# Computing Concept Lattices with Clustering Approaches

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## Abstract

Finding the concepts in formal concept context is time-consuming, which is a NP problem. In this paper, hierarching conceptual clustering analysis based on distance function is investigated, under this clustering, we only check the some subset of feature set instead of all of them, which largely reduces the computing steps in finding the concepts, we pave a way to explore concept lattice with clustering approaches.

**Keywords**: Fuzzy clustering, Concept lattice, Data mining.

## 1. Introduction

Concept Lattices are used to represent conceptual hierarchies which are inherent in data. They are the core of the mathematical theory of *Formal Concept Analysis* (FCA). Introduced in the early 1980s as a formalization of the concept of `concept' [10], FCA has over the years grown to a powerful theory for data analysis, information retrieval, and knowledge discovery [11]. In artificial intelligence (AI), FCA is used as a knowledge representation mechanism [12] and as conceptual clustering method [9] for crisp concepts. In database theory, FCA has been extensively used for class hierarchy design and management [13]- [15]. Its usefulness for the analysis of data stored in relational databases has been demonstrated with the commercially used management system TOSCANA for Conceptual Information Systems [17]. FCA is a branch of lattice theory motivated by the need for a clear mathematization of the notions of concept and *conceptual hierarchy* [6]- [8], and knowledge acquisition [9].

The clustering can be roughly divided into two categories. One category,based on objective functions. Another category is based on a relation matrix such as correlation coefficient, equivalence relation, similarity relation and fuzzy relations, etc, agglomerative hierarchical clustering method belongs to the second type, it is simple and useful in application system. The investigate in this paper focus on this type of clustering, we cluster the objects in concept lattice into groups, thus we avoid the heavy burden computing of every subsets of feature sets with a definite subset of objects, this builds a bridge between concept analysis with clustering.Fuzzy clustering and concept analysis have been explored [1]-[5], this paper explores the connection further.

The paper is organized as follows. Section 2 presents some preliminary notions about formal concept analysis. Some related facts Similarity Measure are proposed in section 3, In section 4, we give an overview about hierarchical conceptual clustering, the main contribution in the concept searching method by similarity measure is show in section 5, Section 6 provides us with an illustrative example. Finally, we conclude this paper with section 7.

## 2. Concept lattice theory

**Definition 1** (B.Ganter,R.Wille,Formal [6]) A formal context is a triple concept $(G,M,I)$ where $G$ is a set of objects, $M$ is a set of attributes, and $I$ is a binary relation from $G$ *to* $M$, i.e. $I \subseteq G \times M$, $gIm$ is also written as $(g, m) \in I$, means that the object g possesses the attribute $m$, $g \in G$, $m \in M$. For a set of objects $A \subseteq G$, $\beta(A)$ is defined as the set of attributes shared by all objects in $A$, that is,

$$\beta(A)=\{m \in M \mid (g, m) \in I, \forall g \in A\}$$

Similarly, for $B \subseteq M$, $\alpha(B)$ is defined as the set of objects possesses all attributes in $B$, that is,

$$\alpha(B)=\{g \in G \mid (g, m) \in I, \forall m \in B\}$$

**Definition 2** (B.Ganter,R.Wille, Formal [6]) A formal concept of the context $(G,M,I)$ is a pair $(A,B)$ with $A \subseteq G, B \subseteq M$ and $\beta(A)=B$, $\alpha(B)=A$. We call $A$ the extent of and $B$ the intent of the concept $(A,B)$, $\underline{B}(G,M,I)$ denotes the set of all concepts of the context $(G,M,I)$

Table 1 shows an example of a context.

| | $m_1$ | $m_2$ | $m_3$ | $m_4$ | $m_5$ |
|---|---|---|---|---|---|
| $g_1$ | 0 | 0 | 1 | 1 | 1 |
| $g_2$ | 1 | 1 | 1 | 1 | 0 |
| $g_3$ | 1 | 1 | 1 | 1 | 0 |
| $g_4$ | 1 | 0 | 0 | 0 | 1 |
| $g_5$ | 1 | 0 | 0 | 0 | 1 |
| $g_6$ | 1 | 0 | 0 | 0 | 1 |

Table 1: Attributes and objects of a context.

**Example 1**
Let $M=\{m_1,m_2,m_3,m_4,m_5\}$, $G=\{g_1,g_2,g_3,g_4,g_5,g_6\}$
As $\alpha(m_1,m_2)=\alpha(m_1)\cap\alpha(m_2)=\{g_2,g_3\}$, and $\beta(g_2,g_3)=\{m_1, m_2, m_3,m_4\}$,thus $(\{g_2, g_3\},\{m_1,m_2\})$ is not a concept of $(G, M, I)$. $\alpha(\{m_3,m_4,m_5\})=\{g_1\}$,$\beta(\{g_1\})=\{m_3,m_4,m_5\}$,therefore the pair $(\{g_1\},\{m_3,m_4,m_5\})$ is a concept.

**Lemma 1** (B. Ganter, R.Wille, Formal [6]) Let $(G, M, I)$ be a context, The following assertions hold:
1) For every $A_1$, $A_2\subseteq G$,$A_1\subseteq A_2$, implies that $\beta(A_2)\subseteq\beta(A_1)$, and for every $B_1$, $B_2\subseteq M$, $B_1\subseteq B_2$, implies $\alpha(B_2)\subseteq\alpha(B_1)$.
2) For any $A\subseteq G$, $A\subseteq\alpha(\beta(A))$ and $\beta(A)=\beta(\alpha(\beta(A)))$; For any $B\subseteq M$, and $B\subseteq\beta(\alpha(B))$ and $\alpha(B)=\alpha(\beta(\alpha(B)))$.

**Definition 3**[3] $(G,M,I)$ is a formal context, for $x, y\in G,B\subseteq M$ ,distance function $d$, induced by $B$, is defined like this;
$d(x, y)=|(((\beta(x)^c\cap(\beta(y)))\cup(\beta(y)^c\cap(\beta(x)))\cap B|/|B|$.

**Theorem 1**[3] The function defined above is a distance function which satisfies;
For all $x, y, z\in G$
(1) $1\geq d(x, y)\geq 0$ and $d(x, y)=0\Rightarrow x\in\alpha(B)$ if and only if $y\in\alpha(B)$.For $x\in\alpha(B)$ and $y\in\alpha(B)$ ,$d(x, y)=0$.
(2) $d(x, y)=d(y, x)$
(3) $d(x, z)\leq d(x, y)+d(y, z)$.

Now, we give an example about this
**Example 2**

Table 2 gives an example of a context.

| | $m_1$ | $m_2$ | $m_3$ | $m_4$ |
|---|---|---|---|---|
| $g_1$ | 1 | 1 | 0 | 0 |
| $g_2$ | 0 | 1 | 1 | 0 |
| $g_3$ | 1 | 1 | 1 | 0 |
| $g_4$ | 0 | 0 | 0 | 1 |

Table 2: Attributes and objects of a context.

Let $B=\{m_2, m_3\}$,By simple computing, we can get that $d(g_1, g_2)=1/2$, $d(g_1, g_3)=1/2$, $d(g_1, g_4)=1/2$, $d(g_2, g_3)=0$, $d(g_2, g_4)=1$, $d(g_3, g_4)=1$ As $(\{g_2, g_3\},\{m_2, m_3\})$ is a concept, thus $d(g_2, g_3)=0$.In fact, we can get all subset of M easily, if for one set $N\subseteq M$, there exists an element $s\in\alpha(B)$ such that $d(s, t)=0$ for some $t\in G$, then all such t and s form the extent with $N$ as the intent, if $d(s, t)\neq 0$ for all $t\in G$, then $N$ is not a intent at all.

# 3. Similarity measure

Distance functions play a major role in pattern recognition and data analysis [1]-[4]**.** When each object is described with respect to quantitative feature variables, Minkowski metrics are often used as convenient dissimilarity measures in classification methods. Distance functions depend on the space which the objects lie in, but in some case, the space is not Euclidean space, thus we should use the key core of distance function, in fact, the main idea of it is that it acts as a "separation" function, for example, when two persons A and B are under consideration, the distance is not a metric one, feature A={***high salary, good health condition, low height, regular education background, mastering of English and Chinese***}, feature B={***low salary, good health condition, low height, irregular education background, mastering of English and Chinese and French***},the we learn that the features set {***good health condition, low height***} is common to all, and it has no difference in separating A and B, {***high salary, regular education background, mastering of English and Chinese***}$\notin$B, but {***high salary, regular education background, mastering of English and Chinese***}$\in$A, while{***low salary, irregular education background, mastering of English and Chinese and French***}$\in$B, but {***low salary, irregular education background, mastering of English and Chinese and French***}$\notin$A.then we can deem that the distance is 6.

**Definition 4** A ***Dissimilarity Measure***(DM) d on X

$$d: X\times X\rightarrow R$$

Where $R$ is the set of real numbers, such that
$\exists d_0\in R$: $d(x, x)=d_0$, and $d(x, y)=d(y, x)$,for $x, y\in X$ ,if in addition
$d(x, y)=d_0$ if and only if $x=y$ and $d(x, y)\leq d(x, z)+d(z, y)$ then $d$ is called a metric *DM*.

**Definition 5** A ***Similarity measure***(*SM*) $s$ on $X$

$$s: X\times X\rightarrow R$$

Where $R$ is the set of real numbers, such that
$\exists d_0\in R$: $s(x, x)=d_0$, and $s(x, y)=s(y, x)$,for $x, y\in X$ ,if in addition $s(x, y)=d_0$ if and only if $x=y$ and

$$s(x, y)\ s(y, z)\leq[s(x, y)+s(y ,z)]\ s(x, z)$$ then $d$ is called a metric *SM*.

We note that if we let $s(x, y)=1/d(x, y)$( $d(x, y)\neq 0$),then it satisfies the above ***triangular inequality***.

The common way of constructing ***Similarity Measure*** is like this, Let $X$ be the object set, $d(x, y)$ is the distance function, let

$$s(x, y)=1-d(x, y)/(\max_{x,y\in X}(d(x, y))).$$

Then it is a Similarity measure.

As to similarity measure and distance function, one derives from the other easily.

## 4. Hierarchical conceptual clustering

The basic idea behind agglomerative clustering is simple. Start each objects in its own separate cluster (i.e.,n clusters of size).At each stage of the process, find two "closest" clusters and join them together. Continue until one cluster of size *n* remains. The algorithm is simple and efficient. There are several kinds clustering

1 ***SINGLE-LINKAGE CLUSTERING***.( Shortest distance)
2 ***COMPLETE LINKAGE*** (Largest distance)
3 ***Average Linkage***(Average distance)
4 ***Centroid Method***(Centroid distance)
5 ***Ward's Method***(Incremental sum of squares)

All of the approaches are agglomerative in nature and produce hierarchical clustering solutions, the main difference among them is that the distance function defined, now we give a general overview of the methods, here we only give the case 1,the rest can be obtained by modifying step 3.

Single-Linkage Clustering

***Step 0***.Let $C_1$ , $C_2$ , $C_3$ , …$C_n$ be the clusters, the distance between two clusters is defined to be the distance between the two objects they contain; that is

$$d_{CiCj}= d_{ij}$$

Let $t$=1 be an index of the iterative process

***Step 1.***Find the smallest distance between any two clusters. Denote these two closest clusters $C_i$ , $C_j$.

***Step 2.***Amalgamate clusters $C_i$ and $C_j$ to form a new cluster denoted $C_{n+I}$

***Step 3.***Denote the distance between new cluster $C_{n+I}$ and all the remaining clusters $C_k$ as follows:

$$d_{C_{n+i}C_k} = \min\{d_{C_iC_k}, d_{C_jC_k}\}$$

***Step 4.***Add cluster $C_{n+I}$ as a new cluster and remove clusters $C_i$ and $C_j$ .Let t=t+1.

***Step 5.***Return to step 1 until one cluster remains.

As to case 2, $d_{Cn+iCk}$ is replaced by $\max\{d_{C_iC_k}, d_{C_jC_k}\}$ , the rest steps remained unchanged.While in case 3, $d_{Cn+iCk}$ is defined as follows

$$d_{C_{n+i}C_k} = \frac{n_i d_{C_iC_k} + n_j d_{C_jC_k}}{n_i + n_j}$$

Where $n_i$ , $n_j$ is the number of elements contained in $C_i$ and $C_j$ respectively. The *Centroid Method* designs the distance function like this

$$d^2(C_k, C_i \cup C_j) = \frac{n_{C_i} d^2{}_{C_kC_i} + n_{C_j} d^2{}_{C_kC_j}}{n_{C_i} + n_{C_j}} - \frac{n_{C_i} n_{C_j} d_{C_iC_j}{}^2}{(n_{C_i} + n_{C_j})^2}$$

## 5 The concept searching method by similarity measure

Not every pair of a set of objects and a set of features defines a concept, suppose (*G, M, I*) is a formal context, and $|G|=n$, $|M|=m$, the non empty subset of $G$ is $2^n-1$, the non empty subset of $M$ is $2^m-1$,for set $A\subseteq G$, $B\subseteq M$, to get a concept ($A$, $B$),we should check $\beta(A)= B$ and $A=\alpha(B)$. Finding all concepts need at least $N(m, n)=2*(2^n-1)*(2^m-1)$ steps, for example, as $m=n=5$, $N(m, n)=1922$, when $m$ and $n$ increase, the computing is time -consuming. Until now, no best way has been designed to cope with problem thoroughly, in [13], some important concepts according to the support and confidence rate are considered, this reduces the computing steps, but it sacrifices the number of all concepts, it means that some concepts has to be omitted, now we proposed a way to consider all concepts by cutting down the unnecessary computing steps beforehand, we first cluster the objects into groups according to some feature set $B$ , then $\alpha(B)$ will fall to the groups, by checking the member of the groups, we get the desired $\alpha(B)$,in this way, we not only reduce the computing steps, but get all the concepts as well. Now we present the algorithms in detail.

**Algorithm**

1. Find all subset of M, denote by $2^{\mathbf{M}}$.
2. Based on Theorem 2,choose feature sets $B$,computing distance matrix $D_B =(d_{ij})$ ,where $d_{ij}=d(x_i, x_j)$,pick the elements from the upper triangle above main diagonal as the distance input.
3. For all $B\in 2^{\mathbf{M}}$.Generate the hierarchical cluster tree, that is the Plot Dendrogram Graph.
4. Discard the unrelated clusters.
5. end
6. Return 3

## 6 Some illustrative examples

Let $G=\{x_1, x_2, x_3, x_4\}$, $M=\{a, b, c, d, e, f, g, h\}$,choose $B=\{ c, e\}$,for the convenience of computing, denote it as vector [0 0 1 0 1 0 0 0].

Table 3 shows an example of a context

| | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | $g$ | $h$ |
|---|---|---|---|---|---|---|---|---|
| $x_1$ | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 1 |
| $x_2$ | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 0 |
| $x_3$ | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 1 |
| $x_4$ | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |

Table 3: Attributes and objects of a context.

The matrix $D_B$ is

$$D_B=\begin{bmatrix} 2 & 6 & 4 & 3 \\ 6 & 2 & 4 & 5 \\ 4 & 4 & 2 & 3 \\ 3 & 5 & 3 & 2 \end{bmatrix}$$

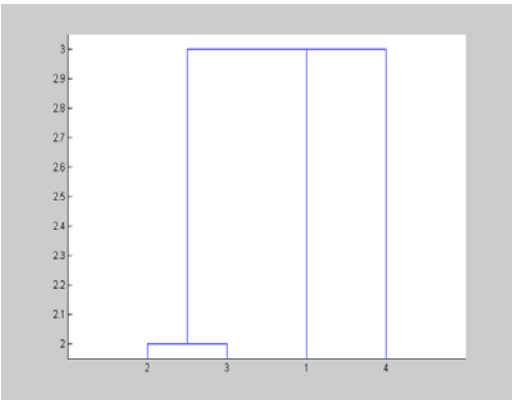


Fig. 1: Extent of $B$={ $c$, $e$}.

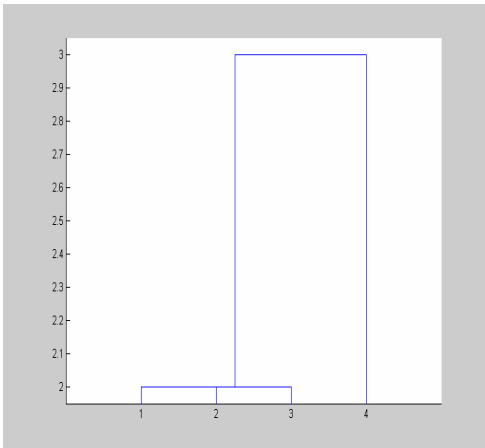


Fig. 2: Extent of $B$={ $a$}.

In Fig 1, we use single-linkage to generate the dendrogram, a vertical line at distance $d$ yields partitions as follows:

$0 \le d < 2 \Rightarrow \{x_2\}, \{x_3\}, \{x_1\}, \{x_4\}$,

$2 \le d < 3 \Rightarrow \{x_2, x_3\}, \{x_1\}, \{x_4\}$,

Here we choose $d$=2.5,it joins objects $x_2$, $x_3$ together, therefore, the candidate sets for feature {$a$, $c$} are {$x_2$, $x_3$},{$x_1$},{$x_4$},it is easy to discard {$x_1$},{$x_4$},thus we get the concept ({ $x_2$, $x_3$}, {$a$, $c$}), if we select $B$={$a$},from Fig 2, we have clusters {$x_1$, $x_2$, $x_3$},{$x_4$}, after discarding {$x_1$, $x_2$, $x_3$}, the concept ({$x_4$},{$a$}) is obtained. Here the "single" distance function is used, if it is replaced by "*complete*" or "*average*", the same results can be followed.

Let start another experiment which appear in [6],it is a context of an educational film "Living Beings and Water".The attributes are a:needs water to live,b:living in water,c:lives on land,d:needs chorophyll to produce food,e:two seed leaves,g:can move around,h:has limbs, i:suckles its offerspring

There are $2^8-1=255$ subsets of objects, $2^9-1=551$ subsets of attributes, there are $(2^8-1)*(2^9-1)=140505$ couples ($A$, $B$) where $A \subseteq G$, $B \subseteq M$, if $\beta(A)=B$, $\alpha(B)=A$ holds,then ($A$, $B$) is a concept in formal context.

Table 4 shows an example of a context.

| | | a | b | c | d | e | f | g | h | i |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | Leech | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| 2 | Bream | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 0 |
| 3 | Frog | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 0 |
| 4 | Dog | 1 | 0 | 1 | 0 | 0 | 0 | 1 | 1 | 1 |
| 5 | Spike-weed | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 0 |
| 6 | Reed | 1 | 1 | 1 | 1 | 0 | 1 | 0 | 0 | 0 |
| 7 | Bean | 1 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 0 |
| 8 | Maize | 1 | 0 | 1 | 1 | 0 | 1 | 0 | 0 | 0 |

Table 4: Attributes and objects of a context

Let B={a,b},then from Plot Dendrogram Graph,we get that

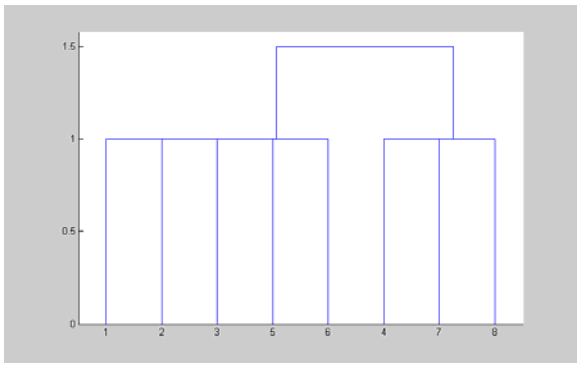


Fig. 3: Extent of $B$={ $a$,$b$}.

By checking,we get two sets {1 2 3 5 6},{4 7 8},it is clear that b∈β({8}),thus ({4 7 8},{a,b}) is not a concept,it is easy to check that ({1 2 3 5 6},{a,b}) is a concept.

Let B={a,c},then from Plot Dendrogram Graph, it follows that

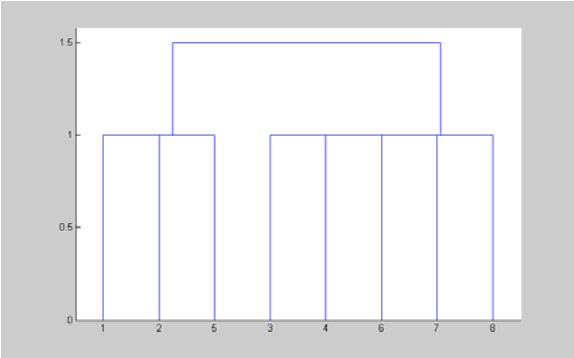


Fig. 4: Extent of $B$={ $a,c$}.

By checking,we have that ({3 4 6 7 8},{a,c}) is a concept
Let B={a},then from Plot Dendrogram Graph,we have that

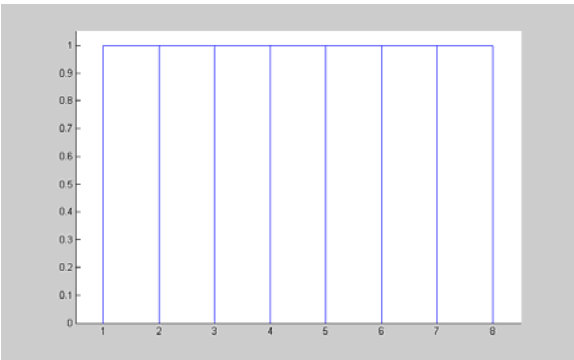


Fig. 5: Extent of $B$={ $a$}.

Only one set {1 2 3 4 5 6 7 8} is obtained,it is easy to see that ({1 2 3 4 5 6 7 8},{a}) is a concept.
Let B={a,b,g},then from Plot Dendrogram Graph,we see that

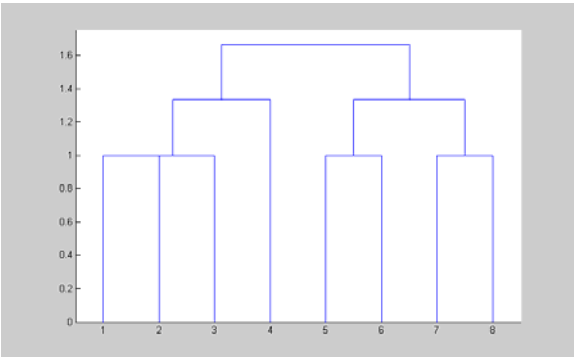


Fig. 6: Extent of $B$={ $a,b,g$}.

Four sets {1 2 3},{4},{5 6},{7 8} are created,it is not difficult to discard {4},{5 6},{7 8},so ({1 2 3},{a,b,g}) Is a concept.
Let B={a,b, d,f},then from Plot Dendrogram Graph, it follows that

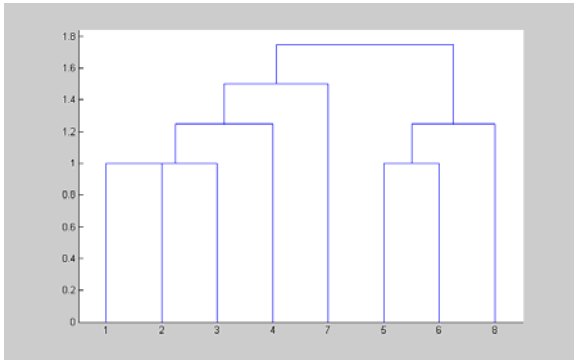


Fig. 7: Extent of $B$={ $a,b,d,f$}.

From the five sets {1 2 3},{4},{5 6},{7},{ 8},it is easy to know that ({5 6},{a,b,d,f}) is a concept.
Let B={a,b, c,d,f},then from Plot Dendrogram Graph,we get that

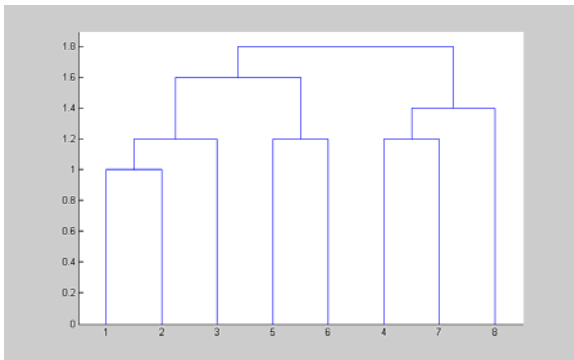


Fig. 8: Extent of $B$={ $a,b,c,d,f$}.

A vertical line at distance $d$ yields partitions as follows:
$1.4 \le d < 1.6 \Rightarrow \{x_1, x_2, x_3\}, \{x_5, x_6\}, \{x_4, x_7, x_8\}$
As $1,2,5 \notin \alpha(c)$ ,therefore,we have
$1.2 \le d < 1.4 \Rightarrow \{x_1, x_2, x_3\}, \{x_5, x_6\}, \{x_4, x_7\}, \{x_8\}$
Further,as $4,7,8 \notin \alpha(b)$,we have
$1.2 \le d < 1 \Rightarrow \{x_1, x_2\}, \{x_3\}, \{x_5\}, \{x_6\}, \{x_4\}, \{x_7\}, \{x_8\}$
Thus,only $\{x_3\}, \{x_6\}$ are considered as the candidate ones, it is verified that ($\{x_6\}$,{a,b, c,d,f}) is the desired concept.
Let B={a,b,d},then from Plot Dendrogram Graph,we have that

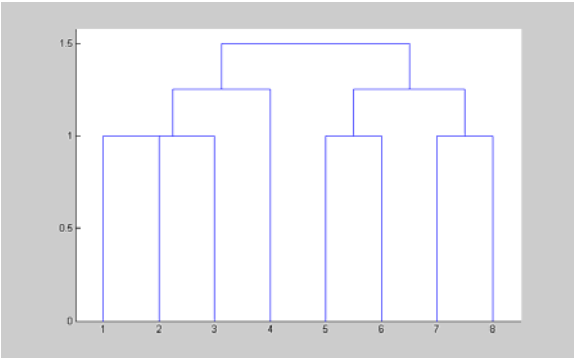


Fig. 9: Extent of $B=\{a,b,d\}$.

A vertical line at distance $d$ yields partitions as follows:
$1 \le d < 1.5 \Rightarrow \{x_1, x_2, x_3\}, \{x_4\}, \{x_5, x_6\}, \{x_7, x_8\}$
As $1,2,3,4 \notin \alpha(d)$ ,therefore,we have
$0 \le d < 1 \Rightarrow \{x_1\}\{x_2\}\{x_3\}, \{x_5\}\{x_6\}, \{x_4\}, \{x_7\}, \{x_8\}$.
Thus,only $\{x_5\}, \{x_6\}, \{x_7\}, \{x_8\}$ are considered as the candidate ones,it is verified that none of them constructs a concept with {a,b,d}.
By computing,all concepts are as follows

Table 5 lists all structures of concepts in context .

| | Extent | Intent |
|---|---|---|
| 1 | {1 2 3 4 5 6 7 8} | {a} |
| 2 | {1 2 3 4 } | {a g} |
| 3 | {3 4 6 7 8} | {a c} |
| 4 | {1 2 3 5 6} | {a b} |
| 5 | {5 6 7 8} | {a d} |
| 6 | {2 3 4} | {a g h} |
| 7 | {3 4} | {a c g h} |
| 8 | {1 2 3} | {a b g} |
| 9 | {2 3} | {a b g h} |
| 10 | {3 6 } | {a b c} |
| 11 | {5 6 8} | {a d f} |
| 12 | {6 7 8} | {a c d} |
| 13 | {5 6} | {a b d f} |
| 14 | {6 8} | {a c d f} |
| 15 | {7} | {a c d e} |
| 16 | {6} | {a b c d f} |
| 17 | {4} | {a c g h i} |
| 18 | {3} | {a b c g h} |

Table 5: Extents and intents of educational film

## 7. Conclusions

In this paper, we created a new clustering based on the distance function defined in formal context, with the new approaches, the computing steps of finding concepts in formal context are reduced, the objects sets are located in the clustering exactly, this makes it possible for us to find all the concepts relatively easily, at the meaning time, a link between concept lattice and fuzzy clustering is established.

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