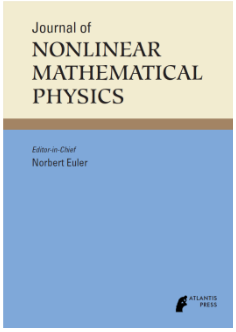

Journal of Nonlinear Mathematical Physics

ISSN (Online): 1776-0852 ISSN (Print): 1402-9251
Journal Home Page: https://www.atlantis-press.com/journals/jnmp

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Article

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Journal of Nonlinear Mathematical Physics, Vol. 19, No. 1 (2012) 1250001 (15 pages)

DOI: 10.1142/S1402925112500015

THE AKNS HIERARCHY REVISITED: A VERTEX OPERATOR APPROACH AND ITS LIE-ALGEBRAIC STRUCTURE

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Received 22 August 2011
Accepted 23 September 2011

A novel approach — based upon vertex operator representation — is devised to study the AKNS hierarchy. It is shown that this method reveals the remarkable properties of the AKNS hierarchy in relatively simple, rather natural and particularly effective ways. In addition, the connection of this vertex operator based approach with Lie-algebraic integrability schemes is analyzed and its relationship with τ-function representations is briefly discussed.

Keywords: AKNS hierarchy; Lax integrability; Lie-algebraic approach; vertex operators.

Mathematics Subject Classification: 37K10, 17B69, 17B80

1. Introduction

The "miraculous" properties of the AKNS hierarchy related to calculations connected with the integrability of nonlinear dynamical systems have, since the early work of their discoverers [1, 20], been the focus of considerable research. These investigations, such as in [8, 9, 11, 14, 15, 19, 21, 24, 26], have produced further insights into the nature of the AKNS hierarchy and several additional methods of construction. In what follows, we devise an alternative approach to exploring the properties of the AKNS hierarchy based upon its generating vector field form and related vertex operator representation. It appears that our formulation offers several advantages over existing methods when it comes to simplicity, effectiveness, flexibility and ease of extension, but more detailed confirmation of these observations must await further investigations.

To set the stage for our approach, we begin with some fundamentals of the remarkable sequence of Lax integrable dynamical systems that is the focus of this study. We shall analyze the AKNS hierarchy of Lax integrable dynamical systems on a complex 2π-periodic functional manifold $M \subset C^\infty(\mathbb{R}/2\pi\mathbb{Z};\mathbb{C}^2)$, which is well known [1, 14, 20, 21] to be related to the following linear differential spectral problem of Lax type:

$$\frac{df}{dx} - \ell(x;\lambda)f = 0, \quad \ell(x;\lambda) := \begin{pmatrix} \lambda/2 & u \\ v & -\lambda/2 \end{pmatrix}. \tag{1.1}$$

Here $x \in \mathbb{R}, f \in L^1(\mathbb{R};\mathbb{C}^2)$, the vector function $(u,v)^\intercal \subset M$, $\top$ denotes the transpose and $\lambda \in \mathbb{C}$ is a spectral parameter. Assume that a vector function $(u,v)^\intercal \subset M$ depends parametrically on the infinite set $t := (t_1, t_2, t_3, \ldots) \in \mathbb{C}^\mathbb{N}$ in such a way that the generalized Floquet spectrum $\sigma(\ell) := \{\lambda \in \mathbb{C} : \sup_{x\in\mathbb{R}} \|f(x;\lambda)\|_1 < \infty\}$ of the problem (1.1) persists in being parametrically iso-spectral, that is $d\sigma(\ell)/dt = 0$. The iso-spectral condition gives rise to the AKNS hierarchy of nonlinear dynamical systems on the functional manifold M in the general form

$$\frac{d}{dt_j}(u(t), v(t))^\intercal = K_j[u(t), v(t)], \tag{1.2}$$

where

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} := \begin{pmatrix} u(x+t_1, t_2, t_3, \ldots) \\ v(x+t_1, t_2, t_3, \ldots) \end{pmatrix} \tag{1.3}$$

for $t \in \mathbb{C}^\mathbb{N}$. The corresponding vector fields $K_j : M \to T(M), j \in \mathbb{N}$, can be constructed [7, 14, 17, 20, 24, 26] via the following Lie-algebraic scheme.

We define the centrally extended affine current $\mathfrak{sl}(2)$- algebra $\hat{\mathcal{G}} := \tilde{\mathcal{G}} \oplus \mathbb{C}$

$$\tilde{\mathcal{G}} := \left\{ a = \sum_{j\in\mathbb{Z},\, j\ll\infty} a^{(j)} \otimes \lambda^j : a^{(j)} \in C^\infty\left(\mathbb{R}/2\pi\mathbb{Z}; \mathfrak{sl}(2;\mathbb{C})\right) \right\}, \tag{1.4}$$

endowed with the Lie commutator

$$[(a_1, c_1), (a_2, c_2)] := ([a_1, a_2], \langle a_1, da_2/dx\rangle) \tag{1.5}$$

with the scalar product

$$\langle a_1, a_2\rangle := \mathrm{res}_{\lambda=\infty} \int_0^{2\pi} \mathrm{tr}(a_1 a_2)dx \tag{1.6}$$

for any two elements $a_1, a_2 \in \tilde{\mathcal{G}}$, where "res" and "tr" are the usual residue and trace maps, respectively. As the spectrum $\sigma(\ell) \subset \mathbb{C}$ is supposed to be parametrically independent, there is a natural association with flows. These flows are generated by the set $I(\hat{\mathcal{G}}^*)$ of Casimir invariants of the coadjoint action of the current algebra $\hat{\mathcal{G}}$ on a given element $\ell(x;\lambda) \in \tilde{\mathcal{G}}_-^* \cong \tilde{\mathcal{G}}_+$ contained in the space $\mathcal{D}(\tilde{\mathcal{G}}^*)$ comprised of smooth functions of the form

$\mathrm{res}_{\lambda=\infty}\mathrm{tr} : \tilde{\mathcal{G}}^* \to \mathbb{C}$. Here we have denoted by $\tilde{\mathcal{G}} := \tilde{\mathcal{G}}_+ \oplus \tilde{\mathcal{G}}_-$ the natural splitting into two affine subalgebras of nonnegative and negative λ-expansions. In particular, a functional $\gamma(\lambda) \in I(\hat{\mathcal{G}})$ if and only if

$$[\tilde{S}(x;\lambda), \ell(x;\lambda)] + \frac{d}{dx}\tilde{S}(x;\lambda) = 0, \tag{1.7}$$

where the gradient $\tilde{S}(x;\lambda) := \mathrm{grad}\,\gamma(\lambda)(\ell) \in \tilde{\mathcal{G}}_-$ is defined with respect to the scalar product (1.6) by means of the variation

$$\delta\gamma(\lambda) := \langle \mathrm{grad}\,\gamma(\lambda)(\ell), \delta\ell \rangle. \tag{1.8}$$

We note here that the determining matrix equation (1.7) in the case of the element $\ell(x;\lambda) \in \tilde{\mathcal{G}}_-^*$, given by the spectral problem (1.1), can be easily solved recursively as $\lambda \to \infty$ in the following asymptotic form as

$$\begin{aligned}
&\tilde{S}(x;\lambda) \sim \sum_{j\in\mathbb{Z}_+} \tilde{S}^{(j)}(x)\lambda^{-(j+1)}, \quad \tilde{S}(x;\lambda) = \begin{pmatrix} \tilde{S}_{11} & \tilde{S}_{12} \\ \tilde{S}_{21} & \tilde{S}_{22} \end{pmatrix}, \\
&\tilde{S}^{(0)}(x) = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}, \quad \tilde{S}^{(1)}(x) = \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix}, \\
&\tilde{S}^{(2)}(x) = \begin{pmatrix} -uv & u_x \\ -v_x & vu \end{pmatrix}, \quad \tilde{S}^{(3)}(x) = \begin{pmatrix} vu_x - uv_x & u_{xx} - 2u^2v \\ v_{xx} - 2v^2u & uv_x - vu_x \end{pmatrix}, \dots,
\end{aligned} \tag{1.9}$$

and so on, based upon the differential relationships

$$\begin{aligned}
\lambda\tilde{S}_{12} &= \tilde{S}_{12,x} + u(\tilde{S}_{11} - \tilde{S}_{22}), \\
-\lambda\tilde{S}_{21} &= \tilde{S}_{21,x} - v(\tilde{S}_{11} - \tilde{S}_{22}), \\
\tilde{S}_{11,x} &= u\tilde{S}_{21} - v\tilde{S}_{12} = -\tilde{S}_{22,x},
\end{aligned} \tag{1.10}$$

following from (1.7).

Now we will take into account that the coadjoint orbits of elements $\ell \in \tilde{\mathcal{G}}_-^*$ with respect to the standard $\mathcal{R}$-structure [14] on the Lie algebra $\hat{\mathcal{G}}$

$$[(a_1, c_1), (a_2, c_2)]_{\mathcal{R}} := ([\mathcal{R}a_1, a_2] + [a_1, \mathcal{R}a_2], \langle \mathcal{R}a_1, da_2/dx \rangle - \langle da_1/dx, \mathcal{R}a_2 \rangle) \tag{1.11}$$

where, by definition, $\mathcal{R} := \frac{1}{2}(P_+ - P_-)$ and $P_\pm\tilde{\mathcal{G}} := \tilde{\mathcal{G}}_\pm$, are Poissonian manifolds [2, 3, 7, 14, 18, 24, 26]. Then the corresponding *a priori* iso-spectral AKNS flows (1.2) can be constructed as the commuting Hamiltonian systems on $\tilde{\mathcal{G}}_-^*$

$$\frac{d\ell}{dt_j} := \{\gamma_j, \ell\} = [(\lambda^{j+1}\tilde{S})_+, \ell] + \frac{d}{dx}(\lambda^{j+1}\tilde{S})_+ \tag{1.12}$$

generated by the Casimir invariants $\gamma_j \in I(\hat{\mathcal{G}}^*), j \in \mathbb{N}$, with respect to the Lie–Poisson structure on $\hat{\mathcal{G}}^*$ defined as

$$\begin{aligned}\{\gamma, \xi\} &:= \langle \ell, [\operatorname{grad}\gamma(\ell), \operatorname{grad}\xi(\ell)]_{\mathcal{R}}\rangle \\ &\quad + \left\langle \mathcal{R}\operatorname{grad}\gamma(\ell), \frac{d}{dx}\operatorname{grad}\xi(\ell)\right\rangle - \left\langle \frac{d}{dx}\operatorname{grad}\gamma(\ell), \mathcal{R}\operatorname{grad}\xi(\ell)\right\rangle \end{aligned} \tag{1.13}$$

for any smooth functionals $\gamma, \xi \in \mathcal{D}(\hat{\mathcal{G}}^*)$. As a result of (1.12), Eq. (1.7) is easily augmented by the commuting hierarchy of evolution equations

$$\frac{d\tilde{S}}{dt_j} = [(\lambda^{j+1}\tilde{S})_+, \tilde{S}] \tag{1.14}$$

for $j \in \mathbb{N}$, including the determining Eq. (1.7) at $j = 1$.

The hierarchy (1.14) can be rewritten with respect to the unique λ-parametric vector field

$$\frac{d}{dt} := \sum_{j\in\mathbb{Z}_+} \lambda^{-j} d/dt_{j+1} \tag{1.15}$$

on the manifold M as the generating flow on $\tilde{\mathcal{G}}_-^*$:

$$\frac{d}{dt}\tilde{S}(x;\mu) = \left[\tilde{S}(x;\mu), \frac{\lambda^3}{\mu-\lambda}\tilde{S}(x;\lambda) + \lambda\tilde{S}_0(x)\right], \tag{1.16}$$

where $\mathbb{Z}_+ := \{0\} \cup \mathbb{N} = \{0, 1, \ldots\}$, and the parameters $\lambda, \mu \to \infty$ in such a way that $|\mu/\lambda| < 1$. We note that the description of $\lambda, \mu \to \infty$ here and $\lambda \to \infty$ in what follows can also prescribed in terms of formal series in μ/λ and $1/\lambda$, respectively. However, we have chosen to use the less elegant notation because we require actual convergence in our approach, it is simpler, and turns out to be equivalent in the context in which it is employed in the sequel. It should be mentioned that an operator similar to our vector field (1.15) was introduced in [12] as an algebraic operator.

Since the flow (1.12) is, by construction, Hamiltonian on the adjoint space $\tilde{\mathcal{G}}_-^*$, it can be represented also as a Hamiltonian flow on the functional manifold M in the form

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \lambda^2\tilde{S}_{12,x} + u\lambda^2(\tilde{S}_{11} - \tilde{S}_{22}) \\ \lambda^2\tilde{S}_{21,x} - v\lambda^2(\tilde{S}_{11} - \tilde{S}_{22}) \end{pmatrix}, \tag{1.17}$$

which establishes a connection with the Eq. (1.2) and will be important for our further analysis. The representation (1.17) will be derived in the next two sections with respect to both the evolution vector field (1.15) and the related vertex vector field mapping $X_\lambda : M \to M$ defined as

$$\begin{aligned} &X_\lambda := (X_\lambda^+, X_\lambda^-), \quad X_\lambda^+ = \exp D_\lambda, \quad X_\lambda^- = \exp(-D_\lambda), \\ &D_\lambda := \sum_{j\in\mathbb{Z}_+} \frac{1}{(j+1)}\lambda^{-(j+1)}\frac{d}{dt_{j+1}}, \end{aligned} \tag{1.18}$$

and satisfying the determining relationship

$$\frac{d}{dt} = \mp\lambda^2 X_\lambda^{\pm,-1}\frac{d}{d\lambda}X_\lambda^{\pm}, \tag{1.19}$$

as $\lambda \to \infty$. These vertex vector field maps and their connections with integrability theory have been studied extensively by a number of researchers, most notably in [11, 20].

2. Hamiltonian Analysis

Consider the Casimir functional $\gamma(\lambda) \in I(\hat{\mathcal{G}})$, $\lambda \in \mathbb{C}$, and its gradient with respect to its dependence on a point $(u,v)^\intercal \in M$ given by

$$\operatorname{grad}\gamma(\lambda)[u,v] = (\tilde{S}_{21}(x;\lambda), \tilde{S}_{12}(x;\lambda))^\intercal \in T^*(M), \tag{2.1}$$

as follows easily from definition (1.8). By introducing on the manifold M the following two skew-symmetric operators

$$\theta := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \eta := \begin{pmatrix} 2u\partial^{-1}u & \partial - 2u\partial^{-1}v \\ \partial - 2v\partial^{-1}u & 2v\partial^{-1}v \end{pmatrix}, \tag{2.2}$$

it follows directly [14, 24, 25] from (1.13) that the relationships (1.10) can be rewritten as

$$\lambda\theta \operatorname{grad}\gamma(\lambda)[u,v] = \eta \operatorname{grad}\gamma(\lambda)[u,v], \tag{2.3}$$

holding for all $\lambda \in \mathbb{C}$. It is easy to verify that owing to (2.3) the Casimir invariant $\gamma(\lambda) \in I(\hat{\mathcal{G}})$ simultaneously satisfies the two involutivity conditions

$$\{\gamma(\lambda), \gamma(\mu)\}_\theta = 0 = \{\gamma(\lambda), \gamma(\mu)\}_\eta \tag{2.4}$$

for all $\lambda, \mu \in \mathbb{C}$ with respect to two Poissonian structures

$$\{\cdot,\cdot\}_\theta := (\operatorname{grad}(\cdot), \theta \operatorname{grad}(\cdot)), \quad \{\cdot,\cdot\}_\eta := (\operatorname{grad}(\cdot), \eta \operatorname{grad}(\cdot)) \tag{2.5}$$

on the manifold M, where $(\cdot,\cdot)$ is the standard convolution on the product bundle $T^*(M) \times T(M)$. As a direct consequence of (2.3) and (2.8), one can readily verify that the operators $\theta, \eta : T^*(M) \to T(M)$, defined by (2.2), are co-symplectic, Nötherian and compatible [7, 17, 24] on M. This, in particular, implies that the Lie derivatives [2, 3, 7, 24]

$$L_{\frac{d}{dt}}\theta = 0 = L_{\frac{d}{dt}}\eta, \quad L_{\frac{d}{dt}}\operatorname{grad}\gamma(\lambda)[u,v] = 0 \tag{2.6}$$

vanish identically on the manifold M.

Taking now into account (1.12) and (2.2), one finds easily that

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = -\left\{\lambda^2\gamma(\lambda), \begin{pmatrix} u \\ v \end{pmatrix}\right\}_\eta = \lambda^2\eta \operatorname{grad}\gamma(\lambda)[u,v] = \begin{pmatrix} \lambda^2\tilde{S}_{12,x} + u\lambda^2(\tilde{S}_{11} - \tilde{S}_{22}) \\ \lambda^2\tilde{S}_{21,x} - v\lambda^2(\tilde{S}_{11} - \tilde{S}_{22}) \end{pmatrix} \tag{2.7}$$

asymptotically as $\lambda \to \infty$, proving the representation (1.17) mentioned in the Introduction. Making use of the expansion (1.9) one easily obtains from (2.7) the first flows of the AKNS hierarchy:

$$\frac{d}{dt_1}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u_x \\ v_x \end{pmatrix} = K_1[u,v], \quad \frac{d}{dt_2}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u_{xx} + 2u^2 v \\ -v_{xx} + 2v^2 u \end{pmatrix} = K_2[u,v], \ldots, \tag{2.8}$$

and so on. Below we will construct this infinite hierarchy of AKNS flows (1.2) by means of a very effective completely algebraic approach based on the vertex operator representation of the solution to the generating flow (2.7).

3. Vertex Operator Structure Analysis

It is well known [14, 17, 21, 24] that the Casimir invariants determining Eq. (1.7) allow general solution representations in the following two important forms:

$$\tilde{S}(x;\lambda) = k(\lambda)S(x;\lambda) - \frac{k(\lambda)}{2}\mathrm{tr}S(x;\lambda) \tag{3.1}$$

and

$$\tilde{S}(x;\lambda) = \tilde{F}(x,x_0;\lambda)\tilde{C}(x_0;\lambda)\tilde{F}^{-1}(x,x_0;\lambda). \tag{3.2}$$

Here $S(x;\lambda) := F(x+2\pi,x;\lambda)$, and $F(y,x;\lambda)$ and $\tilde{F}(y,x;\lambda)$ belong to the space of linear endomorphisms of $\mathbb{C}^2$, End $\mathbb{C}^2$, for all $x, x_0, y \in \mathbb{R}$, and are matrix solutions to the spectral Eq. (1.1) satisfying, respectively, the Cauchy problems

$$\frac{\partial}{\partial y}F(y,x;\lambda) = \ell(y;\lambda)F(y,x;\lambda), \quad F(y,x;\lambda)|_{y=x} = \mathbf{I}, \tag{3.3}$$

and

$$\begin{gathered}\frac{\partial}{\partial y}\tilde{F}(y,x;\lambda) = \ell(y;\lambda)\tilde{F}(y,x;\lambda), \quad \tilde{F}(y,x;\lambda)|_{y=x} = \mathbf{I} + O(1/\lambda), \\ (\tilde{F}(y,x;\lambda)|_{y=x} - \mathbf{I} \neq 0)\end{gathered} \tag{3.4}$$

for all $\lambda \in \mathbb{C}$ and $x \in \mathbb{R}$, where $\mathbf{I} \in$ End $\mathbb{C}^2$ is the identity matrix. The parameters $k(\lambda) \in \mathbb{C}$ and $\tilde{C}(x_0;\lambda) \in$ End $\mathbb{C}^2$ are invariant with respect to the generating vector field (1.15), and chosen in such a way that the asymptotic condition

$$\tilde{S}(x;\lambda) \in \tilde{\mathcal{G}}_- \tag{3.5}$$

holds as $\lambda \to \infty$ for all $x \in \mathbb{R}$.

To construct the solution (3.1) satisfying condition (3.5), we find a preliminary partial solution $\tilde{F}(y,x;\lambda) \in$ End $\mathbb{C}^2$, $x, y \in \mathbb{R}$, to Eq. (3.4) satisfying the asymptotic Cauchy data

$$\tilde{F}(y,x;\lambda)|_{y=x} = \mathbf{I} + O(1/\lambda) \tag{3.6}$$

as $\lambda \to \infty$. It is easy to check that

$$\tilde{F}(y,x;\lambda) = \begin{pmatrix} \tilde{e}_1(y,x;\lambda) & -\tilde{u}(y;\lambda)\lambda^{-1}\tilde{e}_2(y,x;\lambda) \\ \tilde{v}(y;\lambda)\lambda^{-1}\tilde{e}_1(y,x;\lambda) & \tilde{e}_2(y,x;\lambda) \end{pmatrix} \tag{3.7}$$

is an exact functional solution to (3.4) satisfying condition (3.6). Here we have defined

$$\begin{aligned}\tilde{e}_1(y,x;\lambda) &:= \exp\left\{(y-x)\lambda/2 + \lambda^{-1}\int_x^y u\tilde{v}ds\right\},\\ \tilde{e}_2(y,x;\lambda) &:= \exp\left\{(x-y)\lambda/2 - \lambda^{-1}\int_x^y \tilde{u}vds\right\},\end{aligned} \tag{3.8}$$

where the vector-function $(\tilde{u},\tilde{v})^\intercal \in M$ satisfies the determining functional relationships

$$\tilde{u} = u + \tilde{u}_x\lambda^{-1} - \tilde{u}^2 v\lambda^{-2}, \quad \tilde{v} = v - \tilde{v}_x\lambda^{-1} - \tilde{v}^2 u\lambda^{-2}, \tag{3.9}$$

as $\lambda \to \infty$, which were discovered earlier in a very interesting article [22]. There it was also shown that exact asymptotic (as $\lambda \to \infty$) functional solutions of these relationships can be easily constructed by means of the standard iteration procedure.

The fundamental matrix $F(y,x;\lambda) \in \operatorname{End}\mathbb{C}^2$ is represented for all $x,y \in \mathbb{R}$ in the form

$$F(y,x;\lambda) = \tilde{F}(y,x;\lambda)\tilde{F}^{-1}(x,x;\lambda). \tag{3.10}$$

Consequently, if one sets $y = x + 2\pi$ in this formula and defines

$$k(\lambda) := \lambda^{-1}[\tilde{e}_1(x+2\pi,x;\lambda) - \tilde{e}_2(x+2\pi,x;\lambda)]^{-1}, \tag{3.11}$$

it follows from (3.10) that the exact matrix representation

$$\tilde{S}(x;\lambda) = \begin{pmatrix} \dfrac{\lambda^2 - \tilde{u}\tilde{v}}{2\lambda(\lambda^2+\tilde{u}\tilde{v})} & \dfrac{\tilde{u}}{\lambda^2+\tilde{u}\tilde{v}} \\ \dfrac{\tilde{v}}{\lambda^2+\tilde{u}\tilde{v}} & \dfrac{\tilde{u}\tilde{v}-\lambda^2}{2\lambda(\lambda^2+\tilde{u}\tilde{v})} \end{pmatrix} \tag{3.12}$$

satisfies the necessary condition (3.5) as $\lambda \to \infty$.

Remark 3.1. The invariance of the functional (3.11) with respect to the generating vector field (1.15) on the manifold M derives from the representation (3.7), the evolution equations (3.4) and

$$\frac{d}{dt}\tilde{F}(y,x;\mu) = \left(\frac{\lambda^3}{\mu-\lambda}\tilde{S}(x;\lambda) + \lambda\tilde{S}_0(x)\right)\tilde{F}(y,x;\mu), \tag{3.13}$$

which follows naturally from the determining matrix flows (1.12) upon applying the translation $y \to y + 2\pi$.

The matrix expression (3.12) coincides as $\lambda \to \infty$ with the asymptotic expansion (1.9), whose matrix elements satisfy the following important functional relationships:

$$\frac{1-\lambda(\tilde{S}_{11}-\tilde{S}_{22})}{2\tilde{S}_{21}} = \tilde{u}, \quad \frac{1-\lambda(\tilde{S}_{11}-\tilde{S}_{22})}{2\tilde{S}_{12}} = \tilde{v}, \tag{3.14}$$

allowing the introduction in a natural way of the vertex vector field (1.18). To show this, we need to take the preliminary step of deriving the corresponding evolution equation for the vector function $(\tilde{u},\tilde{v})^\intercal \in M$ with respect to the generating vector field (1.15) in the asymptotic form (1.16) as $\lambda \to \infty$. Before doing this we shall find the form of the evolution Eq. (1.17) as $\mu, \lambda \to \infty$:

$$\frac{d}{dt}\tilde{S}(x;\mu) = \left[\lambda^3 \frac{d}{d\lambda}\tilde{S}(x;\mu) - \lambda \tilde{S}_0(x), \tilde{S}(x;\lambda)\right], \tag{3.15}$$

which entails the following differential relationships:

$$\begin{aligned}
\frac{d\tilde{S}_{11}}{dt} &= \lambda^3\left(\frac{\tilde{S}_{21}d\tilde{S}_{12}}{d\lambda} - \frac{\tilde{S}_{12}d\tilde{S}_{21}}{d\lambda}\right),\\
\frac{d\tilde{S}_{22}}{dt} &= \lambda^3\left(\frac{\tilde{S}_{12}d\tilde{S}_{21}}{d\lambda} - \frac{\tilde{S}_{21}d\tilde{S}_{12}}{d\lambda}\right),\\
\frac{d\tilde{S}_{12}}{dt} &= \lambda^3\left(\tilde{S}_{12}\frac{d}{d\lambda}(\tilde{S}_{11} - \tilde{S}_{22}) - (\tilde{S}_{11} - \tilde{S}_{22})\frac{d\tilde{S}_{12}}{d\lambda}\right) - \lambda\tilde{S}_{12},\\
\frac{d\tilde{S}_{21}}{dt} &= \lambda^3\left(\tilde{S}_{21}\frac{d}{d\lambda}(\tilde{S}_{22} - \tilde{S}_{11}) - (\tilde{S}_{22} - \tilde{S}_{11})\frac{d\tilde{S}_{21}}{d\lambda}\right) + \lambda\tilde{S}_{21}.
\end{aligned} \tag{3.16}$$

Using the relationships (3.16), one can easily obtain by means of simple, but rather cumbersome, calculations the evolution equations for the vector function $(\tilde{u}, \tilde{v})^\intercal \in M$ expressed in the form (3.14)

$$\begin{aligned}
\frac{d}{dt}\left[\frac{1 - \lambda(\tilde{S}_{11} - \tilde{S}_{22})}{2\tilde{S}_{21}}\right] &= -\lambda^2\frac{d}{d\lambda}\left[\frac{1 - \lambda(\tilde{S}_{11} - \tilde{S}_{22})}{2\tilde{S}_{21}}\right],\\
\frac{d}{dt}\left[\frac{1 - \lambda(\tilde{S}_{11} - \tilde{S}_{22})}{2\tilde{S}_{12}}\right] &= \lambda^2\frac{d}{d\lambda}\left[\frac{1 - \lambda(\tilde{S}_{11} - \tilde{S}_{22})}{2\tilde{S}_{12}}\right],
\end{aligned} \tag{3.17}$$

which hold as $\lambda \to \infty$. As a direct consequence of the differential relationships (3.17), the following *vertex operator representation* for the vector function $(\tilde{u}, \tilde{v})^\intercal \in M$

$$\begin{aligned}
\tilde{u}(t;\lambda) &:= u^+(t;\lambda) = X_\lambda^+ u(t),\\
\tilde{v}(t;\lambda) &:= v^-(t;\lambda) = X_\lambda^- u(t),
\end{aligned} \tag{3.18}$$

holds. Here we took into account that, owing to the determining functional representations (3.9), the limits

$$\lim_{\lambda\to\infty} \tilde{u}(t;\lambda) = u(t), \quad \lim_{\lambda\to\infty} \tilde{v}(t;\lambda) = v(t), \tag{3.19}$$

exist and the vertex operator $X_\lambda : M \to M$ acts on the functional manifold M via the corresponding shift operators defined above by means of the differential relationships (1.18) and (1.19). Moreover, from (3.9) one obtains that

$$u^+ = u + u_x^+\lambda^{-1} - (u^+)^2 v\lambda^{-2}, \quad v^- = v - v_x^-\lambda^{-1} - (v^-)^2 u\lambda^{-2}, \tag{3.20}$$

The vertex representation (3.20) allows, in particular, to readily construct infinite hierarchies of the conservation laws for the generating AKNS integrable vector field (1.15) as

$$H_+(\lambda) := \int_0^{2\pi} u^+(t;\lambda)v(t)dx, \quad H_-(\lambda) := \int_0^{2\pi} v^-(t;\lambda)u(t)dx, \tag{3.21}$$

which follow from (3.7), (3.8) and reasoning from Remark (3.1). Since the fundamental matrix (3.10) at $y = x + 2\pi$ defines via relationship (3.1) the solution

$$S(x;\lambda) := \tilde{F}(x+2\pi, x;\lambda)\tilde{F}^{-1}(x,x;\lambda) \tag{3.22}$$

to the determining Eqs. (1.7) and (1.10), its determinant $\det S(x;\lambda)$ is invariant with respect to the generating vector field (1.15) and equals $\det S(x;\lambda) = \det \tilde{F}(x + 2\pi, x;\lambda) \det \tilde{F}^{-1}(x,x;\lambda) = 1$ for all $x \in \mathbb{R}$ and $\lambda \in \mathbb{C}$ owing to the condition $\operatorname{tr} \ell(x;\lambda) = 0$. Accordingly, based on the matrix representation (3.7), one finds that the relationships

$$\begin{gathered} \tilde{e}_1(x+2\pi, x;\lambda) := \exp[\pi\lambda + \lambda^{-1}H_+(\lambda)], \\ \tilde{e}_2(x+2\pi, x;\lambda) := \exp[-\pi\lambda - \lambda^{-1}H_-(\lambda)], \\ \tilde{e}_1(x+2\pi, x;\lambda)\tilde{e}_2(x+2\pi, x;\lambda) = 1, \\ \frac{d}{dt}\tilde{e}_1(x+2\pi, x;\lambda) = 0 = \frac{d}{dt}\tilde{e}_2(x+2\pi, x;\lambda) \end{gathered} \tag{3.23}$$

hold for all $x \in \mathbb{R}$ and $\lambda \in \mathbb{C}$. As a consequence of (3.23), we obtain

$$H_+(\lambda) = H_-(\lambda) \tag{3.24}$$

for all $\lambda \in \mathbb{C}$; that is, the two hierarchies of conservation laws (3.21) coincide. Concerning the AKNS hierarchy vector fields (1.15) and the related Hamiltonian flows on the manifold M, we can easily derive them from the canonical vertex representations (3.18), taking into account the recursive functional Eq. (3.9). We obtain from that (3.9) and (3.21) that

$$\begin{aligned} X_\lambda^+ u = u^+ &= u + \lambda^{-1}u_x + \lambda^{-2}[u_{xx}^+ + (u^+)^2 v] + \lambda^{-3}[(u^+)^2 v]_x = \cdots, \\ X_\lambda^- v = v^- &= v - \lambda^{-1}v_x - \lambda^{-2}[v_{xx}^- + (v^-)^2 u] + \lambda^{-3}[(v^-)^2 u]_x = \cdots, \end{aligned} \tag{3.25}$$

which immediately yield the whole AKNS hierarchy of nonlinear Lax integrable dynamical systems on the functional manifold M. For instance, we obtain from (3.25) the AKNS flows

$$\frac{d}{dt_1}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u_x \\ v_x \end{pmatrix}, \quad \frac{d}{dt_2}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u_{xx} + 2u^2 v \\ -v_{xx} + 2v^2 u \end{pmatrix}, \ldots, \tag{3.26}$$

and so on, which coincide with those constructed before in (2.8).

4. The τ-function Representation

The vertex operator representations (3.7), (3.12) and (3.18) can also be naturally associated with the results in [11, 20], based on the generating τ-function approach. This makes extensive use of the versatile dual representation (3.2) for the generating current algebra element $\tilde{S}(x;\lambda) \in \tilde{\mathcal{G}}_-^*$ (as $\lambda \to \infty$) for the AKNS flows with the specially chosen invariant matrix

$$\tilde{C}(x_0;\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \operatorname{End}\mathbb{C}^2. \tag{4.1}$$

In the context of our approach, the relation with the τ-function representation devised in [11, 20] can be based on the matrix solution (3.7) and the simple vertex operator mapping properties

$$X_\lambda \begin{pmatrix} \tilde{e}_1(x,y;\lambda) \\ \tilde{e}_2(x,y;\lambda) \end{pmatrix} = \begin{pmatrix} \tilde{e}_2(y,x;\lambda) \\ \tilde{e}_1(y,x;\lambda) \end{pmatrix}, \tag{4.2}$$

which follow directly from the definitions (3.8) and (3.18). From the functional relationships (3.8) and (3.20), one easily obtains the following differential expressions:

$$\tilde{F}(y,x;\lambda) = \begin{pmatrix} \tilde{e}_1(y,x;\lambda) & -u^+(y;\lambda)\lambda^{-1}\tilde{e}_2(y,x;\lambda) \\ v^-(y;\lambda)\lambda^{-1}\tilde{e}_1(y,x;\lambda) & \tilde{e}_2(y,x;\lambda) \end{pmatrix} \tag{4.3}$$

and

$$\lambda(u^+v^+ - u^-v^-) = \frac{\partial}{\partial y}(u^+v + uv^-), \tag{4.4}$$

where

$$\begin{aligned} \tilde{e}_1(y,x;\lambda) &:= \exp\left\{\frac{(y-x)\lambda}{2} + \lambda^{-1}\int_x^y uv^- ds\right\}, \\ \tilde{e}_2(y,x;\lambda) &:= \exp\left\{\frac{(x-y)\lambda}{2} - \lambda^{-1}\int_x^y u^+ v ds\right\}. \end{aligned} \tag{4.5}$$

Moreover, it follows directly from (4.5) that

$$\lambda\frac{\partial^2}{\partial y^2}\ln\frac{\tilde{e}_1(y,x;\lambda)}{\tilde{e}_2(y,x;\lambda)} = \frac{\partial}{\partial y}(u^+v + uv^-), \tag{4.6}$$

which together with (4.2) leads to the important functional relationship

$$\frac{\partial^2}{\partial y^2}\ln\frac{\tilde{e}_1(y,x;\lambda)}{\tilde{e}_2(y,x;\lambda)} = (u^+v^+ - u^-v^-), \tag{4.7}$$

which allows a natural introduction of a τ-function representation

$$\frac{\tilde{e}_1(y,x;\lambda)}{\tilde{e}_2(y,x;\lambda)} := \frac{\tau^-(y,x;\lambda)}{\tau^+(y,x;\lambda)}\exp[\alpha(x;\lambda) + y\beta(x;\lambda)] \tag{4.8}$$

for some functions $\alpha(\cdot;\lambda), \beta(\cdot;\lambda) : \mathbb{R} \to \mathbb{C}$, where we have defined τ by

$$-\frac{\partial^2}{\partial y^2}\ln\tau := uv, \tag{4.9}$$

which coincides with that in [19, 20]. Taking now into account the relationships (4.2), one easily obtains that

$$\begin{aligned}\tilde{e}_1(y,x;\lambda) &= \exp\left[\alpha(x;\lambda) + \frac{\lambda}{2}(y-x)\right]\frac{\tau^-(y,x;\lambda)}{\tau(y,x;\lambda)},\\ \tilde{e}_2(y,x;\lambda) &= \exp\left[\alpha(x;\lambda) + \frac{\lambda}{2}(x-y)\right]\frac{\tau^+(y,x;\lambda)}{\tau(y,x;\lambda)}\end{aligned} \tag{4.10}$$

for all $x, y \in \mathbb{R}$ and $\lambda \in \mathbb{C}$.

As a result of (3.7) and (4.10), it is straightforward to obtain the crucial expression for the normalized matrix

$$\begin{aligned}\bar{F}(y,x;\lambda) &:= (\det \tilde{F}(x,x;\lambda))^{-1/2}\tilde{F}(y,x;\lambda)\\ &= \begin{pmatrix} \dfrac{\lambda\tilde{e}_2^-(x,y;\lambda)}{[\lambda^2 + u^+(x;\lambda)v^-(x;\lambda)]^{1/2}} & -\dfrac{u^+(y;\lambda)\tilde{e}_1^+(x,y;\lambda)}{[\lambda^2 + u^+(x;\lambda)v^-(x;\lambda)]^{1/2}} \\ \dfrac{v^-(y;\lambda)\tilde{e}_2^+(x,y;\lambda)}{[\lambda^2 + u^+(x;\lambda)v^-(x;\lambda)]^{1/2}} & \dfrac{\lambda\tilde{e}_1^+(x,y;\lambda)}{[\lambda^2 + u^+(x;\lambda)v^-(x;\lambda)]^{1/2}} \end{pmatrix}\\ &= \begin{pmatrix} \dfrac{\tau^-(y,x;\lambda)}{\tau(y,x;\lambda)}\exp\left[\dfrac{\lambda}{2}(y-x)\right] & -\dfrac{u^+(y;\lambda)\tau^+(y,x;\lambda)}{\lambda\tau(y,x;\lambda)}\exp\left[\dfrac{\lambda}{2}(x-y)\right] \\ \dfrac{v^-(y;\lambda)\tau^-(y,x;\lambda)}{\lambda\tau(y,x;\lambda)}\exp\left[\dfrac{\lambda}{2}(y-x)\right] & \dfrac{\tau^+(y,x;\lambda)}{\tau(y,x;\lambda)}\exp\left[\dfrac{\lambda}{2}(x-y)\right] \end{pmatrix},\end{aligned} \tag{4.11}$$

where

$$\begin{aligned}\frac{\tau^-(y,x;\lambda)}{\tau(y,x;\lambda)}\exp\left[\frac{\lambda}{2}(y-x)\right] &:= \frac{\lambda\tilde{e}_2^-(x,y;\lambda)}{[\lambda^2 + u^+(x;\lambda)v^-(x;\lambda)]^{1/2}},\\ \frac{\tau^+(y,x;\lambda)}{\tau(y,x;\lambda)}\exp\left[\frac{\lambda}{2}(x-y)\right] &:= \frac{\lambda\tilde{e}_1^+(x,y;\lambda)}{[\lambda^2 + u^+(x;\lambda)v^-(x;\lambda)]^{1/2}},\end{aligned} \tag{4.12}$$

together with the compatibility relationship

$$\exp[-\alpha(x;\lambda)] := [\lambda^2 + u^+(x;\lambda)v^-(x;\lambda)]^{1/2}.$$

The vertex operator expression (4.11), as is easily checked, can be employed to derive the representation (3.2), where the exact result (3.12) entails the additional application of the useful [20] vertex representation

$$\bar{F}(y,x;\lambda) = \begin{pmatrix} \dfrac{\tau^-(y,x;\lambda)}{\tau(y,x;\lambda)}\exp\left[\dfrac{\lambda}{2}(y-x)\right] & -\dfrac{\sigma^+(y,x;\lambda)}{\lambda\tau(y,x;\lambda)}\exp\left[\dfrac{\lambda}{2}(x-y)\right] \\ \dfrac{\rho^-(y,x;\lambda)}{\lambda\tau(y,x;\lambda)}\exp\left[\dfrac{\lambda}{2}(y-x)\right] & \dfrac{\tau^+(y,x;\lambda)}{\tau(y,x;\lambda)}\exp\left[\dfrac{\lambda}{2}(x-y)\right] \end{pmatrix}, \tag{4.13}$$

which holds as $\lambda \to \infty$ if $\rho(y,x;\lambda) := v(y)\tau(y,x;\lambda), \sigma(y,x;\lambda) := u(y)\tau(y,x;\lambda),\ x, y \in \mathbb{R}$, and the mappings ρ^- and σ^+ are defined in the obvious fashion. In this regard, it should be noted that the vertex operator representation (4.13) for the matrix (4.11) was obtained in [20] as a special normalized solution to the determining Eq. (3.4). Taking into account these dual vertex representations of the AKNS hierarchy of integrable flows on the functional manifold M, one can see that the first one — presented in this work — is both technically simpler and more effective in obtaining exact descriptions of such important functional ingredients as conservation laws, symplectic structures and related commuting vector fields.

5. Concluding Remarks

We have developed a new, essentially analytic approach for clarifying and simplifying many fundamental calculations associated to the AKNS hierarchy. The linchpin of our approach is the AKNS hierarchy generating vector field (1.15) on the functional manifold M and its intrinsic Lie-algebraic structure (1.12). The hub of our method, which is derived from (1.15) and its associated structure, is comprised of the vertex operator functional representations of the matrix solutions (3.7) and (3.12) for the determining Eqs. (3.3) and (1.7), respectively. This leads to the crucial vertex operator relationships (3.18) and (3.21) (which are fundamentally based on the representations (3.14) and Eqs. (3.16)); they provide — as we have shown — a very straightforward and transparent explanation of many of the "miraculous" calculations in [11, 19, 20], including the construction of conserved quantities and the AKNS hierarchy of nonlinear integrable dynamical systems.

The results for the AKNS hierarchy in the above and earlier papers [4, 10, 13, 19, 20, 28, 30] were obtained in a distinctly different manner — making use of direct asymptotic power series expansions of solutions to the determining matrix Eqs. (3.3) and (3.7) and their deep algebraic properties. It is therefore fitting that we present at least a brief historical survey of the more algebraically oriented research on the AKNS hierarchy, which can also be viewed as a system of partial differential equations in functions u and v of ξ comprising the compatibility conditions for

$$\partial_\xi \Psi = U\Psi := \left[\lambda\,\mathrm{diag}(1,-1) + \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix}\right]\Psi, \quad \partial_{t_j}\Psi = V_j\Psi,$$

where Ψ is an invertible matrix function of $\xi := (t_1, t_2, t_3, \ldots)$, and the spectral variable λ is called the Baker (wave) function. Whence, one may define the resolvents $R(j) := \Psi^{-1}\mathrm{diag}(\lambda^j, \lambda^{-j})\Psi$, which in turn complete the above system via $V_j := R(j)_+$, where the subscript denotes the projection on the positive power expansions in λ.

Two fundamental aspects of the Baker function for a general integrable hierarchy are the following: It is intimately related to τ-functions as clearly demonstrated by Sato's school [10, 19, 27], deftly elucidated in [28], and explicitly formulated for the AKNS hierarchy in [20]. Precise algebraic definitions of τ-functions in terms of group and representation theory are given, respectively, in [30] and [4], which are interesting to compare with our analytic formulation in Sec. 4. The Baker function also possesses a factorization of the form

$$\Psi = G_{-}D := \left[\mathbf{I} + \begin{pmatrix} 0 & e \\ f & 0 \end{pmatrix}\right]\mathrm{diag}(h, g),$$

where e, f, g, h are series in positive integral powers of λ^{-1} and e and f can be shown to be differential polynomials in the field u and v, which implies that the same is true of the resolvents $R(j) = G_-^{-1}\mathrm{diag}(\lambda^j, \lambda^{-j})G_-$ and leads to the conclusion that AKNS hierarchy is a collection of nonlinear dynamical systems of the form (1.2). The coefficients g and h, on the other hand, are not differential polynomials in u, v — they are integrals of differential polynomials, and their derivatives are densities of AKNS invariants as shown in [13].

We now return to our work here, which follows the analytical style pioneered by Novikov. It should be noted that, in a certain sense, the effectiveness of our approach to studying the vertex operator representation of the AKNS hierarchy owes a great deal to the important exact representation (3.1) for the solution of the determining Eq. (1.7) for the Casimir invariants. Equation (1.7) is based on the well-known monodromy matrix approach devised by Novikov [21], and it entails the extremely effective AKNS hierarchy representation in the simple recursive form (3.25), which explains several other very interesting results in the literature, such as in [22, 29]. On the other hand, the dual solution representation to (1.7) in the form (3.1), used extensively in [20], led naturally to the introduction of the well-known τ-function that made it possible to present the whole AKNS hierarchy in terms of partial derivatives. In spite of some of the fundamental differences, both our vertex operator approach and the more algebraic τ-function method are intimately related, as was briefly demonstrated.

We should note the following caveat concerning our vertex operator approach: Although we have demonstrated the versatility and simplicity of our approach for several AKNS computations, we make no particular claims that it is superior to other methods such as the r-matrix formalism or pseudo-differentiable algebra in all respects. For example, it may turn out that these other methods are actually more effective for deriving various important tensor invariants of the AKNS and other integrable hierarchies. Naturally, these questions require further investigation, which we hope to undertake in our future work.

Our results in this paper suggest several possible directions for future research. For example, it naturally would be interesting to apply the vertex operator approach to other linear spectral problems such as those related to the generalized Riemann hydrodynamical systems and BSR systems studied recently in [5, 6, 16, 23]. There is also the possibility of simplifying the derivations of the key equations in our approach, which seems well within the realm of possibility given the simplicity of the vertex vector fields.

Acknowledgments

D. Blackmore wishes to thank the National Science Foundation for support from NSF Grant CMMI — 1029809 and his coauthor for enlisting him in the efforts that produced this paper. Special thanks are due to the Scientific and Technology Research Council of Turkey (TUBITAK-2011) for partial support of the research of A. K. Prykarpatsky. A. K. Prykarpatsky cordially thanks Profs. M. V. Pavlov and V. H. Samoylenko for very useful discussions of the results obtained. The authors also wish to express their appreciation to the reviewers whose insightful comments helped to improve the paper.

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