Abstract—A graph is half-arc-transitive if its automorphism group acts transitively on its vertex set, edge set, but not arc set. Let \( n \) be a product of three primes. The problem on classification of the half-arc-transitive graphs of order \( n \) has been considered in [J Algebraic Combin 1(1992) 275-282, Discrete Math 310(2010) 1721-1724, European J Combin 28(2007) 726-733], and it was solved for the cases where \( n \) is a prime cube or twice a product of two primes. In this paper, we give the classification of the tetravalent half-arc-transitive graphs of order \( pqr \), where \( p, q, r \) are distinct odd primes.

Keywords- cayley graph; vertex-transitive graph; half-arc-transitive graph; simple group;quotient graph

I. INTRODUCTION

All graphs considered in this paper are finite, connected, undirected and simple, but with an implicit orientation of the edges when appropriate. Given a graph \( X \), denote by \( V(X) \), \( E(X) \), \( A(X) \) and \( Aut(X) \) the vertex set, edge set, arc set and automorphism group of \( X \), respectively. A graph \( X \) is said to be vertex-transitive, edge-transitive and arc-transitive(symmetric) if \( Aut(X) \) acts transitively on \( V(X) \), \( E(X) \) and \( A(X) \), respectively. The graph \( X \) is said to be half-arc-transitive provided that it is vertex- and edge- but not arc-transitive. More generally, by a half-arc-transitive action of a subgroup \( G \) of \( Aut(X) \) on \( X \) we shall mean a vertex- and edge-, but not arc-transitive action of \( G \) on \( X \). In this case we say that the graph \( X \) is \( G \)-half-arc-transitive.

In 1947, Tutte[1] initiated the investigation of half-arc-transitive graphs by showing that a vertex- and edge-transitive graph with odd valency must be arc-transitive, and few years later, Bouwer[2] gave a construction of \( 2k \)-valent half-arc-transitive graph for every \( k \geq 2 \). Following these two classical articles, half-arc-transitive graphs have been extensively studied from different perspectives over decades by many authors. (for example, see [3,4,5,6]). In fact, constructing and characterizing half-arc-transitive graphs with small valencies is currently an active topic in algebraic graph theory. In view of the fact that 4 is the smallest admissible valency for a half-arc-transitive graph, special attention has rightly been given to the study of tetravalent half-arc-transitive graphs. In particular, constructing and classifying the tetravalent half-arc-transitive graphs is currently one of active topics in algebraic graph theory (for example, see [7-9] and [10-13]).

II. PRELIMINARY RESULTS

For the purpose of this paper, we introduce a result due to Marusic.

Let \( m \geq 3 \) be an integer, \( n \geq 3 \) an odd integer and let \( r \in \mathbb{Z}_n^* \) satisfy \( r^n = \pm 1 \). The graph \( X(r; m, n) \) is defined to have vertex set \( V = \{ u_j \mid j \in \mathbb{Z}_m, j \in \mathbb{Z}_n \} \) and edge set \( E = \{ \{ u_j, u_{j+r} \} \mid j \in \mathbb{Z}_m, j \in \mathbb{Z}_n \} \).

A. Proposition 2.1

[14, Theorem 3.4] A connected tetravalent graph \( X \) is a tightly attached half-arc-transitive graph of odd radius \( n \) if and only if \( X \cong X(r;m,n) \), where \( m \geq 3 \), and \( r \in \mathbb{Z}_n^* \) satisfying \( r^n = \pm 1 \), and moreover none of the following conditions is fulfilled:

1. \( r^n = \pm 1 \);
2. \( (r;m,n) = (2;3,7) \);
3. \( (r;m,n) = (r;6,7k) \), where \( k \geq 1 \) is odd, \( (7,k) = 1 \), \( r^k = 1 \), and there exists a unique solution \( q \in \{ 7-r, 7^{-1}, r^{-1} \} \) of the equation \( x^2 + x - 2 = 0 \) such that \( 7(q-1) = 0 \) and \( q \equiv 5 \pmod{7} \).

Now we state two simple observations about half-arc-transitive graphs.

B. Proposition 2.2

[13, Proposition 2.6] Let \( X \) be a connected half-arc-transitive graph of valency 2n. Let \( A = Aut(X) \) and let \( A_u \) be the stabilizer of \( u \in V(X) \) in \( A \). Then each prime divisor of \( |A_u| \) is a divisor of \( n! \).

C. Proposition 2.3

[9, Propositions 2.1 and 2.2] Let \( X = Cay(G, S) \) be half-arc-transitive. Then \( S \) contains no involutions, and there is no \( a \in Aut(G, S) \) such that \( s^a = s^{-1} \) for some \( s \in S \). In particular, there are no half-arc-transitive Cayley graphs on abelian groups.

The following propositions are some results about group theory. Check the orders of the non-abelian simple groups, we have the following proposition.

D. Proposition 2.4

[15, pp. 12-14, 135-136] Let \( G \) be a non-abelian simple group and let \( p \cdot q \cdot r \) be odd primes. If \( |G| \) has at most three prime divisors then \( G \) is isomorphic to one of the following:

- \( \text{A}_6, \text{A}_5, \text{PSL}(2,7), \text{PSL}(2,8), \text{PSL}(2,17), \text{PSL}(3,3), \text{PSU}(3,3) \), \( \text{PSU}(4,2) \).
- If \( |G| = 2^m \) then \( G \cong \text{Sz}(8) \), \( \text{PSL}(2,p) \), or \( \text{PSL}(2,2^t) \) with an integer \( t \geq 4 \).
Let $p$ be a prime and $G = \text{PSL}(2,p^f)$. Assume that $P$ is a Sylow $p$-subgroup of $G$, and $A$ and $B$ are cyclic subgroups of $G$ of order $(p^f-1)/(2p^f-1)$ and $(p^f+1)/(2p^f-1)$, respectively. It is well known that for any $g \in G$, $P \cap gP = 1$ or $P = gP$, $A \cap gA = 1$ or $A = gA$, and $B \cap gB = 1$ or $B = gB$. Furthermore, $N_G(A) \cong D_{2|A|}$, $N_G(B) \cong D_{2|B|}$, $N_G(A)$ and $N_G(B)$ are maximal subgroups of $G$. Then we have the following proposition

**Proposition 2.5.**

Let $p$ be a prime and $G = \text{PSL}(2,p^f)$. Assume that $P$ is a Sylow $p$-subgroup of $G$ and $H$ is a maximal dihedral subgroup of $G$. Then for any $g \in G$, $P = gP$ or $P \cap gP = 1$, and $H = gH$ or $|H \cap gH| \leq 2$.

### III. Classification

In this section, we determine the classification of tetravalent half-arc-transitive graphs of order $pqr$. The main ideas for the paper come from two situations which named “Primitive” and “Non-Primitive”. Fig. 1 showed the idea for the method.
Then there exists an element \( s \in A \) such that \( |H \cap H^s| = 2 \), \( <H, s>=A \) and \( HsH \neq Hs^2H \). However, by magma, there exists an element \( g \in \text{PGL}(2,11) \) or \( \text{PGL}(2,13) \) such that \( (HsH)g = Hs^2H \). It follows that \( X \) is symmetric, a contradiction. Thus, we may assume that \( N \) is solvable. From the “Primitive” situation we could get the complete graph. Fig. 2 showed the algorithms for “Non-Primitive” situation. We can get the graphs from four different lengths.

![Figure 2. Non-Primitive algorithm process](image)

**D. Claim II**

\( M \) is not isomorphic to \( Z_{qr} \). Suppose that \( M \cong Z_{qr} \). Let \( C = C_A(M) \). Then \( A/C \cong Z_{qr} \). Since \( p > q > r \), we have \( pqr | |C| \), that is \( C > M \). Take a minimal normal subgroup \( B/M \) of \( A/M \) such that \( B/M \leq C/M \). Then \( B/M \cong Z_p \) or \( Z_q \). Note that \( A \) has no non-trivial normal 2-subgroup. Suppose that \( B/M \cong Z_p \). Then the Sylow 2-subgroup of \( B \) is normal in \( A \), a contradiction. Thus \( B/M \cong Z_q \), that is \( B \cong Z_{pq} \). Then \( X \) is a Cayley graph on \( B \), by Proposition 2.3, it is impossible.

**E. Claim III**

If \( M \cong Z_{qr} \) for \( t = q \) or \( r \), then \( X \cong X(s; q, pr) \) or \( X(s; r, pq) \). In this case, \( A/C \leq Z_{q-1}Z_{r-1} \). Consider the quotient graph \( X_M \) of \( X \) corresponding to the orbits of \( M \). Then \( |X_M| = qr/t \). If \( X_M \) has valency 4, then the stabilizer \( K_u \) of \( u \) in \( K \) fixes each neighborhood of \( u \) in \( X \) because \( K \) fixes each orbit of \( M \). It follows that \( K_u = 1 \) and \( K = M \). This implies that \( X_M \) is \( A/M \)-half-arc-transitive, furthermore, \( A/M \) is non-abelian. Thus \( C > M \). If \( X_M \) has valency 2 and \( K_u = 1 \), then \( A \cong \text{D}_{q, r, t} \). In this case we have \( C > M \). Take a minimal normal subgroup \( B/M \) of \( A/M \) such that \( B/M \leq C/M \). Note that \( A \) has no non-trivial normal 2-subgroup. Then \( B/M \cong Z_{qr} \). Thus \( B/M \cong Z_{pq} \), that is \( B \cong Z_{pq} \). Then \( X \) is a Cayley graph on \( B \), a contradiction. Thus \( X_M \) has valency 2 and \( K_u \neq 1 \). It is easy to know that \( X \cong X(s; q, pr) \) or \( X(s; r, pq) \).

By Claim I, \( N \cong Z_p \), \( Z_q \), or \( Z_r \). Let \( C = C_A(N) \). First, we assume that \( N \cong Z_q \). Then \( A/C \leq Z_{q-1} \). Note that \( q > q > r \). That means that \( pq || C | \). Take a minimal normal subgroup \( B/N \) of \( A/N \) such that \( B/N \leq C/N \). Then \( B/N \cong Z_p \) or \( Z_q \). It follows that \( B \cong Z_{pr} \) or \( Z_{qr} \) and \( B \) is a normal subgroup of \( A \). By Claim II and III, we have that \( B \cong Z_{pr} \) and \( X \cong X(s; r, pq) \). Second, we assume that \( N \cong Z_p \). Similarly, we have \( X \cong X(s, r, pq) \). Now we assume that \( N \cong Z_r \). Similarly argument to Claim III, we have that \( X \cong X(s, r, pq) \), \( X(s, q, pr) \) or \( X(s, qr, p) \). By Proposition 2.1, \( X \cong X(s, r, pq) \), \( X(s, q, pr) \) and \( X(s, qr, p) \) are half-arc-transitive graphs.

**F. Remark**

In fact, by [6], we know these graphs are normal Cayley graphs on Frobenius group.

**IV. CONCLUSION**

In the paper, we give the classification of tetravalent half-arc transitive graph of order \( pq \), and proved that tetravalent half-arc transitive graph of order \( pq \) must be tightly attached half-arc-transitive graph. In fact, we know that these graphs must be normal Cayley graphs on meta-cyclic group (see [6]), implying that these graphs must be meta-circulant. So far, the known half-arc transitive graphs are mostly meta-circulant, many scholars focus on judging whether half-arc transitive
graphs is meta-circulant or not and how to form non-meta-circulant half-arc transitive graphs. Next, we want to give the classification of tetravalent half-arc transitive graphs of order square free and give some examples of non-meta-circulant half-arc-transitive graph.

ACKNOWLEDGMENT
This work was supported by the National Natural Science Foundation of China (11201180), the National Natural Science Foundation of Shandong (ZR2012AQ023) and the Doctoral Program of University of Jinan (XBS1212)

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