Improvement of Gradient Projection Algorithm for Nonlinear Programming

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Abstract—The search direction by making use of the matrix LU decomposition gradient projection algorithm for nonlinear programming is given, the stability of sparse and the algorithm of this method can maintain effective constraint matrix, the algorithm can be applied to large sparse nonlinear optimization problem with linear constraints.

Keywords- LU decomposition, large scale sparse, nonlinear optimization.

I. INTRODUCTION

Consider \( \text{min} \{ f(x) \mid x \in S \} \),
\( S \) is a set defined by \( S = \{ x \mid Ax = b, x \geq 0 \} \),
\( S_u = \{ x \mid x \in S, x > 0 \} \).

Structuring gradient direction \( P_{\Delta x} \) in projection algorithm is the same as finding the answer of least-squares of overdetermined equations \( A^T x = c \). Assume \( A \) is Large matrix, since the QR decomposition may not keep the sparsity of large-scale, we cannot use QR decomposition to solve it. In order to solve the above problems, we consider using the matrix LU decomposition to construct descent direction.

II. BASED ON THE STRUCTURE OF VERTICAL SPACE

For a matrix \( A_{m \times n} \) \( (m > n) \), assume the rank of \( A \) is \( n \), according to the theory of Gauss elimination[1], exist the matrix \( M_1, M_2, \ldots, M_n \), subject to

\[
M_n \cdots M_2 M_1 A = \bar{U} = \begin{pmatrix}
* & * & \cdots & * \\
0 & * & \cdots & * \\
0 & 0 & \cdots & * \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\end{pmatrix}
\]  

(1)

\[ MA = \begin{pmatrix}
* & * & \cdots & * \\
0 & * & \cdots & * \\
0 & 0 & \cdots & * \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\end{pmatrix}
\]  

(2)

The (2) both sides of transposition, obtain

\[
A^T M^T = \begin{pmatrix}
* & 0 & 0 & 0 & 0 & \cdots & 0 \\
* & * & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots \\
* & * & * & 0 & 0 & \cdots & 0 \\
\end{pmatrix}
\]  

(3)

Inside \( M_1, M_2, \ldots, M_n \) are lower triangular matrix, \( M^T = (M_n \cdots M_2 M_1)^T = M_1^T M_2^T \cdots M_n^T \) are upper triangular matrix.

Theorem 1.

Assume \( \bar{U} = (u_{n+1}, u_{n+2}, \ldots, u_m) \), then \( \bar{U} \) is a group of base on vertical subspace of \( A \).

Proof. Assume

\[
M^T = (u_1, u_2, \ldots, u_n, \ldots, u_m),
\]
so (3) can be changed to

\[
A^T (u_1, u_2, \ldots, u_n, \ldots, u_m) = \begin{pmatrix}
* & 0 & 0 & 0 & 0 & \cdots & 0 \\
* & * & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots \\
* & * & * & 0 & 0 & \cdots & 0 \\
\end{pmatrix}
\]

Because the rank of \( A \) is \( n \), so the matrix
\[
\begin{pmatrix}
* & 0 & 0 & 0 & 0 & \cdots & 0 \\
* & * & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \cdots & 0 \\
* & * & * & 0 & 0 & \cdots & 0 \\
\end{pmatrix}
\]

From the \((n+1)\)th column is vector 0.

\[
A^T u_{n+1} = 0 \\
A^T u_m = 0
\]

\(\overline{U}\) is a group of base on vertical subspace of \(A\).

According to Theorem 1, we can obtain the corresponding algorithm to solve \(\overline{U}\), denoted Algorithm 1.

Because the effective constraint coefficient matrix of large-scale problems often have some sparse structures, we can make the constraint matrix become a banded structure by adjusting the operative constraint and the order of unknown sequence, so using algorithm 1 calculation \(\overline{U}\), it can not only loosen the sparsity of the constraint matrix requirements, but it can also decrease the storage capacity of new method than the usual LU decomposition.

**III. THE GRADIENT PROJECTION ALGORITHM FOR NONLINEAR PROGRAMMING**

Considering the nonlinear programming problem

\[
\min \{ f(x) \mid x \in S \},
\]

\(S\) is a set defined by

\[
S = \{ x \mid Ax = b, x \geq 0 \}, \quad S_+ = \{ x \mid x \in S, x > 0 \}
\]

Algorithm 2

Step 1. Set the initial point \(x^0 \in S_+, k = 0\).

Step 2. Using \(\overline{x}^k\) the definition of diagonal matrix 

\[X = \text{diag}(\overline{x}_1^k, \cdots, \overline{x}_n^k),\]

do transform

\[
\overline{x} = X^{-1}x, \quad c = Xc, \quad \overline{A} = AX
\]

Step 3. Using algorithm 1 to solve \(u, u\) satisfy:

\[
\overline{A}^T u = 0.
\]

Step 4. Solve the gradient projection direction:

\[
df = dfG + dfb \\
c_p = -uu^T df \\
\overline{d} = \frac{1}{\|c_p\|} c_p
\]

Step 5. Iterative:

\[
x^{k+1} = e + \alpha \overline{d}, 0 < \alpha < 1.
\]

Step 6. If the termination condition is satisfied, then the end; otherwise \(k := k + 1\), return step 2.

**IV. EXAMPLE**

Example 1:

solve the nonlinear programming

\[
\min (x_1^2 + x_2^2 + \cdots + x_{100}^2 + x_{1000}^2) \\
\text{s.t. } x_i + x_{i+1} = 0.1, i = 1, 2, \ldots, 100 \\
x_i \geq 0, i = 1, 2, \ldots, 101
\]

Obviously, the optimal objective function value of the linear programming is 0, one of it is:

\[
x_{2i+1} = 0.1, i = 0, 1, 2, \cdots, 50, \\
x_{2i} = 0.0, i = 1, 2, \cdots, 50.
\]

From the initial point of

\[
x_1 = x_2 = \cdots = x_{100} = x_{101} = 0.05,
\]

after 311 iterations, the optimal value of 4.8640e-013.

Example 2:

solve the nonlinear programming

\[
\min (x_1^2 + x_2^2 + \cdots + x_{100}^2 + x_{1000}^2) \\
\text{s.t. } x_i + x_{i+1} + x_{i+2} = 0.6, \\
i = 1, 2, \cdots, 100 \\
x_i \geq 0, i = 1, 2, \cdots, 102
\]

Obviously, the optimal objective function value of the linear programming is 0, one of it is:

\[
x_{3i+1} = 0.4, \quad x_{3i+2} = 0.2, \quad x_{3i+3} = 0.0, \\
i = 1, 2, \cdots, 33.
\]

From the initial point of

\[
x_1 = x_2 = \cdots = x_{100} = x_{101} = 0.2, \\
after 306 iterations, the optimal value of 4.7609e-013.
\]

**ACKNOWLEDGMENT**

Wang Dan and Zhang Yong-ming thank the Support of Beijing high school youth talent plan (YETP1471) and Beijing Institute of Graphic Communication key projects (221501140121).

**REFERENCES**


