The Existence of the Moore-Penrose Inverse in Symmetrized Max-Plus Algebraic Matrix

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ABSTRACT
In this paper we discuss Moore-Penrose inverse in symmetrized max-plus algebraic matrix. The existence of Moore-Penrose inverse is shown using a link among symmetrized max-plus algebra and conventional algebra. The result is a Moore-Penrose inverse in symmetrized max-plus algebraic matrix exists. Furthermore, the balanced inverse and the max-plus inverse are also the Moore-Penrose inverse in symmetrized max-plus algebraic matrix.

Keywords: Existence, Moore-Penrose inverse, Symmetrized max-plus algebra, Conventional algebra.

1. INTRODUCTION

Max-plus algebra is the set of \( \mathbb{R} \cup \{-\infty\} \) equipped with maximum (simply written "max") as addition and usual addition (simply written "plus") as multiplication, where \( \mathbb{R} \) is the set of all real numbers. Henceforth, the max-plus algebra is denoted by \( \mathbb{R}_{\max} \). It is different from conventional algebra, since there is no inverse element under addition for every element in max-plus algebra, except for zero element [1][2][3]. The symmetrization process can be done to solve the additive inverse problem. This process is carried out using a balance relation (denoted by \( \mathbb{V} \)) in order to obtain the minus and balance of all elements in \( \mathbb{R}_{\max} \). The result of this symmetrization is called symmetrized max-plus algebra and denoted by \( \mathbb{S} \) [4][5].

In conventional algebra, it is known that \( A^{-1} \) denotes the inverse of an invertible square matrix \( A_{n \times n} \) [6]. It is known that there is an inverse for an \( m \times n \) matrix called the Moore-Penrose inverse and usually denoted by \( A^* \) [7]. The concept of the inverse of an \( n \times n \) matrix can be used as an alternative way to find a solution to a system of linear equations in the form \( Ax = b \). If \( A \) is an invertible matrix, then the solution of the system of linear equations can be solved using the formula \( x = A^{-1}b \). If the matrix \( A \) of the system has a size of \( m \times n \), the solution of the system cannot be found using these rules. The discussion about application of the Moore-Penrose inverse in linear equation systems was discussed in [8].

The discussion about the Moore-Penrose inverse on arbitrary ring and integral domain were discussed in [9] and [10], respectively. In this paper, we discuss the Moore-Penrose inverse of matrix over \( \mathbb{S} \). We use a link between \( \mathbb{S} \) and conventional algebra in [11] to show the existence of Moore-Penrose inverse in \( \mathbb{S} \). We adopt the Moore-Penrose inverse in conventional algebra [12] to define the Moore-Penrose inverse in \( \mathbb{S} \), by changing equal relation in the conventional Moore-Penrose inverse into balance relation in symmetrized max-plus algebra. The results in this paper can potentially be used as an alternative tool to solve the solution of the systems of linear balance in \( \mathbb{S} \).

2. BASIC TERMINOLOGY

This section discusses basic terminologies in symmetrized max-plus algebra. Let \( \mathbb{S} \) be the set of all real numbers, \( \mathbb{E} \equiv \{-\infty\} \) and \( \mathbb{E}_{\max} \equiv \mathbb{R} \cup \{-\infty\} \). The basic operations in \( \mathbb{E}_{\max} \) are defined by

\[
\begin{align*}
a \oplus b &= \max\{a, b\} \quad (1) \\
a \oslash b &= a + b \quad (2)
\end{align*}
\]

where \( \max\{a, -\infty\} = a \) and \( a + (-\infty) = -\infty \), for all \( a, b \in \mathbb{E}_{\max} \). The mathematical system \( \mathbb{E}_{\max} = (\mathbb{E}_{\max}, \oplus, \oslash) \) is called the max-plus algebra, with the zero element is \( \mathbb{E} \), the unity element is \( e \) and the zero element \( \mathbb{E} \) is absorbing for \( \oslash \). Furthermore, \( \mathbb{E}_{\max} \) is an idempotent commutative semiring. There is no inverse element under addition for all \( a \) in \( \mathbb{E}_{\max} \) except for \( a = \mathbb{E} \).

Let \( P_{\mathbb{E}} \equiv \mathbb{E}_{\max} \times \mathbb{E}_{\max} \). The basic operations in \( P_{\mathbb{E}} \) are defined by

\[
\begin{align*}
(a, b) \ominus (c, d) &= (a \ominus c, b \ominus d) \quad (3) \\
(a, b) \otimes (c, d) &= (a \otimes c \oplus b \otimes d, a \otimes d \oplus b \otimes c) \quad (4)
\end{align*}
\]

for all \( (a, b) \), \( (c, d) \) \( P_{\mathbb{E}} \). The zero element is \( (\mathbb{E}, \mathbb{E}) \), the unity element is \( (0, \mathbb{E}) \) and \( (\mathbb{E}, \mathbb{E}) \) is absorbing for multiplication. The mathematical system \( P_{\max} = (P_{\mathbb{E}}, \ominus, \otimes) \)
\( \otimes \) is a commutative idempotent semiring and called the algebra of pairs. Some terminologies in algebra of pairs refers to [4]. If \( u = (a, b) \in P_{\text{max}} \), then the absolute value of \( u \) is defined as \( |u|_\oplus = a \oplus b \), the minus of \( u \) is \( u = (b, a) \) and the balance of \( u \) is \( u^* = u \ominus (\ominus u) = ([|u|_\ominus, |u|_\oplus]). \) Furthermore, for all \( u, v \in P_{\text{max}}, \) the following statements are satisfied: \( u^* = (\oplus u)^* = (u^*)^* \), \( u \otimes u^* = (u \otimes v)^* \), \( \ominus (\ominus u) = u \), \( \ominus (u \ominus v) = (\ominus u) \ominus (\ominus v) \) and \( \ominus (u \ominus v) = (\ominus u) \otimes v. \)

In the conventional algebra, for all \( x \in \mathbb{R}, x - x = 0 \), but for all \( u \in P_{\text{max}}, u \ominus u = u^* \neq (E, E), \) except \( u = (E, E). \) It is important to introduce “balance” relation for substituting “equal” relation in conventional algebra. If \( u = (a, b), v = (c, d) \in P_{\text{max}}, \) balance relation (denoted by \( \mathbb{V} \)) in \( P_{\text{max}} \) is defined as follows:

\[
u \mathbb{V} u \text{ if } u \ominus d = b \ominus c. \tag{5}\]

The balance relation reflexive, symmetric but it is not transitive, so that it is impossible to define the quotient set of \( P_\mathbb{V} \) by \( \mathbb{V}. \) For example, \((5,4) \mathbb{V} (5,5) \) and \((5,5) \mathbb{V} (4,5) \) but \((5,4) \mathbb{V} (4,5) \). The new relation will be introduced in order to solve “transitive problem” in balance relation. Let \( u = (a, b), v = (c, d) \in P_{\text{max}}, \) relation \( \mathbb{B} \) in \( P_{\text{max}} \) is defined as follows:

\[
u \mathbb{B} u \text{ if } \begin{cases} (a, b) \mathbb{V} (c, d), & \text{if } a \neq b \text{ and } c \neq d \\ (a, b) = (c, d), & \text{if } a = b \text{ or } c = d. \end{cases} \tag{6}\]

For all \( u \in P_{\text{max}}, u \ominus u \mathbb{B} (E, E) \) except for \( u = (E, E) \) and \( \mathbb{B} \) is an equivalence relation. So, it is possible to obtain a quotient set of \( P_{\text{max}} \) by \( \mathbb{B}. \) The equivalence classes generated by \( \mathbb{B} \) are

1. \((w, -\infty) = \{(w, x) \in P_{\text{max}} | x < w \}\) is called max-positive.
2. \((-\infty, w) = \{(x, w) \in P_{\text{max}} | x < w \}\) is called max-negative,
3. \((w, w) = \{(w, w) \in P_{\text{max}}\}\) is called balanced.

The quotient set of \( P_{\text{max}} \) by \( \mathbb{B} \) is denoted \( P_{\text{max}}/\mathbb{B} \equiv \mathbb{S}. \) Note that \((5,4)\) balance with \((5,5)\) and \((5,5)\) also balance with \((4,5)\), but \((5,4)\) is not \( \mathbb{B} \) relation to \((5,5)\), neither are \((5,5)\) and \((4,5)\).

The mathematical system \( \mathbb{S}_{\text{max}} = (\mathbb{S}, \ominus, \otimes) \) is called the symmetrized max-plus algebra. The zero element is the class \( (E, E) \) the unity element is the class \( 0 = (0, 0) \) and the zero element \((E, E)\) is absorbing for \( \otimes. \) Furthermore, \((w, -\infty), (-\infty, w)\) and \((w, w)\) are sufficiently written by \( w \ominus w \) and \( w \subseteq w \), respectively. The set of all max-positive class or zero class, max-negative class or zero class and balanced class are denoted by \( \mathbb{S}^{\oplus}, \mathbb{S}^{\ominus} \) and \( \mathbb{S}^{\ominus} \), respectively. The set \( \mathbb{S}^{\subseteq} = \mathbb{S}^{\oplus} \cup \mathbb{S}^{\ominus} \) is called and the set of all signed element. Note that \( \mathbb{S}^{\oplus} \cup \mathbb{S}^{\ominus} \subseteq \mathbb{S} \) and \( \mathbb{S}^{\subseteq} \cap \mathbb{S}^{\subseteq} = \{(E, E)\}. \)

The basic operation of matrix over \( \mathbb{S} \) can be done in the usual way as that in conventional algebra. The zero matrix in \( \mathbb{S}^{m \times n} \) with \( E_{ij} = E \) for all \( i = 1, 2, \ldots, m \) and \( j = 1, 2, \ldots, n. \) The identity matrix is \( I = [a] \in \mathbb{S}^{n \times n} \) with \( a_{ij} = e \) if \( i = j \) and \( a_{ij} = E \) if \( i \neq j, \) for \( i, j = 1, 2, \ldots, n. \) For all \( A, B \in \mathbb{S}^{m \times n}, A \mathbb{V} B \) if \( a_{ij} \mathbb{V} b_{ij} \) for \( i = 1, 2, \ldots, m \) and \( j = 1, 2, \ldots, n. \) If \( A = \left[ \begin{array}{c} 1 \otimes 2 \\ 3^* \end{array} \right] \) and \( B = \left[ 2^* \otimes 3 \right], \) then \( A \mathbb{V} B \) since \( a_{ij} \mathbb{V} b_{ij} \) for \( i = 1, 2 \) and \( j = 1, 2. \) Note that the corresponding entries of \( A \) and \( B \) are not always equal.

### 3. A LINK BETWEEN CONVENTIONAL ALGEBRA AND SYMMETRIZED MAX-PLUS ALGEBRA

This section discusses a link between \( \mathbb{S} \) and conventional algebra. It is used to solve the Moore-Penrose inverse in \( \mathbb{S} \) via conventional algebra approach. In this paper, this link is used to show the existence of axioms of the Moore-Penrose inverse in \( \mathbb{S} \) sense. The link is referred to [11].

**Definition 1**

A mapping \( F \) with domain of definition \( \mathbb{S} \times \mathbb{R}_0 \times \mathbb{R}_0^* \) is defined as

\[
F(a, \mu, s) = \begin{cases} |\mu| e^{as}, & \text{if } a \in \mathbb{S}^{\oplus} \\ -|\mu| e^{as}, & \text{if } a \in \mathbb{S}^{\ominus} \\ \mu e^{\langle 0, a \rangle}, & \text{if } a \in \mathbb{S}^{\subseteq} \end{cases} \tag{7}
\]

where \( a \in \mathbb{S}, \mu \in \mathbb{R}_0, s \in \mathbb{R}_0. \)

**Definition 2**

Let \( f(s) = ve^{\langle 0, a \rangle} \) be in the neighbourhood of \( \infty, \) here is the function \( R \) is defined as

\[
R(f) = \begin{cases} |a| & \text{if } v \text{ positive} \\ -|a| & \text{if } v \text{ negative} \end{cases} \tag{8}
\]

The function in (7) and (8) are used to correspond elements in symmetrized max-plus algebra into conventional algebra and otherwise, respectively. If \( \mu = 1 \) then \( F(5,1,1) = e^{5s}, \) \( F(\mathbb{V} 5,1,1) = e^{-5s} \) and \( F(5^*,1,1) = e^{5s}. \) Note that \( R(F(5,1,1)) = R(F(\mathbb{V} 5,1,1)) = R(e^{5s}) = 5 \) and \( R(F(5^*,1,1)) = R(e^{-5s}) = 5. \) Range of \( R \) is \( \mathbb{S}^\subseteq = \mathbb{S}^{\oplus} \cup \mathbb{S}^{\ominus} \) i.e the set of all signed element.

The following theorem explain the correspondence between addition and multiplication in symmetrized max-plus algebra into conventional algebra.

**Theorem 3**

Let \( a, b, c \in \mathbb{S}. \)

1. If \( a \mathbb{B} b = c \) then there are \( \mu_a, \mu_b, \mu_c \in \mathbb{R}_0 \) such that \( F(a, \mu_a, s) + F(b, \mu_b, s) \sim F(c, \mu_c, s), s \to \infty. \)
2. If \( \mu_a, \mu_b, \mu_c \in \mathbb{R}_0 \) such that \( F(a, \mu_a, s) + F(b, \mu_b, s) \sim F(c, \mu_c, s), \) for \( s \to \infty \) then \( a \mathbb{V} b \mathbb{V} c. \)
3. If \( a \mathbb{B} b = c \) then there are \( \mu_a, \mu_b, \mu_c \in \mathbb{R}_0 \) such that \( F(a, \mu_a, s) \times F(b, \mu_b, s) = F(c, \mu_c, s), \) for \( s \to \infty. \)
4. If there are \( \mu_a, \mu_b, \mu_c \in \mathbb{R}_0 \) such that...
This section discusses the existence of the Moore-Penrose inverse in \( S \) sense. The link between \( S \) and conventional algebra is used to derive several properties in determining the existence of the Moore-Penrose inverse on matrix over \( S \).

The following theorems explain the existence of the balance of matrix over \( S \), which is similar to the first and second axioms of the Moore-Penrose in conventional algebra.

**Theorem 8**

Let \( A \) and \( X \) be matrices over \( S \). If there are matrices \( N_m, N_r \) whose entries are in \( \mathbb{R}_0 \) such that

\[
\mathcal{F}(A, N_m) \cdot \mathcal{F}(X, N_r) \cdot \mathcal{F}(A, N_m) \sim \mathcal{F}(A, N_m), \quad \text{for} \quad s \to \infty, \quad \text{then} \quad A \otimes X \otimes A \forall A .
\]

**Proof.** Let \( a \) and \( x \) be entries of \( A \) and \( X \), respectively. By using Definition 1, the exponential form of all entries in \( A \) and \( X \) as in conventional algebra are obtained. All entries in exponential form are in \( S_e \). Therefore, according to Theorem 6 and Theorem 7, all of algebraic operation of those exponential are in \( S_e \). Let \( \mathcal{F}(A, N_m) \) and \( \mathcal{F}(X, N_r) \) be real matrix-valued function for \( A \) and \( X \), with \( N_m \) and \( N_r \) are matrices whose entries are in \( \mathbb{R}_0 \). Suppose there are matrices \( N_m, N_r \) whose entries are in \( \mathbb{R}_0 \) such that

\[
\mathcal{F}(A, N_m) \cdot \mathcal{F}(X, N_r) \cdot \mathcal{F}(A, N_m) \sim \mathcal{F}(A, N_m), \quad \text{for} \quad s \to \infty. \quad \text{If the asymptotic equivalent form in part 4 of Theorem 4 is replaced by}
\]

\[
\mathcal{F}(A, N_m) \cdot \mathcal{F}(X, N_r) \cdot \mathcal{F}(A, N_m) \sim \mathcal{F}(A, N_m), \quad \text{for} \quad s \to \infty \quad \text{then it is obtained that} \quad A \otimes X \otimes A \forall A .
\]

**Theorem 9**

Let \( A \) and \( X \) be matrices over \( S \). If there are matrices \( N_m, N_r \) whose entries are in \( \mathbb{R}_0 \) such that

\[
(X, N_r) \cdot \mathcal{F}(A, N_m) \cdot (X, N_r) \sim (X, N_r), \quad \text{for} \quad s \to \infty \quad \text{then} \quad X \otimes A \otimes X \forall X.
\]

**Proof.** Let \( \mathcal{F}(A, N_m) \) and \( (X, N_r) \) be real matrix-valued function for \( A \) and \( X \), with \( N_m \) and \( N_r \) are matrices whose entries are in \( \mathbb{R}_0 \). Suppose there are matrices \( X, N_r \) whose entries are in \( \mathbb{R}_0 \) such that

\[
(X, N_r) \cdot \mathcal{F}(A, N_m) \cdot (X, N_r) \sim (X, N_r), \quad \text{for} \quad s \to \infty. \quad \text{If the asymptotic equivalent form in part 4 of Theorem 4 is replaced by}
\]

\[
(X, N_r) \cdot \mathcal{F}(A, N_m) \cdot (X, N_r) \sim (X, N_r), \quad \text{for} \quad s \to \infty \quad \text{then it is obtained that} \quad X \otimes A \otimes X \forall X.
\]

The following theorems explain the existence of the balance of matrix over \( S \), which is similar to the third and fourth axioms of the Moore-Penrose in conventional algebra.

**Theorem 10**

Let \( A \) and \( X \) be matrices over \( S \). If there are \( N_m, N_r \) whose entries are in \( \mathbb{R}_0 \) such that

\[
(\mathcal{F}(A, N_m) \cdot (X, N_r)) \sim (\mathcal{F}(A, N_m) \cdot (X, N_r)), \quad \text{for} \quad s \to \infty \quad \text{then} \quad (A \otimes X) \otimes A \forall X.
\]

**Proof.** Let \( \mathcal{F}(A, N_m) \) and \( (X, N_r) \) be real matrix-valued function for \( A \) and \( X \), with \( N_m \) and \( N_r \) are matrices whose entries are in \( \mathbb{R}_0 \). Suppose there are matrices \( N_m, N_r \) whose entries are in \( \mathbb{R}_0 \) such that
Theorem 11
Let A and X be matrices over $\mathbb{S}$. If there are $N_h, N_x$ whose entries are in $\mathbb{R}_0$ such that
\[
(F(X, N_x), s) \cdot (F(A, N_h), s) \cdot (F(X, N_x), s) \cdot (F(A, N_h), s)
\]
for $s \to \infty$. If the asymptotic equivalent form in part 4 of Theorem 4 is replaced by
\[
(F(A, N_h), s) \cdot (F(X, N_x), s)^T \sim (F(A, N_h), s) \cdot (F(X, N_x), s)
\]
for $s \to \infty$ then we have $(A \otimes X)^T \nabla A \otimes X$. □

Proof. Let $F(X, N_x)$ and $F(A, N_h)$ be real matrix-valued function for $X$ and $A$, with $N_x$ and $N_h$ are matrices whose entries are in $\mathbb{R}_0$. Suppose there are matrices $N_x, N_h$ whose entries are in $\mathbb{R}_0$ such that
\[
(F(X, N_x), s) \cdot (F(A, N_h), s)^T \sim (F(X, N_x), s) \cdot (F(A, N_h), s)
\]
for $s \to \infty$. If the asymptotic equivalent form in part 4 of Theorem 4 is replaced by
\[
(F(X, N_x), s) \cdot (F(A, N_h), s)^T \sim (F(X, N_x), s) \cdot (F(A, N_h), s)
\]
for $s \to \infty$ then we have $(X \otimes A)^T \nabla X \otimes A$. □

The following example illustrates the existence of the matrix balance form in Theorem 8 until Theorem 11.

Example 12
Let $A$ and $X$ be matrices over $\mathbb{S}$, respectively where $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$ and $X = \begin{bmatrix} -5 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -3 \end{bmatrix}$.

If $N_h = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $N_x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ then
\[
F(A, N_h) = \begin{bmatrix} e^5 & e^{2s} & e^{3s} \\ e^{-5s} & e^{-2s} & e^{-3s} \\ e^{-5s} & e^{-2s} & e^{-3s} \end{bmatrix}
\]
and $N_x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ such that
\[
F(A, N_h).F(X, N_x), s \cdot F(A, N_h), s \sim \begin{bmatrix} e^5 & e^{2s} & e^{3s} \\ e^{-5s} & e^{-2s} & e^{-3s} \\ e^{-5s} & e^{-2s} & e^{-3s} \end{bmatrix} \cdot \begin{bmatrix} e^5 & e^{2s} & e^{3s} \\ e^{-5s} & e^{-2s} & e^{-3s} \\ e^{-5s} & e^{-2s} & e^{-3s} \end{bmatrix} = (A, N_h), s \to \infty.
\]

Theorem 12
Let $M \in \mathbb{S}^{m \times n}$. The Moore-Penrose inverse of $M$ is an $n \times m$ matrix $M^+ \in \mathbb{S}^{m \times n}$ which satisfies
1. $M \otimes M^+ \otimes M \nabla M$
2. $M^+ \otimes M \otimes M^+ \nabla M^+$
3. $(M \otimes M^+)^T \nabla M \otimes M^+$
4. $(M^+ \otimes M)^T \nabla M^+ \otimes M$

Since $X = \begin{bmatrix} -5 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -3 \end{bmatrix}$ in Example 12 fulfills the axioms in Definition 13, then $X$ is the Moore-Penrose inverse of $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

The discussion in Theorem 8 until Theorem 11 were used to show the existence of the matrix balances in order to define the Moore-Penrose inverse in $\mathbb{S}$.

Definition 13
Let $M \in \mathbb{S}^{m \times n}$. The Moore-Penrose inverse of $M$ is an $n \times m$ matrix $M^+ \in \mathbb{S}^{m \times n}$ which satisfies
1. $M \otimes M^+ \otimes M \subseteq M$
2. $M^+ \otimes M \otimes M^+ \subseteq M^+$
3. $(M \otimes M^+)^T \subseteq M \otimes M^+$
4. $(M^+ \otimes M)^T \subseteq M^+ \otimes M$

According to Definition 14, we can define the Moore-Penrose in max-plus algebra sense.

Definition 14
Let $M \in \mathbb{S}^{m \times n}$. The Moore-Penrose inverse of $M$ is an $n \times m$ matrix $M^+ \in \mathbb{S}^{m \times n}$ which satisfies
1. $A \otimes A^+ \otimes A = A$
2. $A^+ \otimes A \otimes A^+ = A^+$
3. $(A \otimes A^+)^T = A \otimes A^+$
4. $(A^+ \otimes A)^T = A^+ \otimes A$

The balanced inverse of a square symmetrized max-plus algebraic matrix plays a similar role as an inverse in conventional matrix.
Definition 16 (Balanced Inverse)
Let $A \in S^{n \times n}$. If there is $B \in S^{n \times n}$ such that $A \otimes B \mathcal{V} A$ and $B \otimes A \mathcal{V} B$ then $A$ is said to be balanced invertible and $B$ is a balanced inverse of $A$. Furthermore, the balanced inverse of $A$ is denoted by $A^{-b}_b$. The balanced inverse of a square matrix in $S$ can be solved using Definition 1, Definition 2 and the properties of the link between $S$ with conventional algebra. The following example explains the balanced inverse of square matrix over $S$.

Example 17
Let $A = \begin{bmatrix} 1 & 1 \end{bmatrix}_2$ where $\det(A) = 5 \forall E$. The real matrix-valued function which corresponds to $A$ by the function in (7) is $A(s) = \begin{bmatrix} e^{3} & e^{2} \end{bmatrix}$. Therefore $\det(A(s)) = e^{5}s - e^{3}s$ and $\operatorname{cof}(A(s)) = \begin{bmatrix} e^{4}s & -e^3 \\ e^3 & -e^3 \end{bmatrix}$ for $s \in \mathbb{R}^+$. Since $\frac{\operatorname{cof}(A(s))^T}{\det(A(s))} = \begin{bmatrix} e^{5}s - e^{3}s \\ e^{5}s - e^{3}s \\ e^{5}s - e^{3}s \\ e^{5}s - e^{3}s \end{bmatrix}$ then

$s \rightarrow \infty, \begin{bmatrix} 1 & 1 \end{bmatrix} \otimes \begin{bmatrix} -1 \\ \Theta(-3) \\ -4 \\ \Theta(-3) \end{bmatrix} \otimes \begin{bmatrix} 0 \\ E \\ 0 \\ E \end{bmatrix}$

are obtained. Since $\begin{bmatrix} 1 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 2 \end{bmatrix}_4$ where $\begin{bmatrix} 1 & 1 \end{bmatrix}$ is a signed matrix in symmetrized max-plus algebra, then

$\begin{bmatrix} 1 & 1 \end{bmatrix} \otimes \begin{bmatrix} -1 \\ \Theta(-3) \\ -4 \\ \Theta(-3) \end{bmatrix} \otimes \begin{bmatrix} 0 \\ E \\ 0 \\ E \end{bmatrix}$

and

$\begin{bmatrix} -1 \\ \Theta(-3) \\ -4 \\ \Theta(-3) \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}$

are obtained. Furthermore, the balanced inverse $A^{-b}_b = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ such that $A \otimes A^{-b}_b \mathcal{V} I_n$ and $A^{-b}_b \otimes A \mathcal{V} I_n$. Therefore, $A^{-b}_b$ is the Moore-Penrose inverse of $A$.

Theorem 18
A balanced inverse of $A \in S^{n \times n}$ is the Moore-Penrose of $A$.

Proof. Let $A^{-b}_b$ be a balanced inverse of $A$. According to Definition 16, it satisfies $A \otimes A^{-b}_b \mathcal{V} I_n$ and $A^{-b}_b \otimes A \mathcal{V} I_n$. Consequently, it also satisfies

1. $A \otimes A^{-b}_b \mathcal{V} A I_n \otimes A = A$
2. $A^{-b}_b \otimes A \mathcal{V} I_n \otimes A^{-b}_b = A^{-b}_b$
3. $(A \otimes A^{-b}_b)^T \mathcal{V} (I_n)^T = I_n \mathcal{V} A \otimes A^{-b}_b$
4. $(A^{-b}_b \otimes A)^T \mathcal{V} (I_n)^T = I_n \mathcal{V} A^{-b}_b \otimes A$.

According to Example 17, the balanced inverse matrix $A^{-b}_b = \begin{bmatrix} -1 \\ \Theta(-3) \\ -4 \end{bmatrix}$ is the Moore-Penrose inverse of $A$.

Corollary 19
An inverse matrix of $A \in (\mathbb{R}^{\max})^{n \times n}$ is the Moore-Penrose of $A$.

Proof. Let $A^{-b}_b$ be an inverse of $A$ in max-plus algebra sense. It satisfies $A \otimes A^{-b}_b \mathcal{V} I_n$ and $A^{-b}_b \otimes A \mathcal{V} I_n$. Consequently, it also satisfies

1. $A \otimes A^{-b}_b \mathcal{V} I_n \otimes A = A$
2. $A^{-b}_b \otimes A \mathcal{V} I_n \otimes A^{-b}_b = A^{-b}_b$
3. $(A \otimes A^{-b}_b)^T \mathcal{V} (I_n)^T = I_n \mathcal{V} A \otimes A^{-b}_b$
4. $(A^{-b}_b \otimes A)^T \mathcal{V} (I_n)^T = I_n \mathcal{V} A^{-b}_b \otimes A$.

Therefore, $A^{-b}_b$ is the Moore-Penrose inverse of $A$.

5. CONCLUSION

The existence of the Moore-Penrose inverse in symmetrized max-plus algebra can be determined using a link between symmetrized plus algebra and conventional algebra. The Moore-Penrose inverse in symmetrized max-plus algebra can be defined as that in conventional algebra by replacing “equal” relation by “balance” relation. The balanced inverse is the Moore-Penrose inverse in symmetrized max-plus algebra.

The future research potentially can be done in construction Moore-Penrose inverse using matrix decomposition in symmetrized max-plus algebra.

AUTHORS’ CONTRIBUTIONS

S is a researcher whose research object is symmetrized max-plus algebra and the main researcher in this study. NI and R contributed to drafting and editing the manuscript.

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