A Comparison between Lattice-Valued Propositional Logic LP(X) and Gradational Lattice-Valued Propositional Logic L_{vpl}

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Abstract

By comparing the definitions of L_{vpl} with the corresponding definitions of LP(X), the meaning of the parameters of L_{vpl} and the role of inference rule set \( R \) playing in uncertainty reasoning based on L_{vpl} are demonstrated. And to some extent it is shown that the choose of \( R \) determines the semantic and syntactic function of uncertainty reasoning based on L_{vpl}.

Keywords: Lattice-Valued Propositional Logic, Gradational Lattice-Valued Propositional Logic, rule of inference, syntax, semantics

1. Introduction

In order to provide a logical foundation for uncertain information processing theory, especially for the fuzziness, the incomparability uncertain information in the reasoning, Xu presented the lattice implication algebra by combining lattice and implication algebra in 1993 [1]. After then, he and his research group have established the lattice-valued propositional logic LP(X) and gradational lattice-valued propositional logic L_{vpl}, lattice-valued first-order logic and gradational lattice-valued first-order logic and applied them to automated and approximated reasoning [2]-[12]. As we know, during the uncertainty reasoning process based on L_{vpl} the chosen of the value set \( T \) and the implication operator \( \rightarrow \) is very important. Chen researched uncertainty reasoning based on lattice-valued first-order logic L_{vfl} and give some concert methods for selecting appropriate parameters during the uncertainty reasoning process based on L_{vfl} for some representative uncertainty reasoning models[9]. In [9] Chen defined

\[ R^* = \left\{ (r^0_\alpha, t^0_\gamma), (r^1_\alpha, t^1_\gamma) \right\} \cup \left\{ (r^i_\alpha, t^i_\gamma) \mid \theta_\alpha \in L \right\} \]

\[ \cup \left\{ (r^i_\alpha, t^i_\gamma) \mid u \in U \right\} \cup \left\{ (r^i_\alpha, t^i_\gamma) \mid u \in U \right\} \cup \left\{ (r^i_\alpha, t^i_\gamma) \mid u \in U \right\} \subseteq \mathbb{R} \]

and \( I \) is \( \alpha \)-type closed w.r.t. \( R^* \).

where,

\[ r^0_\alpha (\phi, \phi \to \psi) = \psi \]

\[ r^1_\alpha (\phi \to \gamma, \phi \to \psi) = \phi \to (\gamma \land \psi) \]

\[ r^i_\alpha (\phi \to \psi, \phi \to \gamma) = \phi \to \gamma \]

\[ r^i_\alpha (\phi) = \theta_\alpha \to \phi \]

\[ t^i_\alpha (\theta_\alpha) = \theta_\alpha \to \alpha \]

\[ t^i_\alpha (\phi) = (Q_x) \phi \]

\[ t^{\ast}_i (\phi) = (Q_x) \phi \to \psi \]

But we don’t know why \( R^* \) is chosen as this and what the function of \( R^* \) is.

2. Comparing the corresponding definitions

In the following we suppose the sets \( \mathcal{F} \) and \( \mathcal{F}_p \) are the sets of formula of LP(X) and L_{vpl} respectively. By comparing the definitions of LP(X) with the counterpart of L_{vpl}, the meaning of the parameters of L_{vpl} is understood.

**Definition 2.1** [2] A mapping \( v : \text{LP}(X) \to L \) is called a valuation of LP(X), if it is a T-homomorphism.

**Corollary 2.1** [10] Let \( f : \text{LP}(X) \to L \) be a mapping of LP(X), then \( f \) is a valuation of LP(X) if and only if it satisfies

1. \( f(\alpha) = \alpha \) for any \( \alpha \in L \);
2. \( f(p^\prime) = \left( f(p) \right)^\prime \) for any \( p \in \mathcal{F} \);
3. \( f(p \to q) = f(p) \to f(q) \) for any \( p, q \in \mathcal{F} \).
Definition 2.2 [4] Let \( \mathcal{T}_L(\mathcal{F}) \), \( v \) be a valuation of LP(X). It is called that \( v \) satisfies \( A \) if \( A(p) \leq v(p) \) for any \( p \in \mathcal{F} \). \( A \) is called satisfiable if there exists a valuation \( v \), which satisfies \( A \).

Definition 2.3 [6] Let \( X \in \mathcal{T}_L(\mathcal{F}_r) \), \( \delta \in L \), \( T_0 \in \mathcal{T} \). If for any \( p \in \mathcal{F}_r \),

1. \( T_0(p') = (T_0(p))' \),
2. \( X(p) \rightarrow T_0(p) \geq \delta \)

then \( T_0 \) is called \( \delta \)-type closed. \( T_0 \) \( \delta \)-type satisfies \( X \). If there exists \( T_0 \), \( T_0 \) \( \delta \)-type satisfies \( X \), then \( X \) is said to be \( \delta \)-type sufficient, \( i = I, II \).

Definition 2.4 [4] Let \( A \in \mathcal{T}_L(\mathcal{F}) \), \( A \) is called closed if

1. \( A(p \rightarrow q) \otimes A(p) \leq A(q) \),
2. \( \alpha \rightarrow A(p) \leq A(\alpha \rightarrow p) \)

for any \( p, q \in \mathcal{F} \) and \( \alpha \in L \).

Definition 2.5 [6] Let \( X \in \mathcal{T}_L(\mathcal{F}_r) \), \( (r, t) \in \mathcal{R}_n \), \( n \in \mathcal{R} \), \( \alpha \in L \). If

\[
X \circ r \supseteq \alpha \otimes \left( t \circ \prod X \right)
\]

\( r \rightarrow t \) in \( D_n(r) \), then \( X \) is said to be \( \alpha \)-I type closed w.r.t. \( (r, t) \). If

\[
X \circ r \supseteq t \circ \left( \prod \alpha \otimes X \right)
\]

in \( D_n(r) \), then \( X \) is said to be \( \alpha \)-II type closed w.r.t. \( (r, t) \).

In definition 2.5 \( \alpha \) denote the consistency of the semantics and syntax. When \( \alpha = I \), and \( \mathcal{R} = \left\{ \left( t_0^0, l_0^0 \right), \left( t_0^0, l_0^0 \right) \right\} \), the formula (1), (2) are changed into

\[
X(p \rightarrow q) \otimes X(p) \leq X(q) \quad \alpha \rightarrow X(p) \leq X(\alpha \rightarrow p) \text{ respectively, which is the same as Definition 2.4.}
\]

In another word in LP(X) “\( A \) is closed” is to be said for \( \left( r_0^0, l_0^0 \right), \left( r_0^0, l_0^0 \right) \) these two inference rules.

Semantics of LP(X) and L

Definition 2.6 [4] Let \( A \in \mathcal{T}_L(\mathcal{F}) \), \( p \in \mathcal{F} \) and \( \alpha \in L \). \( p \) is called semantically implied from \( A \) with truth value level \( \alpha \) if \( \forall(p) \geq \alpha \) for any valuation \( \nu \), which satisfies \( A \) of LP(X). We write this \( A \models_{\alpha} p \), \( p \) is called valid with truth value level \( \alpha \) (shortly for \( \alpha \)-valid) and denoted by \( \models_{\alpha} p \), if \( \forall(p) \geq \alpha \) for any valuation \( \nu \) of LP(X). \( p \) is called a valid formula, if \( \models p \). \( p \) is called \( \alpha \)-valid in \( A \).

\[
\alpha = \land \left\{ \nu(p) \right\} \nu \text{ is a valuation and satisfies } A \}
\]

We shall write this \( A \models_{\alpha} p \).

Con \( (A(p))) \cap \left\{ \nu(p) \right\} \nu \text{ is a valuation of } L \text{ and satisfies } A \}

Con \( (A)p \) denote the degree of \( p \) can be semantically implied from \( A \).

The corresponding semantics of \( L_{\alpha \beta} \) is given as a mapping \( C_X^\alpha(p) \).

Definition 2.7 [6]

\[
C_X^\alpha(p) \bigcap \left[ \left[ \prod \left[ \left[ \left[ X(q) \rightarrow T(q) \right] \rightarrow T(p) \right] \right] \right] \right] \text{ }
\]

We can see this operator is a semantic operator of \( L_{\alpha \beta} \). \( C_X^\alpha(p) \) denotes the degree of \( p \) can be semantically implied from \( X \).

Theorem 2.1 [10] Each valuation of LP(X) is closed.

Theorem 2.2 Let \( A \) be closed. Then \( A(\alpha) \leq \alpha \), and \( A(\alpha') \leq A(\alpha) \) for any \( \alpha \in L \) and \( p \in \mathcal{F} \).

Proof. \( A \) is closed so for any \( \alpha \in L \) and \( p \in \mathcal{F} \) we have

\[
\alpha \rightarrow A(p) \leq A(\alpha \rightarrow p) \leq A(\alpha) \rightarrow A(p)
\]

Hence \( A(\alpha) \leq \alpha \), and \( A(0) = 0 \).

\[
A(p') = A(p) \rightarrow 0 \leq A(p) \rightarrow A(0)
\]

\[
= A(p) \rightarrow 0 = A(p')
\]

Theorem 2.3 If \( \forall \neq \mathcal{R} \subseteq \left\{ A | A \text{ is closed} \right\} \), then

\( \bigcap_{\alpha \in A} A \) is closed.

Proof. For any \( \alpha \in L \) and \( p \in \mathcal{F} \)

\[
\bigcap_{\alpha \in A} A(q) = \land \left\{ \land \left\{ A(p) \otimes A(p \rightarrow q) \right\} \right\}
\]

\[
\supseteq \land \left\{ A(p) \otimes A(p \rightarrow q) \right\}
\]

\[
= \land \left\{ A(p) \otimes A(p \rightarrow q) \right\}
\]

\[
\left( \bigcap_{\alpha \in A} A \right)(\alpha \rightarrow p) \supseteq \land \left\{ A(p) \right\}
\]

\[
\geq A \rightarrow \land A(p)
\]

\[
= \alpha \rightarrow \left( \bigcap_{\alpha \in A} A \right)(p).
\]

Corollary 2.2 If \( A \models_{\alpha} p \), \( A \models_{\beta} p \rightarrow q \), then

1. \( A \models_{\alpha \land \beta} q \),
2. \( A \models_{\alpha \rightarrow \beta} \alpha_0 \rightarrow p \) for any \( \alpha_0 \in L \).

Proof.

Con \( (A(p)) \cap \left\{ \nu(p) \right\} \nu \text{ is a valuation of } L \text{ and satisfies } A \}

Hence \( Con(A) \cap Con(A) \leq Con(A(q)) \), \( \alpha_0 \rightarrow Con(A)p \leq Con(A)(\alpha_0 \rightarrow p) \) for any
$p,q \in \mathcal{F}$ and $\alpha_0 \in L$. If $A \models_{\omega} p$, $A \models_{\beta} p \rightarrow q$, i.e. $\alpha \otimes \beta \leq \text{Con}(A)(p \rightarrow q) \otimes \text{Con}(A)(p) \leq \text{Con}(A)(q)$, $\alpha_0 \rightarrow \alpha \leq \alpha_0 \rightarrow \text{Con}(A)(p) \leq \text{Con}(A)(\alpha_0 \rightarrow p)$, hence $A \models_{\omega \otimes \beta} q$, $A \models_{\alpha_0 \rightarrow \alpha} \alpha_0 \rightarrow p$ for any $\alpha_0 \in L$.

**Syntax of LP(X) and LVPL**

**Definition 2.8 [5]** Let $A \in \mathcal{F}_L(\mathcal{F})$, $p \in \mathcal{F}$. A formal proof $\omega$ of $p$ from $A$ is a finite sequence as follows:

$(p_1, \alpha_1), ..., (p_n, \alpha_n)$,

where $p_n = p$, and for any $i$, $1 \leq i \leq n$, $p_i \in \mathcal{F}$, $\alpha_i \in L$ and

1. $A_i(p_i) = \alpha_i$ or
2. $A_i(p_i) = \alpha_i$, or
3. there exist $j,k < i$, such that $p_j = p_k \rightarrow p_i$ and $\alpha_i = \alpha_j \otimes \alpha_k$ or
4. there exists $j < i$ and $\alpha \in L$, such that $p_i = \alpha \rightarrow p_j$ and $\alpha_i = \alpha \rightarrow \alpha_j$.

Where the operation $\otimes$ is defined as $(\alpha \rightarrow \beta')'$ in $L$.

**Definition 2.9 [5]** Let $A \in \mathcal{F}_L(\mathcal{F})$, $p \in \mathcal{F}$, $\alpha \in L$. $p$ is called an $\alpha$-theorem of $A$ and written as $A \vdash_{\alpha} p$, if $\alpha \models \{\text{val}(w) | w \text{ is a proof of } p \text{ from } A\}$.

If $\alpha \models \{\text{val}(w) | w \text{ is a proof of } p \text{ from } A\}$, then it is written as $A \vdash_{\alpha} p$.

In [10] define a mapping

\[\text{Ded}: \mathcal{F}_L(\mathcal{F}) \rightarrow \mathcal{F}_L(\mathcal{F})\]

satisfy

\[\text{Ded}(A)(p) = \bigvee \{\text{val}(w) | w \text{ is a proof of } p \text{ from } A\}\]

for any $p \in \mathcal{F}$ and $A \in \mathcal{F}_L(\mathcal{F})$.

\[\text{Ded}(A)(p) = \bigvee \{B(p) | B \supseteq A \cup A, B \text{ is closed}\}\]

So $\text{Ded}(A)(p)$ denote the degree of $p$ syntactically from $A$.

**Definition 2.10 [6]** Let $X \in \mathcal{F}_L(\mathcal{F}_p)$, $\mathcal{T} \subseteq \mathcal{F}_L(\mathcal{F}_p)$, $p \in \mathcal{F}_p$, $\theta, \alpha, \beta \in L$.

$(\pi, (n), X, (p, \theta) - (\alpha, \beta))$ is said to be a $(\alpha, \beta)I$ type proof with the truth degree $\theta$ from $X$ to $p$ (shortly, $\theta - (\alpha, \beta)$-type proof from $X$ to $p$), if the mapping

$P_\iota(n) \rightarrow \mathcal{F}_p \times L ((n) = \{1, 2, ..., n\})$

satisfies

1. $(p_0, \theta_0) = (p, \theta)$ and
2. $\theta_1 = \beta \otimes C_\iota^\theta(p_1)$, or
3. $\theta_1 = \beta \otimes X(p_1)$, or

(4) there exist $i_1, \cdots, i_k \leq i$, and $(r, t) \in \mathbb{R}$ such that

$(p_0, \theta_0) = (r(p_{i_1}, \cdots, p_{i_k}), \alpha \otimes t(\theta_{i_1}, \cdots, \theta_{i_k}))$, where $n$ is said to be the length of $\theta - (\alpha, \beta)$-I type proof from $X$ to $p$ under $P_\iota$, and denoted as $\iota(P)$.

$\theta - (\alpha, \beta)$-II type proof from $X$ to $p$ is the same as definition 2.10 except for (4) changed into

$(p_0, \theta_0) = (r(p_{i_1}, \cdots, p_{i_k}), t(\alpha \otimes \theta_{i_1}, \cdots, \alpha \otimes \theta_{i_k}))$.

**Definition 2.11 [6]**

$C_{\iota, \theta}^{\alpha, \beta}(\mathcal{T}, \mathcal{X}, \mathcal{Y}) \cap \mathcal{Y} | \mathcal{Y} \not\subseteq \beta \otimes (C_\iota^\theta \cup \mathcal{X})$,

$Y$ is $\alpha$-I type closed w.r.t $\mathbb{R}$.

**Theorem 2.4 [6]** Let $X \in \mathcal{F}_L(\mathcal{F}_p)$, $\mathcal{T} \subseteq \mathcal{F}_L(\mathcal{F}_p)$, $\alpha, \beta \in L$, and the truth-valued operations in $\mathbb{R}$ satisfy finite semicontinuity, then for any $p \in \mathcal{F}_p$

$C_{\iota, \theta}^{\alpha, \beta}(\mathcal{T}, \mathcal{X}, \mathcal{Y}) \cap \mathcal{Y} | \mathcal{Y} \not\subseteq \beta \otimes (C_\iota^\theta \cup \mathcal{X})$

for $i = I, II$.

From Theorem 2.4 we know when the truth-valued operations in $\mathbb{R}$ satisfy finite semicontinuity $C_{\iota, \theta}^{\alpha, \beta}(\mathcal{T}, \mathcal{X}, \mathcal{Y})$ is a syntactic operator of LVPL, and $C_{\iota, \theta}^{\alpha, \beta}(\mathcal{T}, \mathcal{X}, \mathcal{Y}) (p)$ denote the degree of $P$ can be syntactically implied from $X$.

### 3. The Relations between LP(X) and LVPL and the role of inference rule set $\mathbb{R}$ playing in uncertainty reasoning based on LVPL

We know any logic system is composed of two parts the semantics and the syntax. So we can discuss the relations of these two logic systems by discussing the relations between the semantics and the syntax of LP(X) and LVPL. In LVPL $\alpha$ denote the consistency of semantics and syntax, and $\beta$ denote the degree of truth value deliver [11]. Firstly, we discuss the semantics of these two logic systems.

$C^X_\iota (p) \bigcap \bigwedge_{t \in T} \{ p(X \leq T) \rightarrow T(p) \}$

$= \bigwedge_{t \in T} \bigwedge_{q \in \tau} (X(q) \rightarrow T(q) \rightarrow T(p))$, when $\alpha = \beta = I$, $\mathbb{R} = \{(r^\alpha_{i, q}, t^\alpha_{i, q}) \in \mathbb{R} \}$, and $\mathcal{T} = \{ T \mid T: \mathcal{F}_p \rightarrow L \text{ is a homomorphic mapping} \}$ and $I-i$ satisfies $X_i$, $i = I, II$.

For any $T \in \mathcal{T}$ $\pi(X \leq T) = I$ i.e. for any $T \in \mathcal{T}$ $T$-I-i type satisfies $X$. Here $C^X_\iota (p) = \bigwedge_{t \in T} \{ T(p) \}$

In LP(X) the semantic operator
\[ Con(A)(p) \]
\[ \square \land \{v(p) \rightarrow v \text{ is a valuation of } LP(X) \text{ and satisfies } A \}. \]
Hence these two semantic operator coincide.

In the following we discuss the syntax of these two logic systems.

By comparing Definition 2.8 with Definition 2.10 we can see that the difference of these two definitions contains these three aspects.

1. In \( L_{\text{VPL}} \) the consistency of semantics and syntax is \( \alpha \), while in \( LP(X) \) the consistency of semantics is \( I \);
2. The degree of truth value deliver is \( \beta \) while in \( LP(X) \) the degree is \( I \);
3. The inference rule set of \( LP(X) \) only contain \( (p^\alpha_1, p^\alpha_2), (p^\alpha_1, p^\alpha_2) \), while in \( L_{\text{VPL}} \) the inference rule set contains more.

From the above discussion we can say that \( L_{\text{VPL}} \) is the generalization of \( LP(X) \), with its semantic and syntax is \( \alpha \) consistent, truth-value transition degree is \( \beta \), and inference rule is much more which determines the syntactic function of \( L_{\text{VPL}} \) is stronger.

The gradational lattice-valued propositional logic \( L_{\text{VPL}} \) denotes a series of logic systems. Different parameters determine different logic system. We discuss the function of inference rule set \( \mathcal{R} \) in \( L_{\text{VPL}} \) as follows.

Theorem 3.1 Let \( \mathcal{R}_1 \subseteq \mathcal{R}_2 \), \( \mathcal{I}_1 = \{T | T \text{ is } \alpha - i \text{ type closed w.r.t. } \mathcal{R}_i \} \)
\( \mathcal{I}_2 = \{T | T \text{ is } \alpha - i \text{ type closed w.r.t. } \mathcal{R}_i \} \). i=I, II. Then
1. \( C^{\beta,X}_{\theta \xi}(c_i^{\beta \alpha}(\alpha-i)) \subseteq C^{\beta,X}_{\theta \xi}(c_i^{\beta \alpha}(\alpha-i)). \)
2. \( C^{\beta X}_{\theta \xi}(c_i^{\beta X}(\alpha-i)) \subseteq C^{\beta X}_{\theta \xi}(c_i^{\beta X}(\alpha-i)). \)

Proof. If \( Y \) is \( \alpha \)-i type closed w.r.t. \( \mathcal{R}_2 \) and \( \mathcal{R}_1 \subseteq \mathcal{R}_2 \), hence If \( Y \) is \( \alpha \)-i type closed w.r.t. \( \mathcal{R}_1 \)
\[ \{Y | Y \text{ is } \alpha \text{-i type closed w.r.t. } \mathcal{R}_i \} \]
\[ \subseteq \{Y | Y \text{ is } \alpha \text{-i type closed w.r.t. } \mathcal{R}_i \} \]
i.e. \( \mathcal{I}_2 \subseteq \mathcal{I}_1 \), hence \( C^{\beta X}_{\theta \xi} \equiv C^{\beta X}_{\theta \xi} \)
\[ C^{\beta,X}_{\theta \xi}(c_i^{\beta \alpha}(\alpha-i)) \equiv C^{\beta,X}_{\theta \xi}(c_i^{\beta \alpha}(\alpha-i)). \]
\[ C^{\beta X}_{\theta \xi} \equiv \bigwedge_{\tau \in \mathcal{I}_1} \left[ (p^{\beta X}_1 \tau X \subseteq T) \rightarrow T(p) \right] \]
\[ \equiv \bigwedge_{\tau \in \mathcal{I}_1} \left[ (p^{\beta X}_1 \tau X \subseteq T) \rightarrow T(p) \right]. \]
\[ C^{\beta X}_{\theta \xi} \]
In fact, to some extent, the inference rule set \( \mathcal{R} \) determine the syntactic function of \( L_{\text{VPL}} \). \( \mathcal{R} \) relates to the valuation set \( \mathcal{T} \) closely, because \( \mathcal{T} \) must be \( \alpha \)-i type closed w.r.t. \( \mathcal{R} \). Furthermore \( \mathcal{T} \) and the implication operator \( \rightarrow \) determine the semantic function of \( L_{\text{VPL}} \). In the above theorem we can see the more of its number is the stronger of the syntactic function and the semantic function of \( L_{\text{VPL}} \) is. Hence the chosen of \( \mathcal{R} \) is very important.

4. Conclusions
By comparing the definitions of \( L_{\text{VPL}} \) with the corresponding definitions of \( LP(X) \) we understand the meaning of the parameters of \( L_{\text{VPL}} \). We conclude that \( L_{\text{VPL}} \) is a generalization of \( LP(X) \). We point out the main three generalized aspects. And we obtain that in \( L_{\text{VPL}} \) the chosen of the inference rule set \( \mathcal{R} \) is very important. The more its number, the stronger the syntactic function and the semantic function of \( L_{\text{VPL}} \) is. As we know in the uncertainty reasoning based on \( L_{\text{VPL}} \), we have to choose proper valuation set \( \mathcal{T} \) s.t. “r: If \( X \), then \( Y \)’s is \( (\alpha, \beta, \tau, \mathcal{J}) \)-i type representable. Our further research work are to choose proper inference rule set \( \mathcal{R} \) and valuation set \( \mathcal{T} \).

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