A New Method for Coupled Best Proximity Point Theorems in Partially Ordered Metric Spaces

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Abstract—Several problems can be changed as equations of the form $Tx = x$, where $T$ is a given self-mapping defined on a subset of a metric space, a normed linear space, a topological vector space or some suitable space. However, if $T$ is a non-self mapping from $A$ to $B$, then the aforementioned equation does not necessarily admit a solution. In this case, it is contemplated to find an approximate solution $x$ in $A$ such that the error $d(x, Tx)$ is minimum, where $d$ is the distance function. In view of the fact that $d(x,Tx)$ is at least $d(A,B)$, a best proximity point theorem guarantees the global minimization of $d(x, Tx)$ by the requirement that an approximate solution $x$ satisfies the condition $d(x, Tx) = d(A,B)$. Such optimal approximate solutions are called best proximity points of the mapping $T$. Interestingly, best proximity point theorems also serve as a natural generalization of fixed point theorems, for a mapping $T$ is a non-self mapping from $A$ to $B$, then the best proximity point becomes a fixed point if the mapping under consideration is a self-mapping. Research on the best proximity point problem is whether we can find an element $x_0 \in A$ such that

$$d(x_0, Tx_0) = \min \{d(x, Tx) : x \in A\}.$$

Since $d(x, Tx) \geq d(A,B)$ for any $x \in A$, in fact, the optimal solution to this problem is the one for which the value $d(A,B)$ is attained.

In [4], the authors give a generalized result by considering a nonself map and they get the following theorem.

**Theorem 1.2** [4] Let $(A,B)$ be a pair of nonempty subsets of a complete metric space $(X,d)$ such that $A_0$ is nonempty. Let $T : A \to B$ be a Geraghty-contraction satisfying $T(A_0) \subseteq B_0$. Suppose that the pair $(A,B)$ has the $P$-property. Then there exists a unique $x^*$ in $A$ such that $d(x^*, Tx^*) = d(A,B)$.

Let us recall the following definitions.

**Definition 1.3** [4] Let $(X,\preceq)$ be a partially ordered set and $F : X \times X \to X$. We say that $F$ has the mixed monotone property if $F(x, y)$ is monotone nondecreasing in $x$ and is monotone nonincreasing in $y$, that is, for any $x, y \in X$.

$P$-property.

**Definition 1.1** [3] Let $(A,B)$ be a pair of nonempty subsets of a metric space $(X,d)$ with $A_0 \neq \emptyset$. Then the pair $(A,B)$ is said to have the $P$-property if and only if for any $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$,

$$d(x_1, y_1) = d(A,B) \Rightarrow d(x_1, x_2) = d(y_1, y_2).$$

Let $A, B$ be two nonempty subsets of a complete metric space and consider a mapping $T : A \to B$. The best proximity point problem is whether we can find an element $x_0 \in A$ such that

$$d(x_0, Tx_0) = \min \{d(x, Tx) : x \in A\}.$$
\[ x_1, x_2 \in X, x_1 \leq x_2 \Rightarrow F(x_1, y) \leq F(x_2, y); \]
\[ y_1, y_2 \in X, y_1 \leq y_2 \Rightarrow F(x, y_1) \geq F(x, y_2). \]

This definition coincides with the notion of a mixed monotone function on \( R^2 \) and \( \leq \) represents the usual total order in \( R \).

**Definition 1.4** [4] We call an element \((x, y) \in X \times X\) a coupled fixed point of the mapping \( F\) if \( F(x, y) = x, F(y, x) = y \).

T. Gnana Bhaskar, V. Lakshmikantham got the following theorems in 2006.

**Theorem 1.5** [4] Let \((X, \leq)\) be a partially ordered set and suppose that there exists a metric \( d \) in \( X \) such that \((X, d)\) is a complete metric space. Let \( F: X \times X \to X \) be a continuous mapping having the mixed monotone property on \( X \). Assume that there exists a \( k \in (0, 1] \) with
\[ d(F(x, y), F(u, v)) \leq \frac{k}{2} [d(x, u) + d(y, v)] \forall x \geq u, y \leq v. \]
If there exists \( x_0, y_0 \in X \) such that \( x_0 \leq F(x_0, y_0) \) and \( y_0 \geq F(y_0, x_0) \)
then, there exist \( x, y \in X \) such that \( x = F(x, y), y = F(y, x) \)

**Theorem 1.6** [4]. Let \((X, \leq)\) be a partially ordered set and suppose that there exists a metric \( d \) in \( X \) such that \((X, d)\) is a complete metric space. Assume that \( X \) has the following property:
(i) if a nonincreasing sequence \( \{x_n\} \to x \), then \( x_n \leq x, \forall n; \)
(ii) if a nonincreasing sequence \( \{y_n\} \to y \), then \( y_n \leq y, \forall n. \)

Let \( F: X \times X \to X \) be a mapping having the mixed monotone property on \( X \). Assume that there exists a \( k \in (0, 1] \) with
\[ d(F(x, y), F(u, v)) \leq \frac{k}{2} [d(x, u) + d(y, v)] \forall x \geq u, y \leq v. \]
If there exists \( x_0, y_0 \in X \) such that \( x_0 \leq F(x_0, y_0) \) and \( y_0 \geq F(y_0, x_0) \)
then, there exist \( x, y \in X \) such that \( x = F(x, y), y = F(y, x) \)

It can be proved that the coupled fixed point is in fact unique, provided that the product space \( X \times X \) endowed with the partial order mentioned above enjoying the following property:
Every pair of elements has either a lower bound or an upper bound.

It is known [5] that this condition is equivalent to:

**Condition (*)**: For every \((x, y), (x^*, y^*) \in X \times X\), there exists a \((z_1, z_2) \in X \times X\) that is comparable to \((x, y), (x^*, y^*)\).

**Theorem 1.7** [4]. Adding condition (*) to the hypothesis of Theorem 1.8, then the uniqueness of the coupled fixed point of \( F \) can be obtained.

**Theorem 1.8** [4]. In addition to the hypothesis of Theorem 1.8, suppose that every pair of elements of \( X \) has an upper bound or a lower bound in \( X \). Then \( x = y \).

**Theorem 1.9** [4]. In addition to the hypothesis of Theorem 1.8 (resp. Theorem 1.9), suppose that \( x_0, y_0 \) in \( X \) are comparable. Then \( x = y \).

We introduce the following definition.

**Definition 1.10** Let \( A, B \) be subsets of a metric space \( X \). An element \((x, y) \in A \times A\) is called a coupled best proximity point of \( F: A \times A \to B \) if
\[ d(F(x, y), x) = d(A, B), d(y, F(y, x)) = d(A, B). \]

The aim of this paper is to obtain the coupled best proximity point theorems for generalized contraction in partially ordered metric spaces by \( P \)-operator technique. An example has also been given to illustrate the theorems. Many recent results in this area have been improved.

II. MAIN RESULTS

**Weak \( P \)-monotone property**: Let \((A, B)\) be a pair of nonempty subsets of a metric space \((X, d)\) with \( A_0 \neq \emptyset \). Then the pair \((A, B)\) is said to have the weak \( P \)-monotone property if and only if for any \( x_1, x_2 \in A_0 \), and \( y_1, y_2 \in B_0 \),
\[ d(x_1, y_1) = d(A, B), \]
\[ d(x_2, y_2) = d(A, B) \]

furthermore, \( y_1 \geq y_2 \) implies \( x_1 \geq x_2 \).

Now we are in a position to give our main results.

**Theorem 2.1**. Let \( X \) be a partially ordered set and \((X, d)\) is a complete metric space. Let \((A, B)\) be a pair of nonempty closed subsets of \( X \) such that \( A_0 \neq \emptyset \). Let \( F: A \times A \to B \) be a continuous mapping with \( F: A_0 \times A_0 \subseteq B_0 \). Suppose that \( F \) has mixed monotone property satisfying
\[ d(F(x, y), F(u, v)) \leq \frac{k}{2} [d(x, u) + d(y, v)] \forall x \geq u, y \leq v, k \in (0, 1] \]

Suppose that the pair \((A, B)\) has the weak \( P \)-monotone property.
property. If there exist \( x_0, y_0 \in A_0 \) such that 
\[
d(x_0, \hat{x}_0) = d(A, B), \quad d(y_0, \hat{y}_0) = d(A, B),
\]
\[
\hat{x}_0 \leq F(x_0, y_0), \quad \hat{y}_0 \geq F(y_0, x_0).
\]
where \( \hat{x}_0, \hat{y}_0 \in B_0 \). Then there exists a \((x^*, y^*) \in A \times A\) such that 
\[
d(x^*, F(x^*, y^*)) = d(A, B), d((y^*, x^*), y^*) = d(A, B).
\]

**Proof.** We first prove that \( B_0 \) is closed. Let \( \{y_n\} \subseteq B_0 \) be a sequence such that \( \{y_n\} \rightarrow y \in B \). It follows from the weak P-monotone property that 
\[
d(y_n, y) \rightarrow 0 \Rightarrow d(x_n, x_m) \rightarrow 0,
\]
as \( n, m \rightarrow \infty \), where \( x_n, x_m \in A_0 \), and 
\[
d(x_n, y_n) = d(A, B), d(x_m, y_m) = d(A, B).
\]
Then \( \{x_n\} \) is a Cauchy sequence so that \( \{x_n\} \) converges strongly to a point \( p \in A \). By the continuity of a metric \( d \), we have 
\[
d(p, q) = d(A, B).
\]
That is, \( q \in B_0 \). Hence, \( B_0 \) is closed.

Let \( \overline{A}_0 \) be the closure of \( A_0 \) we claim that 
\[
F : \overline{A}_0 \times \overline{A}_0 \subseteq B_0 \in B_0. \quad \text{In fact, if } x, y \in \overline{A}_0 \setminus A_0 \text{, then there exist sequences } x_n, y_n \subseteq A_0 \text{ such that } x_n \rightarrow x, y_n \rightarrow y.
\]
By the continuity of \( F \) and the closeness of \( B_0 \), we have 
\[
F(x, y) = \lim_{n \rightarrow \infty} F(x_n, y_n) \subseteq B_0.
\]
That is, \( F(\overline{A}_0 \times \overline{A}_0) \subseteq B_0 \).

Define an operator \( P_{A_0} : F(\overline{A}_0 \times \overline{A}_0) \rightarrow A_0 \) by 
\[
P_{A_0} y = \{ x \in A : d(x, y) = d(A, B) \}.
\]
From the weak \( P \)-monotone property, we can know that \( P_{A_0} \) is single valued. By the definition of \( F \) and the weak \( P \)-monotone property, we have 
\[
d(P_{A_0} F(x, y), P_{A_0} F(u, v)) \leq d(F(x, y), F(u, v)) \leq \frac{k}{2} d(x, u) + d(y, v)
\]
for any \( x \geq u, y \leq v \in \overline{A}_0 \). Let \( x_n, y_n, x, y \in \overline{A}_0 \), \( x_n \rightarrow x, y_n \rightarrow y \). From the above inequality and \( F \) is continuous, we have 
\[
F(x_n, y_n) \rightarrow \text{ as } n \rightarrow \infty.
\]
\[
F(x_n, y_n) \rightarrow F(x, y)
\]
\[
\Rightarrow d(F(x_n, y_n), F(x, y)) \rightarrow 0
\]
\[
\Rightarrow d(P_{A_0} F(x_n, y_n), F(x, y)) \rightarrow 0
\]
\[
\Rightarrow P_{A_0} F(x_n, y_n) \rightarrow P_{A_0} F(x, y) \text{ as } n \rightarrow \infty.
\]
So \( P_{A_0} F \) is continuous. Since \( F \) has the mixed monotone property and \( (A, B) \) has the weak \( P \)-monotone property, we can get 
\[
d(P_{A_0} F(x, y), F(x, y)) = d(A, B)
\]
\[
d(P_{A_0} F(u, v), F(u, v)) = d(A, B)
\]
\[
F(x, y) \geq F(u, v) \subseteq B_0
\]
\[
\Rightarrow P_{A_0} F(x, y) \geq P_{A_0} F(u, v).
\]
As the similar way, we can get 
\[
\begin{align*}
&d(P_{A_0} F(x, y), F(x, y)) = d(A, B) \\
&d(P_{A_0} F(x, v), F(x, v)) = d(A, B) \\
&F(x, y) \geq F(x, v) \subseteq B_0
\end{align*}
\]
\[
\Rightarrow P_{A_0} F(x, y) \geq P_{A_0} F(x, v).
\]
For any \( x \geq u, y \leq v \in \overline{A}_0 \). This shows that \( P_{A_0} F \) is mixed monotone. Because there exist \( x_0, y_0 \in A_0 \) such that 
\[
d(x_0, \hat{x}_0) = d(A, B), d(y_0, \hat{y}_0) = d(A, B),
\]
\[
\hat{x}_0 \leq F(x_0, y_0), \hat{y}_0 \geq F(y_0, x_0).
\]
where \( \hat{x}_0, \hat{y}_0 \in B_0 \). Then we can obtain 
\[
\begin{align*}
&d(P_{A_0} F(x_0, y_0), F(x_0, y_0)) = d(A, B) \\
&d(P_{A_0} F(x_0, v), F(x_0, v)) = d(A, B) \\
&\hat{x}_0 \leq F(x_0, y_0), \hat{y}_0 \geq F(y_0, x_0).
\end{align*}
\]
\[
\Rightarrow x_0 \leq P_{A_0} F(x_0, y_0).
\]
In the same way, we have \( y_0 \leq P_{A_0} F(y_0, x_0) \).

This shows that \( P_{A_0} F : (\overline{A}_0 \times \overline{A}_0) \rightarrow \overline{A}_0 \) is a contraction satisfying all the conditions in Theorem 1.8. Therefore, \( P_{A_0} F \) has a coupled fixed point \((x^*, y^*)\).

That is 
\[
P_{A_0} F(x^*, y^*) = x^* \in A_0, P_{A_0} F(y^*, x^*) = y^* \in A_0,
\]
which implies that 
\[
d(x^*, F(x^*, y^*)) = d(A, B), d(F(y^*, x^*), y^*) = d(A, B).
\]
That is the desired result.

The previous result still hold for \( F \) not necessarily continuous. Instead, we only need to require an additional property on \( X \). We discuss this in the following theorem.

**Theorem 2.2** Let \( X \) be a partially ordered set and \((X, d)\) is a complete metric space. Let \((A, B)\) be a pair of nonempty closed subsets of \( X \) such that \( A_0 \neq \emptyset \).

Let \( F : A \times A \rightarrow B \) be a mapping with \( F : A_0 \times A_0 \subseteq B_0 \). Suppose that \( F \) has mixed monotone property satisfying 
\[
d(F(x, y), F(u, v)) \leq \frac{k}{2} d(x, u) + d(y, v)
\]
\[
\forall x \geq u, y \leq v, k \in (0,1]
\]
Assume that \( \overline{A}_0 \) has the following property:

(i) if a nonincreasing sequence \( \{x_n\} \rightarrow x \), then \( x_n \leq x, \forall n \); 

(ii) if a nonincreasing sequence \( \{y_n\} \rightarrow y \), then \( y_n \leq y, \forall n \).

Suppose that the pair \((A, B)\) has the weak \( P \)-monotone property. If there exist \( x_0, y_0 \in A_0 \) such that 
\[
d(x_0, \hat{x}_0) = d(A, B), d(y_0, \hat{y}_0) = d(A, B),
\]
\[
\hat{x}_0 \leq F(x_0, y_0), \hat{y}_0 \geq F(y_0, x_0).
\]
where \( \hat{x}_0, \hat{y}_0 \in B_0 \). Then there exists a \((x^*, y^*) \in A \times A \)
such that
\[ d(x^*, F(x^*, y^*)) = d(A, B), \]
\[ d((y^*, x^*), y^*) = d(A, B). \]

**Proof:** The proof is the same as Theorem 2.1 without proving the continuity of \( P_{A_0} F \). Then Theorem 2.2 can be got by using Theorem 1.9.

**REFERENCES**


