, \tilde{a}) = \frac{1}{2}(d\tilde{a}\tilde{a}^{-1} \wedge \tilde{k}^{-1} d\tilde{k})_D + \frac{1}{2}((\Xi_L \circ \tilde{\chi})^*\rho_L \triangleright \tilde{a}^{-1} d\tilde{a} + \tilde{a}^{-1}(\tilde{k}^{-1} d\tilde{k}) \tilde{a})_D,$$

where $\rho_R$ and $\rho_L$ are, respectively, the right-invariant Maurer-Cartan forms on $G_R$ and $G_L$. We interpret this phenomenon as the $G_L \leftrightarrow G_R$ duality of the chiral $q \to \infty$ WZW model. This duality relates the Poisson-Lie symmetries of $(M_\infty, \Omega_\infty)$ and $(\hat{M}_\infty, \tilde{\Omega}_\infty)$ in a nice way: we have shown that the left $G_L$ symmetry of $(M_\infty, \Omega_\infty)$ gets $U$-transformed into the right $G_L$ symmetry of $(\hat{M}_\infty, \tilde{\Omega}_\infty)$ and it is not difficult to show that the right $G_R$ symmetry of $(M_\infty, \Omega_\infty)$ gets $U$-transformed into the left $G_R$ symmetry of $(\hat{M}_\infty, \tilde{\Omega}_\infty)$.

Remark. Corollary of Theorem 6 tells us that the right $G_R$ Poisson-Lie symmetry of $M_\infty$ is given by the infinitesimal version of the natural global right $G_R$ action on $M_\infty = G_R \times A_+$, hence we see that the infinitesimal right $G_R$ symmetry can be lifted to the global $G_R$ symmetry. However, the left $G_L$ Poisson-Lie symmetry of $M_\infty$ cannot be lifted to a global action of the group $G_L = L_{pod}K$ on $M_\infty$. This becomes evident upon the $U$-transformation of $M_\infty$ into $\hat{M}_\infty$. Indeed, the manifold $\hat{M}_\infty$ is the proper subset of the direct product $G_L \times A_+$ and the global action of the group $G_L$ on itself does not respect this proper subset. Thus we observe, that a moment map can realize at the same time a global right symmetry and a local left symmetry. In this sense, the duality $G_L \leftrightarrow G_R$ is only local and it cannot be lifted to a global $G_L \leftrightarrow G_R$ duality.

4.7 $q \to \infty$ limit of the exchange relations

Now it is time to work out the dual exchange relations, i.e. to calculate the Poisson brackets of the type $\{\tilde{a}, \tilde{k}(\sigma)\}_\infty, \{\tilde{k}(\sigma) \circ \tilde{k}(\sigma')\}_\infty$. Actually, those two brackets characterize completely the dual symplectic structure $(\hat{M}_\infty, \tilde{\Omega}_\infty)$. It is because $\tilde{k}$ is the element of $G_L = L_{pod}K$ which means that $\tilde{k}(\sigma)^{-1} = \tilde{k}(\sigma)$ and it is not necessary to calculate the brackets $\{\tilde{k}(\sigma) \circ \tilde{k}(\sigma')^{-1}\}_\infty, \{\tilde{k}(\sigma)^{-1} \circ \tilde{k}(\sigma')^{-1}\}_\infty$ like in the $G_R$ case. Our strategy for working out the dual exchange relations will be different than it was for the original relations (4.35-38). We shall not try to invert directly the dual symplectic form $\tilde{\Omega}_\infty$ because the fact that it is not defined everywhere on $G_L \times A_+$ makes the task more involved than the inversion of the form $\Omega_\infty$ in Sec 4.5. Instead, we shall proceed indirectly, by using the duality antisymplectomorphism $U : M_\infty \to \hat{M}_\infty$ and the Poisson-Lie symmetry of $(\hat{M}_\infty, \tilde{\Omega}_\infty)$. Indeed, we take into account the definition (4.41) of the diffeomorphism $U$ to see that, for finding the dual brackets $\{\tilde{a}, \tilde{k}(\sigma)\}_\infty, \{\tilde{k}(\sigma) \circ \tilde{k}(\sigma')\}_\infty$, it is sufficient to calculate the brackets $\{a^{-1}, \Xi_R^{-1}(ka)(\sigma)\}_\infty, \{\Xi_R^{-1}(ka)(\sigma) \circ \Xi_R^{-1}(ka)(\sigma')\}_\infty$ with the help of the original Poisson bivector (4.34).
Consider an element \((k,a) \in G_R \times A_+ = M_\infty\) and write it as \(ka = \Lambda_L(ka)\Xi^{-1}_R(ka)\). Thus we have
\[
\{a^{-1}, \Xi^{-1}_R(ka)(\sigma)\}_\infty = \{a^{-1}, \Lambda^{-1}_L(ka)(\sigma)k(\sigma)a\}_\infty, \tag{4.48}
\]
\[
\{\Xi^{-1}_R(ka)(\sigma) \otimes \Xi^{-1}_R(ka)(\sigma')\}_\infty = \{\Lambda^{-1}_L(ka)(\sigma)k(\sigma)a \otimes \Lambda^{-1}_L(ka)(\sigma')k(\sigma')a\}_\infty. \tag{4.49}
\]
If we take into account the following obvious Poisson matrix relations
\[
\{AB \otimes CD\} = (A \otimes 1)\{B \otimes C\}(1 \otimes D) + \{A \otimes C\}\{B \otimes D\} + \{A \otimes B\}(1 \otimes C)(1 \otimes D)(B \otimes 1),
\]
we realize that the dual exchange relations (4.48) and (4.49) can be worked out from the matrix Poisson brackets of the type \{\otimes ka\}_\infty, \{\otimes ka\}_\infty, \{\Lambda_L(ka) \otimes \Lambda_L(ka)\}_\infty, \{a \otimes \Lambda_L(ka)\}_\infty and \{ka \otimes \Lambda_L(ka)\}_\infty. Let us argue that we already know explicit formulae for all five brackets just listed. Indeed, the first two brackets in the list are given in Eqs. (4.35) and (4.36). Next, the fact that \(\Lambda_L \circ \chi : (M_\infty, \Pi^\infty) \rightarrow (G^*, \Pi^\infty_\text{op})\) is the Poisson map implies that the third bracket is given by the \(\infty\)-current algebra relation (4.11) (cf. also (4.12) and (4.13)):
\[
\{\Lambda_L(ka)(\sigma) \otimes \Lambda_L(ka)(\sigma')\}_\infty = \left(r + C \cotg \frac{\sigma - \sigma'}{2}\right) \times
\]
\[
\left(\Lambda_L(ka)(\sigma) \otimes \Lambda_L(ka)(\sigma')\right)\left(r + C \cotg \frac{\sigma - \sigma'}{2}\right).
\tag{4.52}
\]
Finally, we notice that the brackets \{\otimes \Lambda_L(ka)\}_\infty, \{ka \otimes \Lambda_L(ka)\}_\infty can be worked out as follows:
Consider the vector field \(v_T\) corresponding to the action of \(T \in G_R\) on \(M_\infty\). Corollary of Theorem 6 states that
\[
<v_T, df> = \Pi^\infty(df, (\Lambda_L \circ \chi)^*(\rho, T)_D),
\]
for every function \(f\) defined on \(M_\infty\). Let \(\Upsilon\) be the representation of the group \(K\) introduced in Sec 4.2 and \(T_{\rho\Upsilon}\) be the trace in this representation normalized in such a way that
\[
T_{\rho\Upsilon}(\Upsilon(A)\Upsilon(B)) = (A, B)_{K^C}, \quad A, B \in K^C.
\]
The right-invariant Maurer-Cartan form \(\rho\) on \(G^*\) can be written with the help of the matrix valued functions \(\Upsilon\) defined in Sec 4.2:
\[
\Upsilon(\rho) = d\Upsilon \Upsilon^{-1}, \tag{4.53}
\]
Thus, using (4.3) and (4.4), we can write
\[
<v_T, df> = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\sigma ImT_{\rho\Upsilon}(\Upsilon(T), \{f, (\Lambda_L \circ \chi)^*\Upsilon_\sigma\}_\infty(\Lambda_L \circ \chi)^*\Upsilon_\sigma^{-1}).
\tag{4.54}
\]
Let $T^i_R$ a basis of $G_R$, $t_i$ the basis of $G^*$ verifying the condition (3.7) and $k \in G_R$. We wish to express the quantity $\Upsilon_{\sigma}(T^i_R)\Upsilon_{\sigma}(k) \otimes \Upsilon_{\sigma^\prime}(t_i)$, by using the Poisson brackets on $M_\infty$. By invoking Eqs. (4.10), (4.54) and the notation of the proof of Corollary of Theorem 6, we have

$$\Upsilon_{\sigma}(T^i_R)\Upsilon_{\sigma}(k) \otimes \Upsilon_{\sigma^\prime}(t_i) =$$

$$= \langle \nu_{T^i_R}d\Upsilon_{\sigma}(k) \otimes \Upsilon_{\sigma^\prime}(t_i) = \{\Upsilon_{\sigma}(k) \otimes \Upsilon_{\sigma^\prime}(\Lambda_L^-(ka))\}_\infty 1 \otimes \Upsilon_{\sigma^\prime}(\Lambda_L^-(ka)) \rangle,$$

By suppressing the symbol $\Upsilon$, as usual, we can thus write

$$\{k(\sigma) \otimes \Lambda_L^-(ka)(\sigma^\prime)\}_\infty =$$

$$= (T^i_R(\sigma) \otimes t_i(\sigma^\prime))(k(\sigma) \otimes \Lambda_L^-(ka)(\sigma^\prime)) = (r + C\cotg\frac{\sigma - \sigma^\prime}{2})(k(\sigma) \otimes \Lambda_L^-(ka)(\sigma^\prime)).$$

(4.55)

Note that we have evaluated the expression $T^i_R(\sigma) \otimes t_i(\sigma^\prime)$ in the basis $T^i_R = (T^i_R, I^i_R, C^i_R)$, $\hat{\sigma} > 0$ and $t_i = (b_i, b_i, c_i) + \sigma^\prime > 0$, defined in Sec 4.1.

We know that the coordinates $\phi^\mu$ are invariant with respect to the action of the $G_R$ Poisson-Lie symmetry on $M_\infty$, which, combined with Eqs. (4.29) and (4.53), gives

$$\{\phi^\mu, \Lambda_L^-(ka)\}_\infty \Lambda_L^-(ka) = 0.$$

From this equality we finally infer

$$\{a \otimes \Lambda_L^-(ka)\}_\infty = 0.$$  

(4.56)

We are now ready to write down the seeken dual exchange relations. We take into account Eqs. (4.35), (4.36), (4.52), (4.55), (4.56) as well as Eqs. (4.42), (4.50), (4.51) to find out

$$\{\tilde{a}, \tilde{k}(\sigma)\}_\infty = -i(\tilde{a} \otimes \tilde{k}(\sigma))(H^\mu \otimes H^\mu),$$

(4.57)

$$\{\tilde{k}(\sigma) \otimes \tilde{k}(\sigma^\prime)\}_\infty =$$

$$= (\tilde{k}(\sigma) \otimes \tilde{k}(\sigma^\prime))(r(\tilde{a}^{-1}) + C\cotg\frac{\sigma - \sigma^\prime}{2}) + (r + C\cotg\frac{\sigma - \sigma^\prime}{2})(\tilde{k}(\sigma) \otimes \tilde{k}(\sigma^\prime)).$$

(4.58)

We remind that

$$\tilde{a} = a^{-1} = e^{-\phi^\mu H^\mu}, \quad \tilde{k} = \Lambda_L^-(ka)ka$$

and the minus sign distinguishing the symplectic forms $\Omega_\infty$ and $U^*\tilde{\Omega}_\infty$ has been taken into account.

We stress that the dual $(M_\infty, \tilde{\Omega}_\infty)$-description of the $q \to \infty$ chiral phase space is $U$-equivalent to the original $(M_\infty, \Omega_\infty)$-description, in particular, the dual $G_L$-exchange relations (4.57), (4.58) are equivalent to the $G_R$-exchange relations (4.35-38). From the symmetry point of view, the $(M_\infty, \tilde{\Omega}_\infty)$-formalism is better adapted to the explicit description of the $G_R$-symmetry while the $(M_\infty, \tilde{\Omega}_\infty)$-formalism is better adapted to the
explicit description of the $G_L$-symmetry. Note, however, one important point: the sub-
manifold $M_\infty \subset G_L \times A_-$ is not invariant under the global $G_L$-action while the manifold
$M_\infty = G_R \times A_+$ is invariant under the global $G_R$-action. Said in other words, the $G_R$
symmetry is global while the $G_L$-symmetry is only local. This is the reason why we have
constructed our exposition starting from the $(M_\infty, \Omega_\infty)$-description where the global $G_R$
symmetry looks naturally. However, the $(M_\infty, \Omega_\infty)$-description has also its assets because,
as we are going to see, it is better adapted for the comparison of the finite $q$ chiral WZW
model with the $q \to \infty$ chiral WZW model.

Now we wish to compare the dual exchange relations (4.57) and (4.58) with the exchange
relations of the finite $q$ chiral WZW model of Ref. [9]. To do that we should take
into account that our parameter $q$ is the parameter $1/q$ of Ref.[9], as it follows from the
comparison of Eq. (4.5) of the present paper and of Eq. (4.113) of [9]. In order not to
cause confusion, we shall denote by $q'$ the parameter $q$ of [9] and we set $\epsilon' = \ln q'$. The
exchange relations for finite $q'$ were explicited in Eqs. (5.159) and (5.161) of Ref. [9] only
for $q' > 1$ (or $\epsilon' > 0$) and we have to make them explicit also for $0 < q' < 1$ (since when
our $q$ goes to $\infty$ the "old" $q'$ of [9] goes to $0^+$). By reusing the method of Sec 5.2.4 of [9],
we obtain for $\epsilon' < 0$:

$$\{\tilde{a}, \kappa(\sigma)\}_{\epsilon'} = -i\epsilon'(\tilde{a} \otimes \kappa(\sigma))(H^\mu \otimes H^{\mu}),$$

(4.59)

$$\{\kappa(\sigma) \otimes \kappa(\sigma')\}_{\epsilon'} =$$

$$= (\kappa(\sigma) \otimes \kappa(\sigma'))\hat{r}_{\epsilon'}(\tilde{a}, \sigma - \sigma') + \epsilon'(r + C\cotg \frac{\sigma - \sigma'}{2})(\kappa(\sigma) \otimes \kappa(\sigma')),$$

(4.60)

where $\kappa(\sigma) \in G_L$ and $\hat{r}_{\epsilon'}(\tilde{a}, \sigma)$ is the Felder elliptic dynamical $r$-matrix [7] given by

$$\hat{r}_{\epsilon'}(\tilde{a}, \sigma) =$$

$$= \frac{\epsilon'}{\pi} [\frac{\sigma}{2\pi} - \frac{i k \epsilon'}{\pi}] H^\mu \otimes H^{\mu} + \frac{\epsilon'}{\pi} \sum_{\alpha \in \Phi} |\alpha|^2 \sigma - \frac{\epsilon' k \alpha(\alpha, H^\mu)}{2\pi} \frac{\sigma}{2\pi} - \frac{i k \epsilon'}{\pi}] E^\alpha \otimes E^{-\alpha}.$$

We remind that here $\epsilon' < 0$, $\tilde{a} = e^{k \epsilon' a^\mu H^\mu}$, $k$ is the level of the $q$-WZW model and the
coordinates $a^\mu$ parametrize the so called positive Weyl alcove $A^+_1$ (cf. Sec 5.2 of [9]). Using
the classical formulae from Ref. [20] which define the elliptic functions $\sigma_{-y}(z, \tau)$, $\rho(z, \tau)$

$$\sigma_{-y}(z, \tau) = \pi(\cotg \pi z + \cotg \pi y) + 4\pi \sum_{m, n > 0} \frac{e^{2i\pi mn}}{1 - e^{2i\pi mn}} \sin 2\pi(mz + ny);$$

$$\rho(z, \tau) = \pi \cotg \pi z + 4\pi \sum_{n > 0} \frac{e^{2i\pi nz}}{1 - e^{2i\pi nz}},$$

and keeping fixed the expression $\phi^\mu = -k \epsilon' a^\mu$, we find that

$$\lim_{\epsilon' \to -\infty} \frac{1}{\epsilon'} \hat{r}_{\epsilon'}(\tilde{a}, \sigma) =$$

$$\left( r(e^{\phi^\mu H^\mu}) + C\cotg \frac{\sigma - \sigma'}{2} \right),$$

This gives, in turn, for $\tilde{a} = e^{-\phi^\mu H^\mu}$

$$\lim_{\epsilon' \to -\infty} \frac{1}{\epsilon'} \{\tilde{a}, \kappa(\sigma)\}_{\epsilon'} = -i(\tilde{a} \otimes \kappa(\sigma))(H^\mu \otimes H^{\mu}),$$
\[ \lim_{\epsilon' \to \infty} \frac{1}{\epsilon'} \{ \tilde{k}(\sigma), \tilde{k}(\sigma') \}_{\epsilon'} = \\
= (\tilde{k}(\sigma) \otimes \tilde{k}(\sigma'))(r(\tilde{a}^{-1}) + C\cotg \frac{\sigma - \sigma'}{2}) + (r + C\cotg \frac{\sigma - \sigma'}{2})(\tilde{k}(\sigma) \otimes \tilde{k}(\sigma')). \]

We have just established, as expected, that \( \lim_{\epsilon' \to \infty} \frac{1}{\epsilon'} \{ \ldots \} = \{ \ldots \}_{\infty} \), or, in other words, the finite \( q \) chiral WZW exchange relations (4.59), (4.60) give in the \( q \to \infty \) limit the (dual) \( G_L \)-exchange relations (4.57), (4.58) of the chiral \( \infty \)-WZW model.

The reader may feel intrigued why we did not start our exposition of the \( \infty \)-WZW model by first establishing the \( q \to \infty \) limit of the chiral exchange relations, but, instead, we have first exposed the theory of the affine Poisson groups etc. Actually, the more "theoretical" approach, that we have chosen, have evident benefits. The most important is that it allowed us to identify the singularities of the dual exchange relations (4.57) and (4.58).

Indeed, the following phenomenon takes place: the chiral WZW phase space for finite \( q \) is just the manifold \( G_L \times A^1 \), where \( A^1 \) is the (compact) subset of \( A \) obtained by exponentiating the positive Weyl alcove as follows: \( e^{k'\epsilon'\omega'H^\alpha} \in A^1 \). In the \( q \to \infty \) limit (or the \( \epsilon' \to -\infty \) limit), the set \( A^1 \) gets expanded to the exponentiated negative Weyl chamber \( A_- = \exp A_- \) and the chiral Poisson structure, defined on the whole manifold \( G_L \times A_- \), by the brackets (4.57) and (4.58), becomes invertible only on the submanifold \( \tilde{M}_\infty \subset G_L \times A_- \). Our approach, using the theory of affine Poisson groups, gave as the natural characterization of the regular submanifold \( \tilde{M}_\infty \) in terms of the domain of definition of the maps \( \Sigma_L \) and \( \Lambda_R \).

The general theory of the affine Poisson groups has helped us to clarify one more subtle point. Indeed, as we have already mentioned, there is a price to pay for the restriction to the regular submanifold \( \tilde{M}_\infty \subset G_L \times A_- \), namely, the loop group \( G_L = L_{pol}K \) no longer acts on \( \tilde{M}_\infty \). At the first sight, this may look bad, because the \( q \to \infty \) limit seems to deprive the WZW model from its interesting symmetry structure, however, as we have again learned from Theorems 2 and 7, the \( G_L \)-symmetry does survive the limit in its local \( G_L \)-form. On the top of that, there are further added benifits of our approach: we have discovered the new \( G_R \) symmetry, which emerged in the \( q \to \infty \) limit and also the remarkable \( G_L \leftrightarrow G_R \) duality of the chiral \( \infty \)-WZW model.

### 4.8 \( q \to \infty \) Hamiltonian

A dynamical system is a triple \((M, \Omega, E)\) where \( M \) is a phase space, \( \Omega \) a symplectic form on it and \( E \) is a one-parameter group of symplectomorphisms of \( M \) defining the time evolution. We have often spoken about a particular dynamical system called the chiral \( \infty \)-WZW model but, so far, we have specified only its phase space \( M_\infty = G_R \times A_1 \) and its symplectic form \( \Omega_\infty \) given by Eq. (4.17). In this section, we shall fill the gap and define also the one-parameter group \( E_\infty \) describing the time evolution in the chiral \( \infty \)-WZW model. In order to do that we first define a suitable parametrization of the phase space \( M_q \), for a finite \( q \).

For finite \( q \) (including the standard non-deformed \( q = 1 \) case), the time evolution is most easily described in the so called "monodromic" variables. Let us define them. The phase space \( M_q \) of the chiral \( q \)-WZW model is the direct product \( L_{pol}K \times A^1 \), where \( A^1 \) is the
The usual periodic parametrization of \( L_{\text{pol}} K \times A^1 \) is \((\tilde{k}(\sigma), a^\mu H^\mu)\). In the monodromic parametrization, the points in the phase space \( M_q \) are quasi-periodic maps \( m : \mathbb{R} \to K \), fulfilling the monodromy condition

\[
m(\sigma + 2\pi) = m(\sigma) M, \quad M = \exp(-2\pi i a^\mu H^\mu).
\]

The transformation from the monodromic parametrization into the periodic one is given by the relations

\[
\exp(-2\pi i a^\mu H^\mu) = m^{-1}(\sigma) m(\sigma + 2\pi), \quad \tilde{k}(\sigma) = m(\sigma) \exp(i a^\mu H^\mu \sigma).
\]

The time evolution \( E_q \) of the finite \( q \) chiral WZW model is defined very simply: a point \( m(\sigma) \in M_q \) at the time \( \tau = 0 \) gets evolved to the point \( m(\sigma - \tau) \) at the time \( \tau \). In the periodic parametrization, this time evolution gets translated into

\[
\tilde{k}(\sigma) \to \tilde{k}(\sigma - \tau) \exp(i a^\mu H^\mu \tau), \quad a^\mu \to a^\mu.
\] (4.61)

It is not difficult to see that the transformation (4.61) defines a symplectomorphism. This follows from the fact that the finite \( q \) exchange relations (4.59) and (4.60) (which completely characterize the symplectic structure on \( M_q \)) are invariant with respect to (4.61).

Coming back to the case \( q \to \infty \), we define the time evolution in terms of the dual variables \((\tilde{k}, \tilde{a}) \in \tilde{M}_\infty \) (cf. Eq. (4.41)). The advantage of working with the dual variables in this context is clear: they are the best adapted for the description of the \( q \to \infty \) limit. We remind that \( \tilde{a} = e^{i \epsilon^\mu \phi H^\mu} = e^{-\phi H^\mu} \), which gives \( a^\mu = -\frac{\phi_\mu}{\epsilon^\mu} \). Thus the \( E_q \) evolution transformation (4.61) can be rewritten as

\[
\tilde{k}(\sigma) \to \tilde{k}(\sigma - \tau) \exp\left(-\frac{i \phi^\mu}{\epsilon^\mu} H^\mu \tau\right), \quad \phi^\mu \to \phi^\mu.
\]

The \( \epsilon^\mu \to -\infty \) limit is achieved by keeping \( \phi^\mu \) fixed which suggests the following time evolution \( E_\infty \) of the chiral \( \infty \)-WZW model:

\[
\tilde{k}(\sigma) \to \tilde{k}(\sigma - \tau), \quad \phi^\mu \to \phi^\mu.
\] (4.62)

Let us verify, that the suggested time evolution (4.62) is correctly defined, which means that it leaves invariant the subspace \( \tilde{M}_\infty \subset L_{\text{pol}} K \times A^- \) and it defines the one-parameter group of symplectomorphisms of \( \tilde{M}_\infty \). The latter statement follows from the obvious invariance of the dual exchange relations (4.57), (4.58) with respect to the transformation (4.62). To prove the former statement is also easy. Indeed, let \( l(\sigma) \) be an element of \( L_{\text{pol}} K^C \) which is also in the domain of definition of the maps \( \Lambda_R \) and \( \Xi_L \), i.e. \( l(\sigma) \in S_\infty \). Then it follows that also \( l(\sigma - \tau) \) is in \( S_\infty = G_R G^* \), since the transformation of \( D = L_{\text{pol}} K^C \) defined by \( l(\sigma) \to l(\sigma - \tau) \) leaves the subgroups \( G_R \) and \( G^* \) invariant.

In this section, we have completed the definition of the chiral \( \infty \)-WZW model by defining the consistent time evolution \( E_\infty \) on the phase space \((\tilde{M}, \tilde{\Omega}_\infty)\). However, we may ask another question: is this evolution generated by a Hamiltonian function? The answer to this question is affirmative provided that we enlarge slightly the phase space \( M_\infty \) by working with the smooth loop groups \( L K^C \), \( L K \) etc. rather than with the polynomial
ones $L_{pol}K^C$, $L_{pol}K$ etc. Now recall the result described in Sec 8.9 of [18] that every infinitesimal symplectomorphism of an infinite-dimensional simple connected symplectic manifold has a Hamiltonian function. Let us therefore argue that the chiral $\infty$-WZW phase space $M_\infty$ is simply connected. First of all, we know that $M_\infty$ is diffeomorphic to $M = G_R \times A_-$ which gives the following relation between the fundamental groups:

$$\pi_1(M_\infty) = \pi_1(G_R) \times \pi_1(A_-).$$

Obviously, $A_-$ is simply connected and it remains to show that $G_R$ is simply connected. Denote respectively by $L^0K^C$ and $L^0K$ the subgroups of $LK^C$ and $LK$ formed of the loops verifying $l(0) = e_{K^C}$ and $k(0) = e_K$. Similarly, denote by $G^0_R$ the subgroup of $G_R$ for which $\gamma_0 = e_{K^C}$ (cf. the expansion (4.2)). Clearly, $LK^C$ is diffeomorphic to $K^C \times L^0K^C$, $LK$ is diffeomorphic to $K \times L^0K$ and $G_R$ is diffeomorphic to $K \times G^0_R$. Thus we obtain

$$\pi_1(LK^C) = \pi_1(K^C) \times \pi_1(L^0K^C) = \pi_1(K^C) \times \pi_2(K^C),$$

$$\pi_1(LK) = \pi_1(K) \times \pi_1(L^0K) = \pi_1(K) \times \pi_2(K),$$

$$\pi_1(G_R) = \pi_1(K) \times \pi_1(G^0_R).$$

Note also that the global decomposition $LK^C = G^*LK$ (cf. Lemma 6) implies also a global decomposition $LK^C = G^0_R(AN)LK$, where $G^*_R$ is the subgroup of $G^*$ for which $\gamma_0 = e_{K^C}$. Taking the hermitian conjugation of the global decomposition $LK^C = G^0_R(AN)LK$, we establish that $LK^C$ is diffeomorphic to $LK \times (AN)^\dagger \times G^0_R$. Hence we have

$$\pi_1(LK^C) = \pi_1(G^0_R) \times \pi_1((AN)^\dagger)) \times \pi_1(LK).$$

We have supposed that $K$ is simple connected and simply connected, which means that the fundamental groups $\pi_1(K)$, $\pi_1(K^C)$ and $\pi_1((AN)^\dagger)$ are trivial. Moreover, the classical theorem says (cf. Sec 8.6 of [18]) that the second homotopy groups $\pi_2(K)$ and $\pi_2(K^C)$ are also trivial. Thus we deduce from the relations (4.63-66) the desired result that the fundamental group of $G_R$ is also trivial.

We have established the existence of the hamiltonian generating the simple time evolution (4.62), however, we do not have any explicit formula expressing it as the function of $k$ and $\dot{a}$. This type of situation took place also for the finite $q$ case; it was only for the standard $q = 1$ WZW model that such an explicite (i.e. Sugawara) formula could have been written.

5 Conclusions and outlook

The paper is devoted to the explicit construction of the dynamical system, which can be interpreted as the $q \to \infty$ limit of the $q$-WZW model. The most important feature of this system is its symmetry structure; in fact, the limiting model has two non-isomorphic symmetry groups $G_L$ and $G_R$, which is not true for the finite $q$ WZW model (including the standard case $q = 1$), thus we may even say that the symmetry structure of the $q \to \infty$ WZW model is richer and more intriguing than that of its finite $q$ counterpart. In order to identify the symmetries of the model, we have used in its full richness the concept of the affine Poisson group and of the anomalous Poisson-Lie symmetry. In fact, the formulation
of the $q \to \infty$ WZW model based on the notion of the symplectic groupoid of the affine Poisson group was very insightful; in particular, we would not have suspected the existence of the remarkable duality transformation (4.41) if we did not take the inspiration from Lemma 5.

There are also aspects of the structure of the $q \to \infty$ WZW model which did not follow from the general theory of the affine Poisson groups, like e.g. the chiral decomposability. The possibility to study the simpler chiral $q \to \infty$ WZW model, which also enjoys the anomalous Poisson-Lie symmetry, has lead to very explicit formulae like the exchange relations (4.57) and (4.58) and was also very suggestive on general grounds. In particular, we found remarkable that the $q \to \infty$ trigonometric exchange relations (4.57) and (4.58) are so much simpler than the finite $q$ elliptic exchange relations (4.59) and (4.60).

Finally, we believe that the quantization of the results of the present paper will be doable due to the simplicity of the trigonometric exchange relations (4.57) and (4.58). The eventual quantization will probably make a fruitful use of the theory of twisting of Hopf algebras developed by Majid [17].

References


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