A Generalized Fuzzy Entropy-Based Image Segmentation Method

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Abstract

Image segmentation using fuzzy entropy is an important and common segmentation method. The threshold of fuzzy entropy is mostly selected at the gray value with fuzzy membership degree 0.5. It is a limitation in some cases. In order to solve this problem, we present a new definition of generalized fuzzy entropy and apply it to image segmentation. Compared with the traditional fuzzy entropy-based image segmentation method, the proposed method segments an image using the threshold with membership degree m (0<m<1), and increases the opportunity of choosing appropriate thresholds. Experiments show that our method can obtain better segmentation result than the traditional fuzzy entropy method.

Keywords: Image segmentation, Fuzzy entropy, Generalized fuzzy entropy, Fuzzy complement operator

1. Introduction

Fuzzy entropy describes the fuzziness of a fuzzy set. It is a basic concept in fuzzy set theory and widely used in image processing. For example, fuzzy entropy-based segmentation method is a useful kind of segmentation method [1-3]. The first definition of fuzzy entropy on finite universal set was introduced by De Luca and Termini [4]. They used the basic union, intersection and complement operators $c(x) = 1 - x$ $(x \in [0,1])$ defined by Zadeh [5]. A notable feature of this complement operator is that its fixed point locates at $x = 0.5$, this meaning that the fuzzy set $A = [0.5]$ (i.e., $\mu_A(x) \equiv 0.5$ for all x in the universal set) has the most fuzziness.

Fuzzy set $A = [0.5]$ possessing the most fuzziness is a limitation in real applications. If we use a measure of fuzziness to assess the amount of blur in an image, as pointed out by Pal [6], the use of the fixed peek at 0.5 may lead to inaccurate estimates. Zenzo et al. [7] changed the conditions of fuzzy entropy, replacing the condition that fuzzy set $A = [0.5]$ has the most fuzziness with fuzzy set $A = [m]$ ($m \in (0,1)$) has the most fuzziness. Using a mapping, they constructed a generalized fuzzy entropy formula based on fuzzy entropy formula, and applied it to image segmentation and denoising. Unfortunately, the definition of Zenzo's generalized fuzzy entropy did not involve the complement operator of a fuzzy set, that is, the fourth condition in the definition of fuzzy entropy was abandoned. In this paper we presents a complete definition of generalized fuzzy entropy which is entirely corresponding to the conditions of traditional fuzzy entropy, and give an image segmentation method based on it. The method segments an image using the threshold with membership degree m (0<m<1).

The paper is organized as follows. In Section 2, we briefly state and review the definition of fuzzy entropy. In Section 3, a complete definition of generalized fuzzy entropy is proposed. Section 4 presents an image segmentation method based on the generalized fuzzy entropy. The contrastive segmentation performance of the generalized fuzzy entropy with that of the fuzzy entropy is provided in Section 5. Finally, conclusions and future works are discussed in Section 6.

2. Fuzzy entropy

Throughout this paper, $\mathbf{F}(X)$ expresses the set of all fuzzy sets on universal set X. $\mathbf{P}(X)$ expresses the set of all crisp sets on universal set X. $\mu_A(x)$ is the membership function of $A \in \mathbf{F}(X)$. $[a]$ is the fuzzy set of X for which $\mu_{[a]}(x) = a$, $\forall x \in X$ ($a \in [0,1]$). For fuzzy set A, we use A^c to express the complement of A where $c(x) = 1 - x$ $(x \in [0,1])$. For two fuzzy sets A and B, $A \bigcup B$ the union of A and B is defined as $\mu_{A \bigcup B}(x) = \max(\mu_A(x), \mu_B(x))$, and $A \bigcap B$ the intersection of A and B is defined as $\mu_{A \bigcap B}(x) = \min(\mu_A(x), \mu_B(x))$. A fuzzy set A^* is called a sharpening of A. If $\mu_A(x) \geq 0.5$, then $\mu_{A^*}(x) \geq \mu_A(x)$; if $\mu_A(x) \leq 0.5$, then $\mu_{A^*}(x) \leq \mu_A(x)$.

In order to measure the fuzziness of a fuzzy set on finite universal set $X = \{x_1, x_2, ... x_n\}$, De Luca and

Termini [4] introduced the definition of fuzzy entropy. Fuzzy entropy is a function e: $\mathbf{F}(X) \to R^+$ ($R^+ = [0, +\infty)$), which satisfies the following four conditions:

(EP1) $e(A) = 0$ iff $A \in \mathbf{P}(X)$;

(EP2) $e(A)$ attains its maximum iff $A = [0.5]$;

(EP3) If A^* is a sharpening of A , then $e(A^*) \leq e(A)$;

(EP4) $e(A^c) = e(A)$.

The above four conditions are basic and reasonable for the measure of fuzziness of a fuzzy set. (EP4) requires the fuzzy set A and A^c possess the same fuzziness.

3. Generalized fuzzy entropy

The complement operator $c(x) = 1 - x$ has a unique fixed point $x = 0.5$ .This complement operator is widely used in fuzzy set theory and applications. Many researchers have proposed general fuzzy complement operators [8-11]. For these fuzzy complement operators, the unique fixed point can be any point in the internal $(0,1)$. The general fuzzy complement operator $c : [0,1] \to [0,1]$ must satisfy

(C1) $c(0) = 1$ and $c(1) = 0$;

(C2) For all $a, b \in [0,1]$, if $a \leq b$ then $c(a) \geq c(b)$;

(C3) c is a continuous function;

(C4) For every point $x \in [0,1]$, $c(c(x)) = x$.

Yager [10] gave a fuzzy complement function:

$$c_w(x) = (1 - x^w)^{1/w} \tag{1}$$

where $w \in (0, +\infty)$. When $w = 1$, Yager's complement turns to the fuzzy complement function $c(x) = 1 - x$.

The unique fixed point of $c_w(x)$ is $m = \dfrac{1}{\sqrt[w]{2}}$.

Zenzo et al. [7] had given a definition of generalized fuzzy entropy, but their definition omitted the fourth condition (EP4) making their definition too loose. In order to give a complete definition of generalized fuzzy entropy which is entirely corresponding to the conditions of fuzzy entropy, we propose a new definition of generalized fuzzy entropy on a finite universal set. For $m \in (0,1)$ and a complement operator c_m with the fixed point m , the generalized fuzzy entropy is a function e_m : $\mathbf{F}(X) \to R^+$, which satisfies the following four conditions:

(GEP1) $e_m(A) = 0$ iff $A \in \mathbf{P}(X)$;

(GEP2) $e_m(A)$ attains its maximum iff $A = [m]$;

(GEP3) If A^* is a sharpening of A , then $e_m(A^*) \leq e_m(A)$. Here A^* satisfies that $\mu_{A^*}(x) \geq \mu_A(x)$ when $\mu_A(x) \geq m$; $\mu_{A^*}(x) \leq \mu_A(x)$ when $\mu_A(x) \leq m$;

(GEP4) $e_m(A^{c_m}) = e_m(A)$.

With $m = 0.5$ and fuzzy complement function $c(x) = 1 - x$,the generalized fuzzy entropy comes to the fuzzy entropy.

For the fuzzy set A on a finite universal set $X = \{x_1, x_2, \dots x_n\}$, some generalized fuzzy entropy formulas are given below

$$e_m^1(A) = \frac{M(A \bigcap A^{c_m})}{M(A \bigcup A^{c_m})} \tag{2}$$

$$e_m^2(A) = \sum_{i=1}^{n} \frac{\min(\mu_A(x_i), \mu_{A^{c_m}}(x_i))}{\max(\mu_A(x_i), \mu_{A^{c_m}}(x_i))} \tag{3}$$

$$e_m^3(A) = M(A \bigcap A^{c_m}) \tag{4}$$

$$e_m^4(A) = 1 - \frac{M(A \bigcup A^{c_m}) - M(A \bigcap A^{c_m})}{n} \tag{5}$$

where $M(A) = \sum_{i=1}^{n} \mu_A(x_i)$.

4. Generalized fuzzy entropy-based image segmentation method

Fuzzy entropy-based thresholding method has become one of the frequently used segmentation technique during these years. As described in Section 3, fuzzy entropy is the special case of the generalized fuzzy entropy, so it is natural to apply the generalized fuzzy entropy to image segmentation problems. We sample the parameter m in the internal $(0,1)$ with a step of 0.1,then for each image, nine segmentation results can be obtained. When $m = 0.5$, the result is the very one of fuzzy entropy-based method. This means that the generalized fuzzy entropy-based method can obtain performance not worse than that of the fuzzy entropy, because it involves the result of the fuzzy entropy. Among the nine segmentation results, we then select the one which has the best visual effect as the final result of the generalized fuzzy entropy.

Let Q denote an image of size $M \times N$ with L levels, and q_{xy} is the gray level of (x, y) pixel in Q. Let $\mu_Q(q_{xy})$ denote the membership value which represents the degree of brightness of (x, y) pixel in Q. In the notation of fuzzy set, the image set Q can be written as $Q = \{(q_{xy}, \mu_Q(q_{xy}))\}$, where $0 \leq \mu_Q(q_{xy}) \leq 1$.

In this paper, we use the S-function to represent the brightness of gray level. It is defined as [12]

$$S(x;a,b,d) = \begin{cases} 0, & x \leq a \\ \dfrac{(x-a)^2}{(b-a)(d-a)}, & a \leq x \leq b \\ 1 - \dfrac{(x-d)^2}{(d-b)(d-a)}, & b \leq x \leq d \\ 1, & x \geq d \end{cases} \quad (6)$$

where x is the gray level in Q, and a, b, d are the parameters determining the shape of S-function. Parameter b can be any point between a and d.

Fuzzy entropy-based segmentation method always regard the maximum fuzzy entropy as the threshold selecting principle. The shape of S-function is determined by parameter a, b and d. Therefore, the threshold selecting problem becomes to find a combination of the parameters such that the corresponding event has the maximum fuzzy entropy [2,13]. After obtaining the best combination of the parameters (a^*, b^*, d^*), the gray level with membership value 0.5 is selected as the threshold. We use $G = \{g(x,y)\}$ to denote the segmented image, then the optimal threshold value is selected at

$$(a^*, b^*, d^*) = Arg \max_{0 \leq a < b < d \leq L-1} e(a,b,d) \quad (7)$$

$$T^* = t, \;\; iff \;\; \mu_Q(t) = 0.5 \quad (8)$$

$$g(x,y) = \begin{cases} 0 & q(x,y) \leq T^* \\ 1 & q(x,y) > T^* \end{cases} \quad (9)$$

As described in Introduction, selecting the threshold value with membership degree 0.5 is a limitation in real applications. Next we give the generalized fuzzy entropy-based threshold method. The optimal threshold value is selected at

$$(a^*, b^*, d^*) = Arg \max_{0 \leq a < b < d \leq L-1} e_m(a,b,d) \quad (10)$$

$$T^* = t, \;\; iff \;\; \mu_Q(t) = m \quad (11)$$

$$g(x,y) = \begin{cases} 0 & q(x,y) \leq T^* \\ 1 & q(x,y) > T^* \end{cases} \quad (12)$$

When we use Yager's complement function to substitute the complement operator $c(x) = 1 - x$ in (2)~(5), parameters w is included in these formulas. Because m is the fixed point, thus the relation between m and w is:

$$w = -\frac{1}{\log_2 m} \quad (13)$$

Formula (13) maps $m \in (0,1)$ onto $w \in (0, +\infty)$. Using formula (13), Yager's complement can be expressed as

$$c_m(x) = (1 - x^{-\frac{1}{\log_2 m}})^{-\log_2 m} \quad (14)$$

By formula (14), the generalized entropy formulas (2)~(5) contained the parameter m ($m \in (0,1)$).

We sample the parameter m in the internal (0,1) with a step of 0.1. So for each image, nine segmentation results can be obtained. Among the nine segmentation results, we select the one which has the best visual effect as the final result of the generalized fuzzy entropy.

5. Experimental results

In this section, we compare the segmentation performance of the generalized fuzzy entropy with that of the traditional fuzzy entropy-based method. The generalized fuzzy entropy formulas are formula (2) ~ (5). The complement function is Yager's complement. The experimental images are Rice, Circuit board and Car number. Table 1~3 list the thresholds under the four generalized fuzzy entropy formulas, when m equals $0.1, 0.2, \cdots, 0.9$, respectively. When m=0.5, the threshold is the threshold of traditional fuzzy entropy-based method. In Table 1~3, under each generalized fuzzy entropy formula, every overstriking data is the final threshold selected from the nine thresholds with the best visual effect.

m	e_m^1	e_m^2	e_m^3	e_m^4
0.1	92	102	100	92
0.2	108	105	105	113
0.3	105	105	105	105
0.4	101	102	101	101
0.5	88	100	88	88
0.6	101	101	101	101
0.7	114	**110**	114	114
0.8	**131**	107	**131**	**131**
0.9	152	152	`152	152

Table1. The threshold of Rice image

m	e_m^1	e_m^2	e_m^3	e_m^4
0.1	71	199	71	71
0.2	100	199	100	100
0.3	**122**	**122**	**122**	**122**
0.4	141	141	141	141
0.5	158	158	158	158
0.6	173	173	173	173
0.7	186	186	186	186
0.8	124	124	124	124
0.9	153	153	153	153

Table2. The threshold of Circuit board image

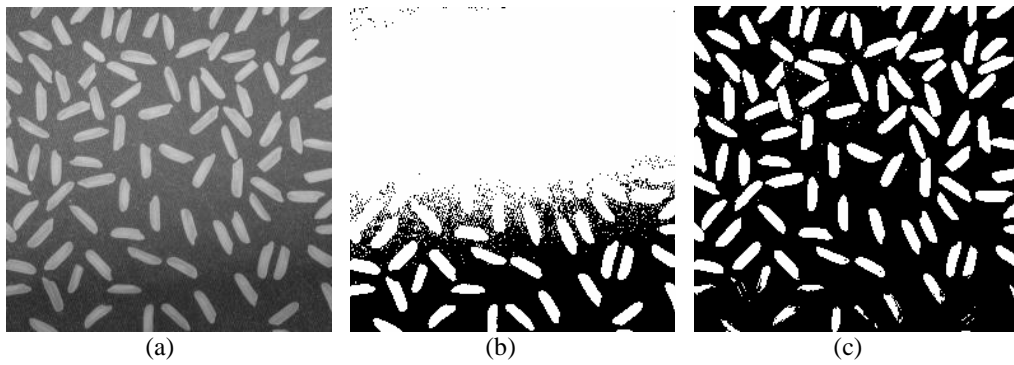| m	e_m^1	e_m^2	e_m^3	e_m^4
0.1	108	119	116	83
0.2	115	116	115	115
0.3	128	127	128	129
0.4	128	128	128	128
0.5	116	119	116	116
0.6	101	108	102	102
0.7	117	117	117	117
0.8	142	142	142	142
0.9	**175**	**175**	**175**	**175**

Table3. The threshold of Car number image

Figure.1 The result of Rice image: (a) original image ; (b) segmentation by fuzzy entropy ; (c) segmentation by generalized fuzzy entropy.
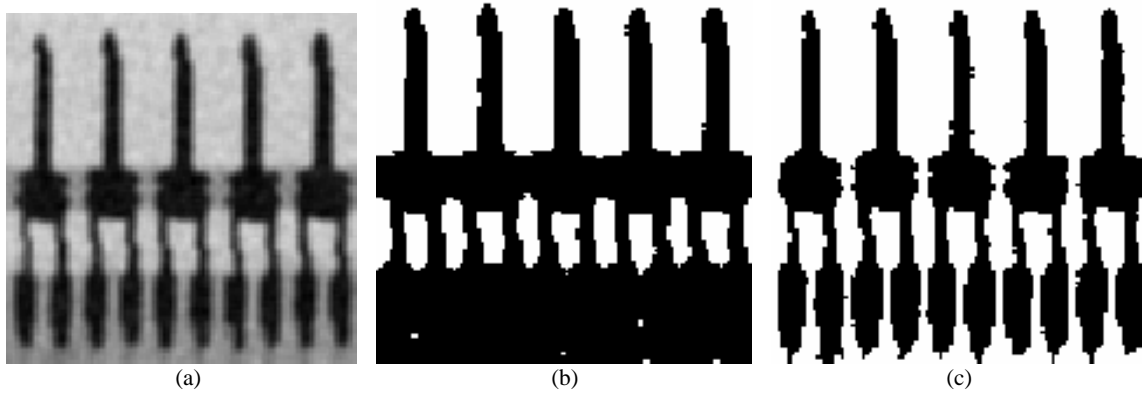

Figure.2 The result of Circuit board image: (a) original image ; (b) segmentation by fuzzy entropy ; (c) segmentation by generalized fuzzy entropy.

Figure.3 The result of Car number board image: (a) original image ; (b) segmentation by fuzzy entropy ; (c) segmentation by generalized fuzzy entropy.

From Table.1 ~ 3, we can see that for each image, the thresholds of the four generalized fuzzy entropy formulas are almost the same under the same m value. So we only present the segmentation results of the formula e_m^1, listed in Fig.1 ~ 3.

For Rice image, under the generalized fuzzy entropy formula e_m^1, the threshold of fuzzy entropy is 88, and the final threshold of the generalized fuzzy entropy is located at $m = 0.8$ (T=131) (see Table 1). Fig.1.(b) and (c) list the two segmentation results. We

can see that the generalized fuzzy entropy obtains more satisfactory result than the fuzzy entropy. Similar analysis of other images is omitted.

6. Conclusions

Generalized fuzzy entropy-based method segments an image using the threshold with membership value m ($m = 0.1, 0.2...0.9$). This method increases the opportunity of choosing appropriate thresholds, thus it

obtains better performance than the fuzzy entropy-based method. However, the results of generalized fuzzy entropy are not always satisfactory. Take Rice image for example (see Fig. 1), there are some object pixels having been wrong classified into background. This means that the step 0.1 for m might be too coarse. However, too small step for m would cause larger computation and increase the difficulty to select the result with the best visual effect. Therefore, how to automatically determine the parameter m and evaluate the segmentation results are problems worth to research further.

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