A Comparison between Lattice-Valued Propositional Logic LP(X) and Gradational Lattice-Valued Propositional Logic L_{vpl}

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Abstract

By comparing the definitions of L_{vpl} with the corresponding definitions of LP(X), the meaning of the parameters of L_{vpl} and the role of inference rule set $\mathfrak{R}$ playing in uncertainty reasoning based on L_{vpl} are demonstrated. And to some extent it is shown that the choose of $\mathfrak{R}$ determines the semantic and syntactic function of uncertainty reasoning based on L_{vpl}.

Keywords: Lattice-Valued Propositional Logic, Gradational Lattice-Valued Propositional Logic, rule of inference, syntax, semantics

1. Introduction

In order to provide a logical foundation for uncertain information processing theory, especially for the fuzziness, the incomparability uncertain information in the reasoning, Xu presented the lattice implication algebra by combining lattice and implication algebra in 1993 [1]. After then, he and his research group have established the lattice-valued propositional logic LP(X) and gradational lattice-valued propositional logic L_{vpl}, lattice-valued first-order logic and gradational lattice-valued first-order logic and applied them to automated and approximated reasoning [2]-[12]. As we know, during the uncertainty reasoning process based on L_{vpl} the chosen of the value set $\mathcal{T}$ and the implication operator $\to$ is very important. Chen researched uncertainty reasoning based on lattice-valued first-order logic L_{vfl} and give some concert methods for selecting appropriate parameters during the uncertainty reasoning process based on L_{vfl} for some representative uncertainty reasoning models[9]. In [9] Chen defined

$$\mathfrak{R}^* = \left\{\left(r_2^0, t_2^*\right), \left(r_2^*, t_2^*\right), \left(r_2^{\square}, t_2^*\right)\right\} \cup \left\{\left(r_1^{\theta_0}, t_1^{\theta_0}\right) \mid \theta_0 \in L\right\}$$
$$\cup \left\{\left(r_1^u, t_1\right) \mid u \in U\right\} \cup \left\{\left(r_2^u, t_1\right) \mid u \in U\right\}$$
$$\cup \left\{\left(r_3^u, t_1\right) \mid u \in U\right\} \cup \left\{\left(r_4^u, t_1\right) \mid u \in U\right\} \subseteq \mathfrak{R}$$

and $I_i = \left\{T \mid T \text{ is } \alpha\text{-i type closed w.r.t. } \mathfrak{R}^*\right\}$.

where,

$r_2^0\left(\varphi, \varphi \to \psi\right) = \psi$, $t_2^*\left(\theta, \beta\right) = \theta \wedge \beta$,

$r_2^*\left(\varphi \to \gamma, \varphi \to \psi\right) = \varphi \to \left(\gamma \wedge \psi\right)$,

$r_2^{\square}\left(\varphi \to \psi, \psi \to \gamma\right) = \varphi \to \gamma$,

$r_1^{\theta_0}\left(\varphi\right) = \theta_0 \to \varphi$, $t_1^{\theta_0}\left(\alpha\right) = \theta_0 \to \alpha$

$r_1^u\left(\varphi\right) = \left(Q_u x\right)\varphi$, $t_1\left(\theta\right) = \theta$,

$r_2^u\left(\varphi \to \psi\right) = \varphi \to \left(Q_u x\right)\psi$, x is not free in φ,

$r_3^u\left(\varphi \to \psi\right) = \left(Q_u x\right)\varphi \to \psi$, x is not free in ψ,

$r_4^u\left(Q_u x\right)\left(\varphi \otimes \psi\right) = \left(Q_u x\right)\varphi \otimes \psi$, x is not free in ψ,

But we don't know why $\mathfrak{R}^*$ is chosen as this and what the function of $\mathfrak{R}^*$ is.

2. Comparing the corresponding definitions

In the following we suppose the sets $\mathcal{F}$ and $\mathcal{F}_p$ are the sets of formula of LP(X) and L_{vpl} respectively. By comparing the definitions of LP(X) with the counterpart of L_{vpl}, the meaning of the parameters of L_{vpl} is understood.

Definition 2.1 [2] A mapping v : LP(X) $\to$ L is called a valuation of LP(X), if it is a T-homomorphism.

Corollary 2.1 [10] Let f : LP(X) $\to$ L be a mapping of LP(X), then f is a valuation of LP(X) if and only if it satisfies

(1). $f\left(\alpha\right) = \alpha$ for any $\alpha \in L$;

(2). $f\left(p'\right) = \left(f\left(p\right)\right)'$ for any $p \in \mathcal{F}$;

(3). $f\left(p \to q\right) = f\left(p\right) \to f\left(q\right)$ for any $p, q \in \mathcal{F}$.

Definition 2.2 [4] Let $\mathscr{F}_L(\mathscr{F})$, *v* be a valuation of LP(X). It is called that *v* satisfies *A* if $A(p) \le v(p)$ for any $p \in \mathscr{F}$. *A* is called satisfiable if there exists a valuation *v*, which satisfies *A*.

Definition 2.3 [6] Let $X \in \mathscr{F}_L(\mathscr{F}_P)$, $\delta \in L$, $T_0 \in \mathscr{T}$. If for any $P \in \mathscr{F}_P$,

(1). $T_0(p') = (T_0(p))'$,

(2). $X(p) \to T_0(p) \ge \delta$

then T_0 is called δ-i type satisfies *X*. If there exists T_0 , T_0 δ-i type satisfies *X*, then X is said to be δ-i type satisfiable, i = I, II.

Definition 2.4 [4] Let $A \in \mathscr{F}_L(\mathscr{F})$, *A* is called closed if

(1). $A(p \to q) \otimes A(p) \le A(q)$,

(2). $\alpha \to A(p) \le A(\alpha \to p)$.

for any $p, q \in \mathscr{F}$ and $\alpha \in L$.

Definition 2.5 [6] Let $X \in \mathscr{F}_L(\mathscr{F}_P)$, $(r,t) \in \Re_n$, $n \in \Box$, $\alpha \in L$. If

$$X \circ r \supseteq \alpha \otimes \left(t \circ \prod^{n} X \right) \qquad (1)$$

in Dn(r), then *X* is said to be α-I type closed w.r.t. (r, t). If

$$X \circ r \supseteq t \circ \left(\prod^{n} \alpha \otimes X \right) \qquad (2)$$

in Dn(r), then *X* is said to be α-II type closed w.r.t. (r, t).

In definition 2.5 α denote the consistency of the semantics and syntax When $\alpha = I$, and $\Re = \left\{ (r_2^0, t_2^0), (r_1^\theta, t_1^\theta) \right\}$, the formula (1), (2) are changed into $X(p \to q) \otimes X(p) \le X(q)$, $\alpha \to X(p) \le X(\alpha \to p)$ respectively, which is the same as Definition 2.4. In another word in LP(X) "*A* is closed" is to be said for $(r_2^0, t_2^0), (r_1^\theta, t_1^\theta)$ these two inference rules.

Semantics of LP(X) and L$_{VPL}$

Definition 2.6 [4] Let $A \in \mathscr{F}_L(\mathscr{F})$, $p \in \mathscr{F}$ and $\alpha \in L$. *p* is called sematically implied from *A* with truth value level α if $v(p) \ge \alpha$ for any valuation *v*, which satisfies *A* of LP(X). We write this $A \models_\alpha p$. *p* is called valid with truth value level α (shortly for α -valid) and denoted by $\models_\alpha p$, if $v(p) \ge \alpha$ for any valuation *v* of LP(X). *p* is called a valid formula, if $\models_I p$. *p* is called α -valid in *A*, If

$$\alpha = \wedge \{ v(p) | \text{ v is a valuation and satisfies A} \}$$

We shall write this $A \Vdash_\alpha p$.

$Con(A)(p) \Box \wedge \{ v(p) | \ v$ is a valuation of $LP(X)$ and satisfies $A \}$. $Con(A)(p)$ denote the degree of p can be semantically implied from *A*.

The corresponding semantics of L_{vpl} is given as a mapping $C_{\mathscr{T}}^X(p)$.

Definition 2.7 [6]

$$C_{\mathscr{T}}^X(p) \Box \bigwedge_{T \in \mathscr{T}} \left[\pi(X \subseteq T) \to T(p) \right]$$

$$= \bigwedge_{T \in \mathscr{T}} \left[\bigwedge_{q \in \mathscr{F}_P} \left(X(q) \to T(q) \right) \to T(p) \right].$$

We can see this operator is a semantic operator of L$_{vpl}$. $C_{\mathscr{T}}^X(p)$ denotes the degree of p can be semantically implied from *X*.

Theorem 2.1 [10] Each valuation of LP(X) is closed.

Theorem 2.2 Let *A* be closed. Then $A(\alpha) \le \alpha$, and $A(p') \le A(p)'$ for any $\alpha \in L$ and $p \in \mathscr{F}$.

Proof. *A* is closed so for any $\alpha \in L$ and $p \in \mathscr{F}$ we have

$$\alpha \to A(p) \le A(\alpha \to p) \le A(\alpha) \to A(p) \quad .$$

Hence $A(\alpha) \le \alpha$, and $A(0) = 0$.

$$A(p') = A(p \to 0) \le A(p) \to A(0)$$

$$= A(p) \to 0 = A(p)' .$$

Theorem 2.3 If $\varnothing \ne \mathfrak{A} \subseteq \{ \text{A} \mid \text{A is closed} \}$, then $\cap_{A \in \mathfrak{A}} A$ is closed.

Proof. For any $\alpha \in L$ and $p \in \mathscr{F}$

$$\cap_{A \in \mathfrak{A}} A(q) = \bigwedge_{A \in \mathfrak{A}} A(q) \ge \bigwedge_{A \in \mathfrak{A}} \left(A(p) \otimes A(p \to q) \right)$$

$$\ge \bigwedge_{A \in \mathfrak{A}} A(p) \otimes \bigwedge_{A \in \mathfrak{A}} A(p \to q)$$

$$= \cap_{A \in \mathfrak{A}} A(p) \otimes \cap_{A \in \mathfrak{A}} A(p \to q).$$

$$(\cap_{A \in \mathfrak{A}} A)(\alpha \to p) \ge \bigwedge_{A \in \mathfrak{A}} \left(\alpha \to A(p) \right)$$

$$\ge \alpha \to \bigwedge_{A \in \mathfrak{A}} A(p)$$

$$= \alpha \to (\cap_{A \in \mathfrak{A}} A)(p).$$

Corollary 2.2 If $A \models_\alpha p$, $A \models_\beta p \to q$, then

(1). $A \models_{\alpha \otimes \beta} q$,

(2). $A \models_{\alpha_0 \to \alpha} \alpha_0 \to p$ for any $\alpha_0 \in L$.

Proof.

$Con(A)(p) \Box \wedge \{ v(p) | \ v$ is a valuation of $LP(X)$ *and satisfies* $A \}$, so $Con(A)$ is closed.

Hence $Con(A)(p \to q) \otimes Con(A)(p) \le Con(A)(q)$, $\alpha_0 \to Con(A)(p) \le Con(A)(\alpha_0 \to p)$ for any

$p,q \in \mathscr{F}$ and $\alpha_0 \in L$. If $A \models_\alpha p$, $A \models_\beta p \to q$, i.e. $\alpha \otimes \beta \le Con(A)(p \to q) \otimes Con(A)(p) \le Con(A)(q)$, $\alpha_0 \to \alpha \le \alpha_0 \to Con(A)(p) \le Con(A)(\alpha_0 \to p)$ hence $A \models_{\alpha\otimes\beta} q$, $A \models_{\alpha_0 \to \alpha} \alpha_0 \to p$ for any $\alpha_0 \in L$.

Syntax of LP(X) and L_{VPL}

Definition 2.8 [5] Let $A \in \mathscr{F}_L(\mathscr{F})$, $p \in \mathscr{F}$. A formal proof ω of p from A is a finite sequence as follows:

$(p_1,\alpha_1),...,(p_n,\alpha_n)$,

where $p_n = p$, and for any i, $1 \le i \le n$, $p_i \in \mathscr{F}$, $a_i \in L$ and

(1). $A_L(p_i) = \alpha_i$ or

(2). $A(p_i) = \alpha_i$, or

(3). there exist $j,k < i$, such that $p_j = p_k \to p_i$ and $a_i = a_j \otimes a_k$ or

(4). there exists $j < i$ and $\alpha \in L$, such that $p_i = \alpha \to p_j$ and $\alpha_i = \alpha \to \alpha_j$.

Where the operation $\otimes$ is defined as $(\alpha \to \beta')'$ in *L***.**

Definition 2.9 [5] Let $A \in \mathscr{F}_L(\mathscr{F})$, $p \in \mathscr{F}$, $\alpha \in L$. p is called an α-theorem of A and written as $A \vdash_\alpha p$, if $\alpha \le \vee\{val(w) \mid w \text{ is a proof of p from } A\}$.

If $\alpha = \vee\{val(w) \mid w \text{ is a proof of p from } A\}$,

then it is written as $A \Vdash_\alpha$ p.

In [10] define a mapping

Ded: $\mathscr{F}_L(\mathscr{F}) \to \mathscr{F}_L(\mathscr{F})$ satisfy

$Ded(A)(p) = \vee\{val(w) \mid w \text{ is a proof of p from } A\}$

for any $p \in \mathscr{F}$ and $A \in \mathscr{F}_L(\mathscr{F})$.

$Ded(A)(p) = \vee\{B(p) \mid \mathrm{B} \supseteq \mathrm{A} \cup \mathrm{A}_L,\ B \text{ is closed}\}$.

So $Ded(A)(p)$ denote the degree of p syntactically from A.

Definition 2.10 [6]
Let $X \in \mathscr{F}_L(\mathscr{F}_P)$, $\mathfrak{T} \subseteq \mathscr{F}_L(\mathscr{F}_P)$, $p \in \mathscr{F}_P$, $\theta,\alpha,\beta \in L$. $(P^I,\ (n),\ X,\ (p,\theta)-(\alpha,\beta))$ is said to be a $(\alpha,\beta)-I$ type proof with the truth degree θ from X to p (shortly, θ − (α, β)-I type proof from X to p), if the mapping

P^I: (n) $\to \mathscr{F}_P \times$ L ((n) = {1, 2, · · · , n}),

i $\to$ (p_i, θ_i),

satisfies

(1). (p_n, θ_n) = (p, θ) and

(2). $\theta_i = \beta \otimes C_{\mathfrak{T}}^{\varnothing}(p_i)$, or

(3). $\theta_i = \beta \otimes X(p_i)$, or

(4). there exist $i_1, \cdots, i_k \le i$, and (r, t) $\in \Re_k$ such that (p_i, θ_i) = (r(p_{i1} , · · · , p_{ik}), $\alpha \otimes$ t(θ_{i1} , · · · ,θ_{ik})),

where n is said to be the length of θ −(α, β)-I type proof from X to p under P^I, and denoted as $l(P^I)$.

θ − (α, β)-II type proof from X to p is the same as definition 2.10 except for (4) changed into

(p_i, θ_i) = (r(p_{i1} , · · · , p_{ik}), t($\alpha \otimes \theta_{i1}$, · · · , $\alpha \otimes \theta_{ik}$)).

Definition 2.11 [6]

$C^{\beta,X}_{(C_{\mathfrak{T}}^{\varnothing},\Re(\alpha-i))} \triangleq \wedge\{Y(p) \mid Y \supseteq \beta \otimes (C_{\mathfrak{T}}^{\varnothing} \cup X),$

$Y \text{ is } \alpha\text{-}i \text{ type closed w.r.t.} \Re\}$ **.**

Theorem 2.4 [6] Let $X \in \mathscr{F}_L(\mathscr{F}_P)$, $\mathfrak{T} \subseteq \mathscr{F}_L(\mathscr{F}_P)$, $\alpha,\beta \in L$, and the truth-valued operations in $\Re$ satisfy finite semicontinuity, then for any $p \in \mathscr{F}_P$,

$C^{\beta,X}_{(C_{\mathfrak{T}}^{\varnothing},\Re(\alpha-i))} = \vee\{\theta \text{ there exists } (P^i,(n),X,(p,\theta)-(\alpha,\beta))\}$

for i = I, II.

From Theorem 2.4 we know when the truth-valued operations in $\Re$ satisfy finite semicontinuity $C^{\beta,X}_{(C_{\mathfrak{T}}^{\varnothing},\Re(\alpha-i))}$ is a syntactic operator of L_{VPL}, and $C^{\beta,X}_{(C_{\mathfrak{T}}^{\varnothing},\Re(\alpha-i))}(p)$ denote the degree of P can be syntactically implied from X.

3. The Relations between LP(X) and L_{VPL} and the role of inference rule set $\Re$ playing in uncertainty reasoning based on L_{vpl}

We know any logic system is composed of two parts the semantics and the syntax. So we can discuss the relations of these two logic systems by discussing the relations between the semantics and the syntax of LP(X) and L_{VPL}. In L_{VPL} α denote the consistency of semantics and syntax, and β denote the degree of truth value deliver [11]. Firstly, we discuss the semantics of these two logic systems.

$$C_{\mathfrak{T}}^{X}(p) \triangleq \underset{T\in\mathfrak{T}}{\wedge}\left[\pi(X \subseteq T) \to T(p)\right]$$

$$= \underset{T\in\mathfrak{T}}{\wedge}\left[\underset{q\in\mathscr{F}_P}{\wedge}(X(q) \to T(q)) \to T(p)\right],$$

when $\alpha = \beta = I$, $\Re^0 = \{(r_2^0,t_2^0),(r_1^\theta,t_1^\theta)\}$, and

$\mathfrak{T}^0 = \{T \mid T : \mathscr{F}_P \to L \text{ is a homomorphic mapping and } I\text{-}i \text{ satisfies } X\}$, i = I, II .

For any $T \in \mathfrak{T}^0$ $\pi(X \subseteq T) = I$ i.e. for any $T \in \mathfrak{T}$ T I-i type satisfies X. Here $C_{\mathfrak{T}}^{X}(p) = \underset{T\in\mathfrak{T}}{\wedge}\{T(p)\}$

In LP(X) the semantic operator

$$Con(A)(p)$$

$$\triangleq \wedge\left\{v(p) \middle| \text{ v is a valuation of LP(X) and satisfies A}\right\}.$$

Hence these two semantic operator coincide.

In the following we discuss the syntax of these two logic systems.

By comparing Definition 2.8 with Definition 2.10 we can see that the difference of these two definitions contains these three aspects.

(1) In L_{VPL} the consistency of semantics and syntax is α , while in LP(X) the consistency of semantics is I;

(2) The degree of truth value deliver is β while in LP(X) the degree is I ;

(3) The inference rule set of LP(X) only contain $(r_2^0, t_2^0), (r_1^\theta, t_1^\theta)$, while in L_{VPL} the inference rule set contains more.

From the above discussion we can say that L_{VPL} is the generalization of LP(X), with its semantic and syntax is α consistent, truth-value transition degree is β , and inference rule is much more which determines the syntactic function of L_{VPL} is stronger.

The gradational lattice-valued propositional logic L_{VPL} denotes a series of logic systems. Different parameters determine different logic system. We discuss the function of inference rule set $\Re$ in L_{VPL} as follows.

Theorem 3.1 Let $\Re_1 \subseteq \Re_2$, $\mathfrak{T}_i^1 = \{T \mid T \text{ is } \alpha - i \textit{ type closed w.r.t. } \Re_1\}$ $\mathfrak{T}_i^2 = \{T \mid T \text{ is } \alpha - i \textit{ type closed w.r.t. } \Re_2\}$. i=I, II. Then

(1). $C^{\beta,X}_{\left(C^{\varnothing}_{\mathfrak{T}_i^2}, \Re_2(\alpha-i)\right)} \supseteq C^{\beta,X}_{\left(C^{\varnothing}_{\mathfrak{T}_i^1}, \Re_1(\alpha-i)\right)}$,

(2). $C^{\beta\otimes X}_{\mathfrak{T}_i^2} \supseteq C^{\beta\otimes X}_{\mathfrak{T}_i^1}$.

Proof. If Y is α -i type closed w.r.t. $\Re_2$ and $\Re_1 \subseteq \Re_2$, hence If Y is α -i type closed w.r.t. $\Re_1$

$$\{Y \mid Y \text{ is } \alpha\text{-i type closed w.r.t.} \Re_2\}$$

$$\subseteq \{Y \mid Y \text{ is } \alpha\text{-i type closed w.r.t.} \Re_1\}$$

i.e. $\mathfrak{T}_i^2 \subseteq \mathfrak{T}_i^1$, hence $C^{\varnothing}_{\mathfrak{T}_i^2} \supseteq C^{\varnothing}_{\mathfrak{T}_i^1}$

$$C^{\beta,X}_{\left(C^{\varnothing}_{\mathfrak{T}_i^2}, \Re_2(\alpha-i)\right)} \supseteq C^{\beta,X}_{\left(C^{\varnothing}_{\mathfrak{T}_i^1}, \Re_1(\alpha-i)\right)}.$$

$$C^{\beta\otimes X}_{\mathfrak{T}_i^2} = \underset{T\in\mathfrak{T}_i^2}{\wedge}\left[\pi(\beta\otimes X \subseteq T) \to T(p)\right]$$

$$\supseteq \underset{T\in\mathfrak{T}_i^1}{\wedge}\left[\pi(\beta\otimes X \subseteq T) \to T(p)\right].$$

$$= C^{\beta\otimes X}_{\mathfrak{T}_i^1}$$

In fact, to some extent, the inference rule set $\Re$ determine the syntactic function of L_{vpl}. $\Re$ relates to the valuation set $\mathfrak{T}$ closely, because $\mathfrak{T}$ must be α -i type closed w.r.t. $\Re$. Furthermore $\mathfrak{T}$ and the implication operator $\to$ determine the semantic function of L_{vpl}. In the above theorem we can see the more of its number is the stronger of the syntactic function and the semantic function of L_{vpl} is. Hence the chosen of $\Re$ is very important.

4. Conclusions

By comparing the definitions of L_{vpl} with the corresponding definitions of LP(X) we understand the meaning of the parameters of L_{vpl}. We conclude that L_{vpl} is a generalization of LP(X). We point out the main three generalized aspects. And we obtain that in L_{vpl} the chosen of the inference rule set $\Re$ is very important. The more its number, the stronger the syntactic function and the semantic function of L_{vpl} is. As we know in the uncertainty reasoning based on L_{vpl}, we have to choose proper valuation set $\mathfrak{T}$ s.t. "r: If *X*, then *Y*" is $(\alpha, \beta, \tau, \mathfrak{T})$-i type representable. Our further research work are to choose proper inference rule set $\Re$ and valuation set $\mathfrak{T}$.

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