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Developing Mathematical Conceptual Understanding through Problem-Solving: The Role of Abstraction Reflective

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***Abstract*---Reflective abstraction is a mechanism that moves individuals from one level to a higher level of knowledge. Reflective abstraction is a mechanism that builds novelty. Therefore, the study of reflective abstraction is dominant in the process of how reflective abstraction forms new knowledge or understanding. For example, Piaget, Dubinsky, David Tall, Mitchelmore, are some researchers who focus on the process of reflective abstraction in concept formation. The resulting mastery of the students' concepts played a lot in the problem-solving process. A good understanding of concepts, students will be able to reason, comprehend, operate, and connect the mathematics idea that will play a role in problem-solving. However, when students have to solve assignments or problems that are not routine, this problem-solving process also contributes to the development of understanding mathematical concepts. The problem-solving process will result in understanding a new concept if there is a reflective abrasion in it. This paper is the result of a literature review that will describe the role of reflective abstraction in problem-solving so that students can get new concepts.**

Keywords: reflective abstraction, conceptual understanding, problem-solving.

I. INTRODUCTION

One of Piaget's phenomenal works is Genetic Psychology which talks about what knowledge consists of and how knowledge develops. Assimilation and accommodation are the keywords in the process of cognitive development. Piaget believed that assimilation and accommodation occur naturally and that the development of cognition is driven by a tilted process towards equilibration between assimilation and accommodation.

How a person constructs a new cognitive structure from a pre-existing structure is described in reflective abstraction which consists of two phases, namely *reflechissement* (projecting a structure at a lower level to a higher level) and reflexion (rearranging a higher structure). 1]. This may be the first study of reflective abstraction and is a very important part of how mathematical knowledge is formed. Beth & Piaget explicitly states that reflective abstraction is very important for the development of advanced mathematical concepts because mathematical constructs are processed through reflective abstraction [2]. Dubinsky also stated that mathematics is a product of reflective abstraction [3]. Simon et.al stated that reflective abstraction is also a method that supports and animates large buildings of mathematical logic construction [4]. Arnon et. al. emphasized that reflective abstraction is concerned with the extraction of basic structures by considering the relationship between actions or actions, and is a mental mechanism where all mathematical logic structures are developed in the thinking of an individual [5].

Piaget's study of reflective abstraction was continued by Dubinsky who explained the mental mechanism as a reflective abstraction in the formation of mental structures [5]. Also, Dreyfus describes the processes of representation, generalization, and synthesis required in reflective abstraction [6]. Meanwhile, according to Hershkowitz, the abstraction process occurs through recognition, building-with, and construction [7]. The three studies form a new family in the study of reflective abstraction.

However, reflective abstraction as a means of developing cognition does not occur only in the formation or understanding of concepts. Conversely, with a proper reflective abstraction concept understanding can develop in the problem-solving process.

II. THEORITICAL BACKGROUND

REFLECTIVE ABSTRACTION

Reflective abstraction is one of the three types of abstraction mentioned by Piaget. The other two are empirical abstraction and pseudo empirical abstraction. Compared to the other two types, reflective abstraction is a type of abstraction that is closely related to mathematical knowledge. According to Piaget (1980) reflective abstraction is a general coordination of actions, and reflective abstraction takes place entirely internally [8]. This type of abstraction leads to constructive generalizations and results in a new synthesis which Damerow calls a feature by which the level of intelligence has increased [9]. Thus the result of reflective abstraction - in Piaget's paradigm - is the logical structure of mathematics that specifically distinguishes human thought from previous forms of intelligence.

The reflective abstraction process involves two inseparable elements, namely refechissement and reflexion. Reflechissement is a projection of something borrowed from a previous level to a higher level, and reflexion is an awareness of cognitive reconstruction or reorganization of what has been transferred. This two-component abstraction reflection can be observed at all stages, from sensory motor [10].

The process that is characterized by reflective abstraction is the process of constructing the structure. Thus, the emergence of reflective abstraction can be identified in the form of developmental psychology, in which reflective abstraction evokes a transition period from the sensory-motor intelligence stage to the concrete operation stage, or in all subsequent transitions in the development of intelligence. According to Piaget, the process of reflective abstraction takes place during cognitive development and does not have an absolute beginning, and has appeared since the earliest stages in motor sensory [2]. This process lasted until mathematics advanced and formed a history of the development of mathematics [8].

Piaget distinguished various types of constructs in reflective abstraction, namely interiorization, coordination, encapsulation, and generalization [1]. Meanwhile, Dreyfus (2002) states that abstraction requires a process of representation, generalization and synthesis [11]. Meanwhile, Hershkowitz et.al (2001) stated that the abstraction process occurs through the process of recognition, building-with, and construction [7]. This model is hereinafter known as the RBC model.

CONCEPTUAL UNDERSTANDING

Skemp stated that understanding something means assimilating it into a suitable schema [12]. Harel & Sowder stated that understanding mathematical activities refers to (1) certain interpretations or meanings of concepts, relationships between concepts, statements, or problems; (2) a particular solution offered by an individual to a problem; and (3) certain evidence offered by an individual to build or reject a mathematical statement [13].

As for concepts, Gray & Tall argues that there are at least three types of mathematical concepts, namely (1) concepts based on perceptions of objects, (2) concepts based on processes that are symbolized and understood as processes and objects (procept), and (3) a concept based on a set of properties acting as a concept definition for constructing axiomatic systems in advanced mathematical thinking [14]. Each of these concepts, according to Gray & Tall, is an abstraction, namely a mental image of an object received (for example a triangle), a mental process that becomes a concept (such as counting into numbers), and a formal system (such as a permutation group). which is based on its properties with a concept built through deductive logic [15].

The need for conceptual understanding in mathematics learning is emphasized by the National Mathematics Advisory Panel which states that learning mathematics requires three types of knowledge, namely factual, procedural and conceptual knowledge. NCTM also states that conceptual understanding is one of the five indicators of math proficiency. The other four indicators are problem-solving, reasoning, connection, representation and communication [16]. Operationally, indicators of understanding the concept are described in various versions. Engelbrecht, Harding & Potgier also stated that understanding operations and relationships is part of understanding concepts [17]. Concept understanding consists of relationships that are built internally and relate to pre-existing ideas; and it will be necessary when an individual identifies and applies principles, knows and applies facts and definitions, and compares and contrasts concepts.

The existence of a connection in conceptual understanding is also emphasized by Hiebert and Lefevre [18]. They describe conceptual understanding as knowledge that is rich in connectedness, so that all pieces of information are linked into some information. Hiebert and Lefevre also made a distinction between what is called the ground-level conceptual understanding relationship and what they call the reflective level. Basic level refers to pieces of knowledge that are at the same level of abstraction. The reflective level refers to the higher level of abstraction of two pieces of knowledge that were originally conceived as separate pieces of knowledge. The National Assessment of Educational Progress shows that there is a slice in the definition of conceptual understanding between those used by NCTM and those used by the National Research Council (NRC), namely that students have demonstrated understanding of mathematical concepts when they are proven to be able to (1) recognize, label, and generate examples of concepts, (2) using and interpreting various models, diagrams, manipulations and representations of concepts, (3) identifying and applying principles, (4) knowing and applying facts and definitions, (5) comparing, contrasting and integrating related concepts and principles, and (6) recognizing, interpreting the signs, symbols, and forms used to represent concepts. Meanwhile, the Mathematics Core Curriculum document issued by New York Education Development (NYED) states that conceptual understanding consists of relationships that are built internally and are connected to existing ideas. to this indicator slice used in this study. The indicators of conceptual understanding set out by NYED are identifying and applying principles, knowing and applying facts and definitions, and comparing and contrasting related concepts [19].

PROBLEM-SOLVING

In general, the mathematics curriculum differentiates assignments or questions given to students into the form of excercises and problems. Exercise is a question whose solution requires a routine procedure. Meanwhile, the problem is a question or assignment that is not an exercise. In other words, a problem is a question whose resolution process is not clear. However, a question cannot be separated into exercise or problem categories, because it depends on the child's ability. Training for one student may be a problem for another student. This is also conveyed by Stanic and Kilpatrick who define a problem as a condition in which a person does a task that was not found in the previous time [20] [21]. This means, a task is a problem or does not depend on the individual and time. So that a task is a problem for someone, but maybe not a problem for someone else. Likewise, a task is a problem for someone at one time, if that person already knows how or the process of getting a solution to the problem.

The characteristic that distinguishes between practice and problem is novelty, which has an impact on the need for creativity to answer. Some of the novelties that can arise in the problem are [20] [22]:

(1) novelty in problem formulation, so it requires careful interpretation

(2) novelty in the type of strategy for finding solutions to problems

(3) novelty of the concept used

This novelty is in line with the opinion of NCTM which states that problem-solving means involving students in tasks whose solving methods were not previously known [23].

In general, when researchers use the term problem-solving they refer to tasks that provide intellectual challenges that can encourage students' mathematical development. This task, which is a problem, can encourage conceptual understanding, reasoning and communication skills and capture their mathematical interest and curiosity [23] [16] [18]. Even according to badger, problem-solving is a student skill that will be most useful if they graduate [20].

REFLECTIVE ABSTRACTION IN PROBLEM-SOLVING

The reflective abstraction that occurs in problem-solving, Piaget hinted at when Piaget stated that when a problem is raised or confronted, the individual can go beyond the things that can be observed and put them into relationships, producing logico-mathematical knowledge or endogenous knowledge. That reflective abstraction occurs when there is a confrontation, this problem is related to the idea of equilibration from Piaget's constructivism theory. Equilibration itself is defined as a process where the subject tries to understand a concept by placing the concept in the context of the cognitive system as a whole [24].

Reflective abstraction is a linking mechanism in equilibration that moves the individual to a higher level, and is a mechanism that builds novelty [25]. This novelty is what distinguishes problem-solving from ordinary math practice questions. The novelty possessed by problem-solving problems includes novelty in terms of problem formulation, novelty in terms of solving strategies, or novelty of concepts discussed in the problem [20]. Therefore reflective abstraction will be more likely to occur when students work on problem-solving problems than practice questions.

Conjectures about the use of reflective abstraction in problem-solving were hypothesized by researchers in Geneva in 1983 who suggested that students might use reflective abstraction in problem-solving to explain the process of development [26]. In addition, Cohen also stated that reflective abstraction occurs when a new problem is confronted [25]. The discussion of reflective abstraction in problem-solving is further found by turning to Cohen [25] and Cifarelli [27].

In explaining the relationship between reflective abstraction and problem-solving, Cohen departed from the concept of equilibration, which is the means by which reflective abstraction emerges. Through equilibration, reflective abstraction is also a way of forming something new, be it relationships, links, or correspondences. There are six stages to bring up this reflective abstraction, namely encoding, conflict or contradiction, coordination, destructuring, and followed by two processes of reflective abstraction, reflecting and reflection.

Encoding is the process of identifying each element of the problem and taking attributes from long-term memory that are relevant to the solution. Individuals with good problem-solving abilities can be identified through attention at this stage. Encoding in problem-solving is an individual attempt to assimilate what can be observed into various schemes.

Conflict or Contradiction occurs when when building a basic information some elements cannot be assimilated into an existing structure so that there is a discrepancy. The individual will experience disturbances, gaps, or other types of contradiction. Coordination is an effort to unite elements from various schemes or structures that can bear problems in order to further build relationships between (among) and between them.

Destructuring. In order for an individual to place new constructs into an existing system, the individual must break the system back to the decision-making node. This destruction doesn't happen entirely, it's just sort of unzipping the structure to the knot where the problem started and starting to rebuild the structure of the knot.

Reflecting and reflection is the last stage which is the hallmark of the reflective abstraction process. Reflecting and reflection is the translation of reflechissement and reflexion in French, where Piaget wrote his theory. Reflecting or projection is the process of projecting a structure at a lower level of development to a higher level, while reflection is the process of rearranging structures at a higher level. In reflection, there is an integration between the old structures [1].

Meanwhile, Cifarelli considers what is meant by reflective abstraction in problem-solving is about how to construct knowledge through problem-solving. Every student who solves a problem will try to recall the knowledge they already have. This is a reflection process, according to Cifarelli. The potential outcome for this reflective activity is that students achieve a deeper understanding of their previous activities.

Similar to Cohen, Cifarelli no longer looked at whether students could find out the structure of the problem or not, but rather the students' responses, how the students interpreted and interpreted the problem. The difference with Cohen, if Cohen is more focused and detailed in describing the process of reflective abstraction occurs when students solve problems, Cifarelli looks more at the awareness and anticipation made by students.

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