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Research Article

Analytical Properties for the Fifth Order Camassa-Holm (FOCH) Model

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ABSTRACT

This paper devotes to present analysis work on the fifth order Camassa-Holm (FOCH) model which recently proposed by Liu and Qiao. Firstly, we establish the local and global existence of the solution to the FOCH model. Secondly, we study the property of the infinite propagation speed. Finally, we discuss the long time behavior of the support of momentum density with a compactly supported initial data.

1. INTRODUCTION

In this paper, we consider the following fifth order Camassa-Holm (FOCH) model [29]:

$$\begin{cases} m_t + um_x + bu_x m = 0, & t > 0, x \in \mathbb{R}, \\ m = (1-\alpha^2\partial_x^2)(1-\beta^2\partial_x^2)u, & t > 0, x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $b \in \mathbb{R}$ is a constant , $\alpha, \beta \in \mathbb{R}$ are two parameters, $\alpha \neq \beta, \alpha\beta \neq 0$. Without loss of generality, we only consider the case $\alpha > 0, \beta > 0$. When $\alpha < 0, \beta < 0$, one can get the similar results by using the corresponding absolute values $|\alpha|$ and $|\beta|$ instead of α, β.

In what follows, we present some mathematical results related to the topic of this paper. Liu and Qiao [29] obtained some interesting solutions including explicit single pseudo-peakons, two-peakon, and N-peakon solutions. Detailed dynamical interactions for two-pseudo-peakons and three-pseudo-peakons were also investigated in their paper with numerical simulations. There have been extensive studies on high order Camassa-Holm type equations in the mathematics physics fields [5,16,17,20,22,31,38–41]. For the case $\alpha = \beta = 1$, on the circle, McLachlan and Zhang [31] established the local well-posedness of the solution in H^s with $s > \frac{7}{2}$, it was shown that system (1.1) with $\alpha = \beta = 1$ does't admit finite time blow-up solutions. Tang and Liu [38] proved that the Cauchy problem for this equation is locally well-posed in the critical Besov space $B^{7/2}_{2,1}$ or $B^s_{p,r}(1 \leq p, r \leq +\infty$ and $s > \max\{3 + \frac{1}{p}, \frac{7}{2}\})$. The peakon-like solution and ill-posedness was also studied in [38]. For the case $b = 2$, $m = u - u_{xx} + u_{xxxx}$, by using the Kato's theory, the local well-posedness [39] was studied in the Sobolev space H^s with $s > \frac{9}{2}$. Ding [16,17] investigated the stationary solution, generality mild traveling solutions and conservative solution. Coclite, Holden and Karlsen [5] established the existence of global weak solutions. They also presented some invariant spaces under the action of the equation. In [20], the infinite propagation speed was considered for the case $m = 4u - 5u_{xx} + u_{xxxx}$. They also proved asymptotic behavior of the solution under the condition that the initial data decays exponentially and algebraically.

When $\beta = 0$ (or $\alpha = 0$), it means $m = u - \alpha^2 u_{xx}$. The Camassa-Holm equation, the Degasperis-Procesi equation, and the Holm-Staley b-family equations are the special cases of equation (1.1) with $b = 2$, $b = 3$ and $b \in \mathbb{R}$, respectively. These equations arise at various levels

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of approximation in shallow water theory, and possess a physics background with shallow water propagation, the bi-Hamiltonian structure, Lax pair, and explicit solutions including classical soliton, cuspon, and peakon solutions.

In 1993, Camassa and Holm [3] derived an integrable shallow water equation with peaked solitons, which was called the Camassa-Holm equation. In 1999, Degasperis and Procesi [15] extended the Camassa-Holm equation to a new water wave equation (Degasperis-Procesi equation). Both Camassa-Holm equation and Degasperis-Procesi equation have attracted much attention. They are completely integrable [11,12,14,35]. Infinitely many conservation laws have been shown in [14,27,37]. For the Camassa-Holm equation, in [9,28], They proved the local well-posedness for the initial datum in H^s with $s > 3/2$. There were many works to study the blow-up phenomenon, such as [8–10,24,28,32]. McKean [32] (See also [24] for a simple proof) proved that if and only if some portion of the positive part of $y_0 = u_0 - u_{0xx}$ lies to the left of some portion of its negative part, then the Camassa-Holm equation blow-up in finite time. The hierarchy properties, related finite-dimensional constrained flows, and algebro-geometric solutions of the Camassa-Holm equation were proposed in [34]. In [1], they studied the global conservative solution for the Camassa-Holm equation. Global dissipative solution have been shown in [2]. Constantin and Strauss [13] studied the orbital stability of the peakons. Himonas and his collaborators [21] obtained the persistence properties and unique continuation of solutions of the Camassa-Holm equation. In [25], the authors deduced the limit of the support of momentum density as t goes to $+\infty$. In [4,6,7,23,26,30,33,35,36,42], they have investigated some mathematical properties for the Degasperis-Procesi equation. For the Holm-Staley b-family equation, mathematical studies have also been presented in [18,19,43].

The paper is organized as follows. In Section 2, we establish the local well-posedness and blow up scenario for the FOCH model. Conditions for global existence are found in Section 3. In Section 4, we establish the property of the infinite propagation speed for the FOCH model. In Section 5, we discuss the long time behavior for the support of momentum density of the FOCH model.

2. LOCAL WELL-POSEDNESS AND BLOW UP SCENARIO

Similar to the Camassa-Holm equation [9], we can establish the following local well-posedness theorem for the FOCH model (1.1).

Theorem 2.1. *Let $u_0 \in H^s(\mathbb{R})$ with $s > \frac{7}{2}$. Then there exist a $T > 0$ depending on $\|u_0\|_{H^s}$, such that the FOCH model (1.1) has a unique solution*

$$u \in C([0,T); H^s(\mathbb{R})) \cap C^1([0,T); H^{s-1}(\mathbb{R})).$$

Morever, the map $u_0 \in H^s \to u \in C([0,T); H^s(\mathbb{R})) \cap C^1([0,T); H^{s-1}(\mathbb{R}))$ is continuous.

The proof is similar to that of Theorem 2.1 in [9,39]. To make the paper concise, we would like to omit the detail proof here. The maximum value of T in Theorem 2.1 is called the lifespan of the solution, in general. If $T < \infty$, that is

$$\lim_{t \to T^-} \|u\|_{H^s} = \infty,$$

we say the solution blows up in finite time.

Before going to the blow up scenario, we have the following Lemma.

Lemma 2.2. *As $m = (1 - \alpha^2\partial_x^2)(1 - \beta^2\partial_x^2)u$, then*

$$u = p * m, \quad p = \frac{\alpha^2}{\alpha^2 - \beta^2} p_1 - \frac{\beta^2}{\alpha^2 - \beta^2} p_2,$$

where $p_1 = \frac{1}{2\alpha} e^{-\frac{|x|}{\alpha}}$, $p_2 = \frac{1}{2\beta} e^{-\frac{|x|}{\beta}}$, $\alpha \neq \beta$, $\alpha > 0$, $\beta > 0$.

Proof. Taking fourier transform to $m = (1 - \alpha^2\partial_x^2)(1 - \beta^2\partial_x^2)u$, we have

$$\hat{m} = (1 + \alpha^2\xi^2)(1 + \beta^2\xi^2)\hat{u}.$$

Notice that when $f(x) = e^{-a|x|}$, $a > 0$ then

$$\hat{f}(\xi) = \frac{2a}{\xi^2 + a^2}.$$

It follows that

$$\hat{u}(\xi) = \frac{1}{1 + \alpha^2\xi^2} \cdot \frac{1}{1 + \beta^2\xi^2} \cdot \hat{m}(\xi) = \hat{p_1}(\xi) \cdot \hat{p_2}(\xi) \cdot \hat{m}(\xi),$$

where $p_1 = \frac{1}{2\alpha} e^{-\frac{|x|}{\alpha}}$, $p_2 = \frac{1}{2\beta} e^{-\frac{|x|}{\beta}}$. Then,

$$u(x) = \mathcal{F}^{-1}\left(\hat{p_1}(\xi) \cdot \hat{p_2}(\xi) \cdot \hat{m}(\xi)\right) = p_1 * p_2 * m(x).$$

Let $p = p_1 * p_2$, we have

$$\begin{aligned} p(x) &= \frac{1}{4\alpha\beta}\int_{\mathbb{R}} e^{-\frac{|x-y|}{\alpha}} e^{-\frac{|y|}{\beta}} dy \\ &= \frac{1}{2(\alpha^2-\beta^2)}\left[\alpha e^{-\frac{|x|}{\alpha}} - \beta e^{-\frac{|x|}{\beta}}\right] \\ &= \frac{\alpha^2}{(\alpha^2-\beta^2)} p_1 - \frac{\beta^2}{(\alpha^2-\beta^2)} p_2. \end{aligned}$$

□

By Lemma 2.2, we can rewrite $u(x,t)$ as

$$\begin{aligned} u &= \left[\frac{\alpha^2}{(\alpha^2-\beta^2)} p_1 - \frac{\beta^2}{(\alpha^2-\beta^2)} p_2\right] * m \\ &= \frac{\alpha}{2(\alpha^2-\beta^2)}\left[e^{-\frac{x}{\alpha}}\int_{-\infty}^{x} e^{\frac{\xi}{\alpha}} m(\xi,t)d\xi + e^{\frac{x}{\alpha}}\int_{x}^{+\infty} e^{-\frac{\xi}{\alpha}} m(\xi,t)d\xi\right] \\ &\quad - \frac{\beta}{2(\alpha^2-\beta^2)}\left[e^{-\frac{x}{\beta}}\int_{-\infty}^{x} e^{\frac{\xi}{\beta}} m(\xi,t)d\xi + e^{\frac{x}{\beta}}\int_{x}^{+\infty} e^{-\frac{\xi}{\beta}} m(\xi,t)d\xi\right]. \end{aligned} \tag{2.1}$$

Differentiating u with respect to x, we have

$$\begin{aligned} u_x &= \frac{1}{2(\alpha^2-\beta^2)}\left[-e^{-\frac{x}{\alpha}}\int_{-\infty}^{x} e^{\frac{\xi}{\alpha}} m(\xi,t)d\xi + e^{\frac{x}{\alpha}}\int_{x}^{+\infty} e^{-\frac{\xi}{\alpha}} m(\xi,t)d\xi\right] \\ &\quad + \frac{1}{2(\alpha^2-\beta^2)}\left[e^{-\frac{x}{\beta}}\int_{-\infty}^{x} e^{\frac{\xi}{\beta}} m(\xi,t)d\xi - e^{\frac{x}{\beta}}\int_{x}^{+\infty} e^{-\frac{\xi}{\beta}} m(\xi,t)d\xi\right]. \end{aligned}$$

Then, we present the precise blow-up scenario.

Theorem 2.3. *Assume that $u_0 \in H^4(\mathbb{R})$ and let T be the maximal existence time of the solution $u(x,t)$ to equation (1.1), $\alpha \neq \beta$, $\alpha > 0$, $\beta > 0$ with the initial data $u_0(x)$.*
***(1).** If $b > \frac{1}{2}$, then the corresponding solution of the FOCH model (1.1) blows up in finite time if and only if*

$$\lim_{t\to T}\inf_{x\in\mathbb{R}}\{u_x(x,t)\} = -\infty.$$

***(2).** If $b < \frac{1}{2}$, then the corresponding solution of the FOCH model (1.1) blows up in finite time if and only if*

$$\lim_{t\to T}\sup_{x\in\mathbb{R}}\{u_x(x,t)\} = +\infty.$$

Proof. By direct calculation, we have

$$\begin{aligned} \|m\|_{L^2}^2 &= \int_{\mathbb{R}} [u - (\alpha^2+\beta^2)u_{xx} + \alpha^2\beta^2 u_{xxxx}]^2 dx \\ &= \int_{\mathbb{R}} u^2 + (\alpha^2+\beta^2)^2 u_{xx}^2 - 2(\alpha^2+\beta^2)uu_{xx} + \alpha^4\beta^4 u_{xxxx}^2 + 2\alpha^2\beta^2 uu_{xxxx} - 2(\alpha^2+\beta^2)\alpha^2\beta^2 u_{xx}u_{xxxx} dx \\ &= \int_{\mathbb{R}} u^2 + (\alpha^2+\beta^2)^2 u_{xx}^2 + 2(\alpha^2+\beta^2)u_x^2 + \alpha^4\beta^4 u_{xxxx}^2 + 2\alpha^2\beta^2 u_{xx}^2 + 2(\alpha^2+\beta^2)\alpha^2\beta^2 u_{xxx}^2 dx. \end{aligned}$$

Hence

$$c\|u\|_{H^4}^2 \leq \|m\|_{L^2}^2 \leq C\|u\|_{H^4}^2,$$

where c and C are positive constants depending on α and β. If $b > \frac{1}{2}$, direct calculation we have

$$\frac{d}{dt}\int_{\mathbb{R}} m^2 dx = (1-2b)\int_{\mathbb{R}} u_x m^2 dx \leq (1-2b)\inf_{x\in\mathbb{R}}\{u_x(x,t)\}\int_{\mathbb{R}} m^2 dx.$$

If

$$\inf_{x\in\mathbb{R}}\{u_x(x,t)\} \geq -M,$$

then

$$\frac{d}{dt}\int_{\mathbb{R}} m^2 dx \leq -(1-2b)M\int_{\mathbb{R}} m^2 dx.$$

By using the Gronwall inequality,

$$\|m\|_{L^2}^2 = \int_{\mathbb{R}} m^2 dx \le e^{-(1-2b)M} \int_{\mathbb{R}} m_0^2 dx = e^{-(1-2b)M} \|m_0\|_{L^2}^2.$$

Therefore the H^4-norm of the solution is bounded on $[0, T)$.

On the other hand,

$$\begin{aligned} u &= \frac{\alpha^2}{(\alpha^2 - \beta^2)} p_1 * m - \frac{\beta^2}{(\alpha^2 - \beta^2)} p_2 * m \\ &= \frac{\alpha^2}{(\alpha^2 - \beta^2)} \int_{\mathbb{R}} p_1(x-\xi) m(\xi) d\xi - \frac{\beta^2}{(\alpha^2 - \beta^2)} \int_{\mathbb{R}} p_2(x-\xi) m(\xi) d\xi. \end{aligned}$$

By the Sobolev's embedding $\|u_x\|_\infty \le \|u\|_{H^4}$, it tells us if H^4-norm of the solution is bounded, then the L^∞-norm of the first derivative is bounded.

By the same argument, we can get the similar result for $b < \frac{1}{2}$. So, we omit the details and complete the proof of Theorem 2.3. □

3. GLOBAL EXISTENCE

In this section, we study the global existence. Before going to our main results, we give the particle trajectory as

$$\begin{cases} q_t = u(q, t), & 0 < t < T, x \in \mathbb{R}, \\ q(x, 0) = x, & x \in \mathbb{R}, \end{cases} \tag{3.1}$$

where T is the lifespan of the solution. Taking derivative (3.1) with respect to x, we obtain

$$\frac{dq_t}{dx} = q_{tx} = u_x(q, t) q_x, \qquad t \in (0, T).$$

Therefore

$$\begin{cases} q_x = exp\{\int_0^t u_x(q, s) ds\}, & 0 < t < T, \quad x \in \mathbb{R}, \\ q_x(x, 0) = 1, & x \in \mathbb{R}, \end{cases}$$

which is always positive before the blow-up time. Therefore, the function $q(x, t)$ is an increasing diffeomorphism of the line before blow-up. In fact, direct calculation yields

$$\frac{d}{dt}(m(q) q_x^b) = [m_t(q) + u(q, t) m_x(q) + b u_x(q, t) m(q)] q_x^b = 0.$$

Hence, we have the following identity

$$m(q) q_x^b = m_0(x), \qquad 0 < t < T, x \in \mathbb{R}. \tag{3.2}$$

Theorem 3.1. *Assume that $u_0 \in H^4(\mathbb{R})$, $\alpha \neq \beta$, $\alpha > 0$, $\beta > 0$, if $b = \frac{1}{2}$ or $b = 2$, then the corresponding solution of FOCH model (1.1) will exist globally in time.*

Remark 3.1. *If $\alpha = 0$ or $\beta = 0$, system (1.1) reduce to the well-known b-family equation. The global existence for $b = \frac{1}{2}$ and Theorem 3.2 can be reduce to the results for b-family equation [18]. The global existence for $b = 2$ is the new discovery compared to the b-family equation.*

Proof. Let

$$E(t) = \int_{\mathbb{R}} u^2 + (\alpha^2 + \beta^2) u_x^2 + \alpha^2 \beta^2 u_{xx}^2 dx.$$

Differentiating $E(t)$, we have

$$\begin{aligned} \frac{d}{dt} E(t) &= \int_{\mathbb{R}} 2uu_t + 2(\alpha^2 + \beta^2) u_x u_{xt} + 2\alpha^2 \beta^2 u_{xx} u_{xxt} dx \\ &= \int_{\mathbb{R}} 2uu_t - 2(\alpha^2 + \beta^2) u u_{xxt} + 2\alpha^2 \beta^2 u u_{xxxxt} dx \\ &= 2 \int_{\mathbb{R}} u m_t dx \\ &= (b-2) \int_{\mathbb{R}} u^2 m_x dx. \end{aligned}$$

It yields that $E(t) = E(0)$ when $b = 2$. By the Sobolev's imbedding, we have

$$\|u_x\|_{L^\infty} \le \|u\|_{H^2}^2 \le CE(t) = CE(0).$$

The global existence for $b = 2$ is completed by Theorem 2.3. Applying m on (1.1) and integration by parts, we obtain

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}} m^2 dx &= -2\int_{\mathbb{R}} bu_x m^2 + mm_x u dx\\ &= -2\int_{\mathbb{R}} bu_x m^2 - \frac{m^2}{2}u_x dx\\ &= (1-2b)\int_{\mathbb{R}} u_x m^2 dx.\end{aligned}$$

If $b = \frac{1}{2}$, then $\frac{d}{dt}\int_{\mathbb{R}} m^2 dx = 0$. Hence,

$$\|u\|_{H^4}^2 \le \|m\|_{L^2}^2 = \|m_0\|_{L^2}^2.$$

It follows that the corresponding solution of FOCH model (1.1) exists globally when $b = \frac{1}{2}$. □

Theorem 3.2. *Supposing that $u_0 \in H^4$, $\alpha \neq \beta$, $\alpha > 0$, $\beta > 0$, $m_0 = (1-\alpha^2\partial_x^2)(1-\beta^2\partial_x^2)u_0$ does not change sign. Then the corresponding solution to (1.1) exists globally.*

Proof. We can assume that $m_0 \ge 0$. It is sufficient to prove u_x is bounded for all t. In fact,

$$\begin{aligned}u_x = &\frac{1}{2(\alpha^2-\beta^2)}\left[e^{\frac{x}{\alpha}}\int_x^{+\infty} e^{-\frac{\xi}{\alpha}} m(\xi,t)d\xi - e^{-\frac{x}{\alpha}}\int_{-\infty}^x e^{\frac{\xi}{\alpha}} m(\xi,t)d\xi\right]\\ &+\frac{1}{2(\alpha^2-\beta^2)}\left[e^{-\frac{x}{\beta}}\int_{-\infty}^x e^{\frac{\xi}{\beta}} m(\xi,t)d\xi - e^{\frac{x}{\beta}}\int_x^{+\infty} e^{-\frac{\xi}{\beta}} m(\xi,t)d\xi\right].\end{aligned}$$

If $m_0 \ge 0, \alpha > \beta > 0$, then

$$\begin{aligned}u_x &= \frac{1}{2(\alpha^2-\beta^2)}\left[e^{\frac{x}{\alpha}}\int_x^{+\infty} e^{-\frac{\xi}{\alpha}} m(\xi,t)d\xi - e^{-\frac{x}{\alpha}}\int_{-\infty}^x e^{\frac{\xi}{\alpha}} m(\xi,t)d\xi\right]\\ &\quad+\frac{1}{2(\alpha^2-\beta^2)}\left[e^{-\frac{x}{\beta}}\int_{-\infty}^x e^{\frac{\xi}{\beta}} m(\xi,t)d\xi - e^{\frac{x}{\beta}}\int_x^{+\infty} e^{-\frac{\xi}{\beta}} m(\xi,t)d\xi\right]\\ &\le \frac{1}{2(\alpha^2-\beta^2)}\left[e^{\frac{x}{\alpha}}\int_x^{\infty} e^{-\frac{\xi}{\alpha}} m(\xi,t)d\xi + e^{-\frac{x}{\beta}}\int_{-\infty}^x e^{\frac{\xi}{\beta}} m(\xi,t)d\xi\right]\\ &\le \frac{1}{2(\alpha^2-\beta^2)}\left[e^{\frac{x}{\alpha}}e^{-\frac{x}{\alpha}}\int_x^{\infty} m(\xi,t)d\xi + e^{-\frac{x}{\beta}}e^{\frac{x}{\beta}}\int_{-\infty}^x m(\xi,t)d\xi\right]\\ &\le \frac{1}{2(\alpha^2-\beta^2)}\left[\int_{\mathbb{R}} m(\xi,t)d\xi + \int_{\mathbb{R}} m(\xi,t)d\xi\right]\\ &= \frac{1}{(\alpha^2-\beta^2)}\int_{\mathbb{R}} m_0(\xi,t)d\xi\end{aligned}$$

and

$$\begin{aligned}u_x &= \frac{1}{2(\alpha^2-\beta^2)}\left[e^{\frac{x}{\alpha}}\int_x^{\infty} e^{-\frac{\xi}{\alpha}} m(\xi,t)d\xi - e^{-\frac{x}{\alpha}}\int_{-\infty}^x e^{\frac{\xi}{\alpha}} m(\xi,t)d\xi\right]\\ &\quad+\frac{1}{2(\alpha^2-\beta^2)}\left[e^{-\frac{x}{\beta}}\int_{-\infty}^x e^{\frac{\xi}{\beta}} m(\xi,t)d\xi - e^{\frac{x}{\beta}}\int_x^{+\infty} e^{-\frac{\xi}{\beta}} m(\xi,t)d\xi\right]\\ &\ge \frac{1}{2(\alpha^2-\beta^2)}\left[-e^{-\frac{x}{\alpha}}\int_{-\infty}^x e^{\frac{\xi}{\alpha}} m(\xi,t)d\xi - e^{\frac{x}{\beta}}\int_x^{+\infty} e^{-\frac{\xi}{\beta}} m(\xi,t)d\xi\right]\\ &\ge -\frac{1}{(\alpha^2-\beta^2)}\int_{\mathbb{R}} m(\xi,t)d\xi\\ &= -\frac{1}{(\alpha^2-\beta^2)}\int_{\mathbb{R}} m_0(\xi,t)d\xi.\end{aligned}$$

That is

$$|u_x| \le \frac{1}{(\alpha^2-\beta^2)}\int_{\mathbb{R}} m_0(\xi,t)d\xi.$$

If $m_0 \geq 0, 0 < \alpha < \beta$, then

$$\begin{aligned}
u_x &= \frac{1}{2(\alpha^2-\beta^2)}\left[e^{\frac{x}{\alpha}}\int_x^{\infty} e^{-\frac{\xi}{\alpha}} m(\xi,t)d\xi - e^{-\frac{x}{\alpha}}\int_{-\infty}^{x} e^{\frac{\xi}{\alpha}} m(\xi,t)d\xi\right] \\
&\quad + \frac{1}{2(\alpha^2-\beta^2)}\left[e^{-\frac{x}{\beta}}\int_{-\infty}^{x} e^{\frac{\xi}{\beta}} m(\xi,t)d\xi - e^{\frac{x}{\beta}}\int_x^{+\infty} e^{-\frac{\xi}{\beta}} m(\xi,t)d\xi\right] \\
&\leq \frac{1}{2(\alpha^2-\beta^2)}\left[-e^{-\frac{x}{\alpha}}\int_{-\infty}^{x} e^{\frac{\xi}{\alpha}} m(\xi,t)d\xi - e^{\frac{x}{\beta}}\int_x^{+\infty} e^{-\frac{\xi}{\beta}} m(\xi,t)d\xi\right] \\
&\leq \frac{1}{2(\alpha^2-\beta^2)}\left[-e^{-\frac{x}{\alpha}}e^{\frac{x}{\alpha}}\int_{-\infty}^{x} m(\xi,t)d\xi - e^{\frac{x}{\beta}}e^{-\frac{x}{\beta}}\int_x^{\infty} m(\xi,t)d\xi\right] \\
&\leq \frac{1}{2(\beta^2-\alpha^2)}\left[\int_{\mathbb{R}} m(\xi,t)d\xi + \int_{\mathbb{R}} m(\xi,t)d\xi\right] \\
&= \frac{1}{(\beta^2-\alpha^2)}\int_{\mathbb{R}} m_0(\xi,t)d\xi
\end{aligned}$$

and

$$\begin{aligned}
u_x &= \frac{1}{2(\alpha^2-\beta^2)}\left[e^{\frac{x}{\alpha}}\int_x^{\infty} e^{-\frac{\xi}{\alpha}} m(\xi,t)d\xi - e^{-\frac{x}{\alpha}}\int_{-\infty}^{x} e^{\frac{\xi}{\alpha}} m(\xi,t)d\xi\right] \\
&\quad + \frac{1}{2(\alpha^2-\beta^2)}\left[e^{-\frac{x}{\beta}}\int_{-\infty}^{x} e^{\frac{\xi}{\beta}} m(\xi,t)d\xi - e^{\frac{x}{\beta}}\int_x^{+\infty} e^{-\frac{\xi}{\beta}} m(\xi,t)d\xi\right] \\
&\geq \frac{1}{2(\alpha^2-\beta^2)}\left[e^{\frac{x}{\alpha}}\int_x^{\infty} e^{-\frac{\xi}{\alpha}} m(\xi,t)d\xi + e^{-\frac{x}{\beta}}\int_{-\infty}^{x} e^{\frac{\xi}{\beta}} m(\xi,t)d\xi\right] \\
&\geq -\frac{1}{(\beta^2-\alpha^2)}\int_{\mathbb{R}} m(\xi,t)d\xi \\
&= -\frac{1}{(\beta^2-\alpha^2)}\int_{\mathbb{R}} m_0(\xi,t)d\xi.
\end{aligned}$$

That is

$$|u_x| \leq \frac{1}{(\beta^2-\alpha^2)}\int_{\mathbb{R}} m_0(\xi,t)d\xi.$$

When $m_0 \leq 0$, via the similar approach that is used above, we could also obtain the global existence result. So, we omit the details and complete the proof of Theorem 3.2. □

4. INFINITE PROPAGATION SPEED

The main theorem reads as follows:

Theorem 4.1. *Assume that the initial datum $u_0(x) \in H^4(\mathbb{R})$ is compactly supported in $[a,c]$, then for $t \in (0,T)$, the corresponding solution $u(x,t)$ to the FOCH model (1.1) $\alpha \neq \beta, \alpha > 0, \beta > 0$ has the following property:*

$$u(x,t) = \begin{cases} \frac{\alpha}{2(\alpha^2-\beta^2)}e^{-\frac{x}{\alpha}}E_1(t) - \frac{\beta}{2(\alpha^2-\beta^2)}e^{-\frac{x}{\beta}}E_2(t), & \text{as } x > q(c,t), \\ \frac{\alpha}{2(\alpha^2-\beta^2)}e^{\frac{x}{\alpha}}F_1(t) - \frac{\beta}{2(\alpha^2-\beta^2)}e^{\frac{x}{\beta}}F_2(t), & \text{as } x < q(a,t), \end{cases}$$

where

$$E_1(t) = \int_{\mathbb{R}} e^{\frac{x}{\alpha}} m(x,t)dx, \qquad F_1(t) = \int_{\mathbb{R}} e^{-\frac{x}{\alpha}} m(x,t)dx,$$

and

$$E_2(t) = \int_{\mathbb{R}} e^{\frac{x}{\beta}} m(x,t)dx, \qquad F_2(t) = \int_{\mathbb{R}} e^{-\frac{x}{\beta}} m(x,t)dx,$$

denote continuous nonvanishing functions.

Furthermore, if $\alpha > 0, 0 < \beta \leq \sqrt{\frac{3}{2}}\alpha, 0 \leq b \leq \min\{3 - \frac{2\beta^2}{\alpha^2}, \frac{5}{3}\}$, $E_1(t)$ is strictly increasing function, while $F_1(t)$ is strictly decreasing function. Similarly, if $\beta > 0, 0 < \alpha \leq \sqrt{\frac{3}{2}}\beta, 0 \leq b \leq \min\{3 - \frac{2\alpha^2}{\beta^2}, \frac{5}{3}\}$, $E_2(t)$ is strictly increasing function, while $F_2(t)$ is strictly decreasing function.

Remark 4.1. *Theorem 4.1 implies that the strong solution $u(x,t)$ doesn't have compact x-support for any $t > 0$ in its lifespan, although the corresponding $u_0(x)$ is compactly supported.*

Proof. Since $u_0(x)$ has a compact support in the interval $[a,c]$, so does $m_0(x) = (1-\alpha^2\partial_x^2)(1-\beta^2\partial_x^2)u_0(x)$. Equation (3.2) tells us that $m(x) = (1-\alpha^2\partial_x^2)(1-\beta^2\partial_x^2)u(x)$ is compactly supported in the interval $[q(a,t), q(c,t)]$ in its lifespan. Hence the following functions are well-defined

$$E_1(t) = \int_{\mathbb{R}} e^{\frac{x}{\alpha}} m(x,t)dx, \qquad F_1(t) = \int_{\mathbb{R}} e^{-\frac{x}{\alpha}} m(x,t)dx,$$
$$E_2(t) = \int_{\mathbb{R}} e^{\frac{x}{\beta}} m(x,t)dx, \qquad F_2(t) = \int_{\mathbb{R}} e^{-\frac{x}{\beta}} m(x,t)dx.$$

Using (3.2),

$$m(q(x,t),t) \equiv 0, \qquad x < a \quad or \quad x > c,$$

we know

$$\begin{aligned} u(x,t) &= \left(\frac{\alpha^2}{(\alpha^2-\beta^2)}p_1 - \frac{\beta^2}{(\alpha^2-\beta^2)}p_2\right) * m(x,t) \\ &= \frac{\alpha}{2(\alpha^2-\beta^2)}\int_{\mathbb{R}} e^{-\frac{|x-\xi|}{\alpha}} m(\xi)d\xi - \frac{\beta}{2(\alpha^2-\beta^2)}\int_{\mathbb{R}} e^{-\frac{|x-\xi|}{\beta}} m(\xi)d\xi \\ &= \frac{\alpha}{2(\alpha^2-\beta^2)}\int_{q(a,t)}^{q(c,t)} e^{-\frac{|x-\xi|}{\alpha}} m(\xi)d\xi - \frac{\beta}{2(\alpha^2-\beta^2)}\int_{q(a,t)}^{q(c,t)} e^{-\frac{|x-\xi|}{\beta}} m(\xi)d\xi. \end{aligned}$$

Then, for $x > q(c,t)$, we have

$$\begin{aligned} u(x,t) &= \frac{\alpha}{2(\alpha^2-\beta^2)}\int_{q(a,t)}^{q(c,t)} e^{-\frac{x-\xi}{\alpha}} m(\xi)d\xi - \frac{\beta}{2(\alpha^2-\beta^2)}\int_{q(a,t)}^{q(c,t)} e^{-\frac{x-\xi}{\beta}} m(\xi)d\xi \\ &= \frac{\alpha}{2(\alpha^2-\beta^2)}e^{-\frac{x}{\alpha}}\int_{q(a,t)}^{q(c,t)} e^{\frac{\xi}{\alpha}} m(\xi)d\xi - \frac{\beta}{2(\alpha^2-\beta^2)}e^{-\frac{x}{\beta}}\int_{q(a,t)}^{q(c,t)} e^{\frac{\xi}{\beta}} m(\xi)d\xi \\ &= \frac{\alpha}{2(\alpha^2-\beta^2)}e^{-\frac{x}{\alpha}}E_1(t) - \frac{\beta}{2(\alpha^2-\beta^2)}e^{-\frac{x}{\beta}}E_2(t). \end{aligned} \tag{4.1}$$

Similarly, when $x < q(a,t)$, we have

$$\begin{aligned} u(x,t) &= \frac{\alpha}{2(\alpha^2-\beta^2)}\int_{q(a,t)}^{q(c,t)} e^{\frac{x-\xi}{\alpha}} m(\xi)d\xi - \frac{\beta}{2(\alpha^2-\beta^2)}\int_{q(a,t)}^{q(c,t)} e^{\frac{x-\xi}{\beta}} m(\xi)d\xi \\ &= \frac{\alpha}{2(\alpha^2-\beta^2)}e^{\frac{x}{\alpha}}\int_{q(a,t)}^{q(c,t)} e^{-\frac{\xi}{\alpha}} m(\xi)d\xi - \frac{\beta}{2(\alpha^2-\beta^2)}e^{\frac{x}{\beta}}\int_{q(a,t)}^{q(c,t)} e^{-\frac{\xi}{\beta}} m(\xi)d\xi \\ &= \frac{\alpha}{2(\alpha^2-\beta^2)}e^{\frac{x}{\alpha}}F_1(t) - \frac{\beta}{2(\alpha^2-\beta^2)}e^{\frac{x}{\beta}}F_2(t). \end{aligned} \tag{4.2}$$

On the other hand,

$$\frac{dE_1(t)}{dt} = \int_{\mathbb{R}} e^{\frac{\xi}{\alpha}} m_t(\xi,t)d\xi.$$

It is easy to get

$$\begin{aligned} m_t &= -m_x - bmu_x \\ &= [(\alpha^2+\beta^2)u_{xxx} - \alpha^2\beta^2 u_{xxxxx} - u_x]u - b[u - (\alpha^2+\beta^2)u_{xx} + \alpha^2\beta^2 u_{xxxx}]u_x \\ &= (\alpha^2+\beta^2)u_{xxx}u - \alpha^2\beta^2 u_{xxxxx}u - (b+1)uu_x + b(\alpha^2+\beta^2)u_{xx}u_x - b\alpha^2\beta^2 u_{xxxx}u_x. \end{aligned} \tag{4.3}$$

Taking (4.3) into $\frac{dE_1(t)}{dt}$, we obtain

$$
\begin{aligned}
\frac{dE_1(t)}{dt} &= \int_{\mathbb{R}} e^{\frac{x}{\alpha}} m_t dx \\
&= \int_{\mathbb{R}} e^{\frac{x}{\alpha}} \left[(\alpha^2+\beta^2)u_{xxx}u - \alpha^2\beta^2 u_{xxxxx}u - (b+1)uu_x + b(\alpha^2+\beta^2)u_{xx}u_x - b\alpha^2\beta^2 u_{xxxx}u_x\right] dx \\
&= (\alpha^2+\beta^2)\int_{\mathbb{R}} e^{\frac{x}{\alpha}} u_{xxx}udx - \alpha^2\beta^2\int_{\mathbb{R}} e^{\frac{x}{\alpha}} u_{xxxxx}udx - (b+1)\int_{\mathbb{R}} e^{\frac{x}{\alpha}} uu_xdx \\
&\quad + b(\alpha^2+\beta^2)\int_{\mathbb{R}} e^{\frac{x}{\alpha}} u_{xx}u_xdx - b\alpha^2\beta^2\int_{\mathbb{R}} e^{\frac{x}{\alpha}} u_{xxxx}u_xdx \\
&= \sum_{i=1}^{5} I_i.
\end{aligned}
\tag{4.4}
$$

I_1-I_5 can be estimated as follows:

$$
\begin{aligned}
I_1 &= (\alpha^2+\beta^2)\int_{\mathbb{R}} e^{\frac{x}{\alpha}} u_{xxx}udx \\
&= -\frac{\alpha^2+\beta^2}{2\alpha^3}\int_{\mathbb{R}} e^{\frac{x}{\alpha}} u^2dx + \frac{3(\alpha^2+\beta^2)}{2\alpha}\int_{\mathbb{R}} e^{\frac{x}{\alpha}} u_x^2dx,
\end{aligned}
\tag{4.5}
$$

$$
\begin{aligned}
I_2 &= -\alpha^2\beta^2\int_{\mathbb{R}} e^{\frac{x}{\alpha}} u_{xxxxx}udx \\
&= \frac{\beta^2}{2\alpha^3}\int_{\mathbb{R}} e^{\frac{x}{\alpha}} u^2dx - \frac{5\beta^2}{2\alpha}\int_{\mathbb{R}} e^{\frac{x}{\alpha}} u_x^2dx + \frac{5\alpha\beta^2}{2}\int_{\mathbb{R}} e^{\frac{x}{\alpha}} u_{xx}^2dx,
\end{aligned}
\tag{4.6}
$$

$$
\begin{aligned}
I_3 &= -(b+1)\int_{\mathbb{R}} e^{\frac{x}{\alpha}} uu_xdx \\
&= \frac{b+1}{2\alpha}\int_{\mathbb{R}} e^{\frac{x}{\alpha}} u^2dx,
\end{aligned}
\tag{4.7}
$$

$$
\begin{aligned}
I_4 &= b(\alpha^2+\beta^2)\int_{\mathbb{R}} e^{\frac{x}{\alpha}} u_{xx}u_xdx \\
&= -\frac{b(\alpha^2+\beta^2)}{2\alpha}\int_{\mathbb{R}} e^{\frac{x}{\alpha}} u_x^2dx,
\end{aligned}
\tag{4.8}
$$

$$
\begin{aligned}
I_5 &= -b\alpha^2\beta^2\int_{\mathbb{R}} e^{\frac{x}{\alpha}} u_{xxxx}u_xdx \\
&= \frac{b\beta^2}{2\alpha}\int_{\mathbb{R}} e^{\frac{x}{\alpha}} u_x^2dx - \frac{3b\alpha\beta^2}{2}\int_{\mathbb{R}} e^{\frac{x}{\alpha}} u_{xx}^2dx.
\end{aligned}
\tag{4.9}
$$

Combining (4.5)–(4.9) to (4.4), we have

$$
\begin{aligned}
\frac{dE_1(t)}{dt} &= \int_{\mathbb{R}} e^{\frac{x}{\alpha}} m_t dx \\
&= \frac{b}{2\alpha}\int_{\mathbb{R}} e^{\frac{x}{\alpha}} u^2dx - \frac{(b-3)\alpha^2+2\beta^2}{2\alpha}\int_{\mathbb{R}} e^{\frac{x}{\alpha}} u_x^2dx + \frac{(5-3b)\alpha\beta^2}{2}\int_{\mathbb{R}} e^{\frac{x}{\alpha}} u_{xx}^2dx.
\end{aligned}
\tag{4.10}
$$

For $\alpha > 0, 0 < \beta \le \sqrt{\frac{3}{2}}\alpha, 0 \le b \le \min\{3 - \frac{2\beta^2}{\alpha^2}, \frac{5}{3}\}$, from (4.10), $E_1(t)$ is strictly increasing for nontrivial solution.

Similary,

$$
\begin{aligned}
\frac{dF_1(t)}{dt} &= \int_{\mathbb{R}} e^{-\frac{x}{\alpha}} m_t dx \\
&= -\frac{b}{2\alpha}\int_{\mathbb{R}} e^{-\frac{x}{\alpha}} u^2dx + \frac{(b-3)\alpha^2+2\beta^2}{2\alpha}\int_{\mathbb{R}} e^{-\frac{x}{\alpha}} u_x^2dx - \frac{(5-3b)\alpha\beta^2}{2}\int_{\mathbb{R}} e^{-\frac{x}{\alpha}} u_{xx}^2dx.
\end{aligned}
\tag{4.11}
$$

For $\alpha > 0, 0 < \beta \le \sqrt{\frac{3}{2}}\alpha, 0 \le b \le \min\{3 - \frac{2\beta^2}{\alpha^2}, \frac{5}{3}\}$, from (4.11), $F_1(t)$ is strictly decreasing for nontrivial solution.

$$
\begin{aligned}
\frac{dE_2(t)}{dt} &= \int_{\mathbb{R}} e^{\frac{x}{\beta}} m_t dx \\
&= \frac{b}{2\beta}\int_{\mathbb{R}} e^{\frac{x}{\beta}} u^2dx - \frac{(b-3)\beta^2+2\alpha^2}{2\beta}\int_{\mathbb{R}} e^{\frac{x}{\beta}} u_x^2dx + \frac{(5-3b)\beta\alpha^2}{2}\int_{\mathbb{R}} e^{\frac{x}{\beta}} u_{xx}^2dx.
\end{aligned}
\tag{4.12}
$$

For $\beta > 0, 0 < \alpha \le \sqrt{\frac{3}{2}}\beta, 0 \le b \le \min\{3 - \frac{2\alpha^2}{\beta^2}, \frac{5}{3}\}$, from (4.12), $E_2(t)$ is strictly increasing for nontrivial solution.

$$\begin{aligned}\frac{dF_2(t)}{dt} &= \int_{\mathbb{R}} e^{-\frac{x}{\beta}} m_t dx \\ &= -\frac{b}{2\beta}\int_{\mathbb{R}} e^{-\frac{x}{\beta}} u^2 dx + \frac{(b-3)\beta^2 + 2\alpha^2}{2\beta}\int_{\mathbb{R}} e^{-\frac{x}{\beta}} u_x^2 dx - \frac{(5-3b)\beta\alpha^2}{2}\int_{\mathbb{R}} e^{-\frac{x}{\beta}} u_{xx}^2 dx.\end{aligned} \tag{4.13}$$

For $\beta > 0, 0 < \alpha \le \sqrt{\frac{3}{2}}\beta, 0 \le b \le \min\{3 - \frac{2\alpha^2}{\beta^2}, \frac{5}{3}\}$, from (4.13), $F_2(t)$ is strictly decreasing for nontrivial solution.

This complete the proof of Theorem 4.1. □

Remark 4.2. *Let*

$$u'(x,t) = \begin{cases} \frac{\alpha}{2(\alpha^2-\beta^2)} e^{-\frac{x}{\alpha}} E_1(t), & \text{as } x > q(c,t), \\ \frac{\alpha}{2(\alpha^2-\beta^2)} e^{\frac{x}{\alpha}} F_1(t), & \text{as } x < q(a,t), \end{cases} \qquad u''(x,t) = \begin{cases} \frac{\beta}{2(\alpha^2-\beta^2)} e^{-\frac{x}{\beta}} E_2(t), & \text{as } x > q(c,t), \\ \frac{\beta}{2(\alpha^2-\beta^2)} e^{\frac{x}{\beta}} F_2(t), & \text{as } x < q(a,t), \end{cases}$$

We rewrite $u = u' - u''$, as consequences of (4.1) and (4.2), we have

$$\begin{aligned} u'(x,t) &= -\alpha u'_x(x,t) = \alpha^2 u'_{xx}(x,t) = \frac{\alpha}{2(\alpha^2-\beta^2)} e^{-\frac{x}{\alpha}} E_1(t), && \text{as } x > q(c,t), \\ u'(x,t) &= \alpha u'_x(x,t) = \alpha^2 u'_{xx}(x,t) = \frac{\alpha}{2(\alpha^2-\beta^2)} e^{\frac{x}{\alpha}} F_1(t), && \text{as } x < q(a,t). \end{aligned}$$

and

$$\begin{aligned} u''(x,t) &= -\beta u''_x(x,t) = \beta^2 u''_{xx}(x,t) = \frac{\beta}{2(\alpha^2-\beta^2)} e^{-\frac{x}{\beta}} E_2(t), && \text{as } x > q(c,t), \\ u''(x,t) &= \beta u''_x(x,t) = \beta^2 u''_{xx}(x,t) = \frac{\beta}{2(\alpha^2-\beta^2)} e^{\frac{x}{\beta}} F_2(t), && \text{as } x < q(a,t). \end{aligned}$$

Theorem 4.2. *Suppose the initial value $u_0(x) \in H^4(\mathbb{R})$, $m_0 = (1-\alpha^2\partial_x^2)(1-\beta^2\partial_x^2)u_0$, $\alpha > \beta > 0$, m_0 doesn't change sign on $\mathbb{R}$ and u_0 has compact support in the interval $[a,c]$. Then for $t \in (0,T)$, the corresponding solution $u(x,t)$ of equation (1.1) satisfies*

$$\begin{aligned} \frac{1}{2(\alpha+\beta)} e^{-\frac{x}{\alpha}} |E_1(t)| \le u(x,t) &\le \frac{\alpha}{2(\alpha^2-\beta^2)} e^{-\frac{x}{\alpha}} |E_1(t)|, && \text{as } x > q(c,t), \\ \frac{1}{2(\alpha+\beta)} e^{\frac{x}{\alpha}} |F_1(t)| \le u(x,t) &\le \frac{\alpha}{2(\alpha^2-\beta^2)} e^{\frac{x}{\alpha}} |F_1(t)|, && \text{as } x < q(a,t). \end{aligned}$$

where

$$E_1(t) = \int_{\mathbb{R}} e^{\frac{\xi}{\alpha}} m(\xi,t) d\xi, \qquad F_1(t) = \int_{\mathbb{R}} e^{-\frac{\xi}{\alpha}} m(\xi,t) d\xi,$$

denote continuous nonvanishing functions.

Remark 4.3. *We assume $\alpha > \beta > 0$ to get the above conclusion in Theorem 4.2, because the position of α, β is symmetric, then $\beta > \alpha > 0$, we have results similar to the above conclusions about $E_2(t) = \int_{\mathbb{R}} e^{\frac{\xi}{\beta}} m(\xi,t)d\xi$, $F_2(t) = \int_{\mathbb{R}} e^{-\frac{\xi}{\beta}} m(\xi,t)d\xi$,*

$$\begin{aligned} \frac{1}{2(\alpha+\beta)} e^{-\frac{x}{\beta}} |E_2(t)| \le u(x,t) &\le \frac{\beta}{2(\beta^2-\alpha^2)} e^{-\frac{x}{\beta}} |E_2(t)|, && \text{as } x > q(c,t), \\ \frac{1}{2(\alpha+\beta)} e^{\frac{x}{\beta}} |F_2(t)| \le u(x,t) &\le \frac{\beta}{2(\beta^2-\alpha^2)} e^{\frac{x}{\beta}} |F_2(t)|, && \text{as } x < q(a,t). \end{aligned}$$

Theorem 4.2 can be seem as a generalization of the result in [20]. Comparing with Theorem 4.1, it show more detailed estimation by adding the additional condition on m_0.

Proof. If u_0 has a compact support set $[a,c]$, then the corresponding m_0 also has a corresponding compact support set $[a,c]$. It is known from (3.2) that m has the same compact support set $[q(a,t), q(c,t)]$. We define

$$\begin{aligned} u_1 &= \left(1 - \frac{\beta}{\alpha}\right) \cdot \frac{\alpha}{2(\alpha^2-\beta^2)} \int_{\mathbb{R}} e^{-\frac{|x-\xi|}{\alpha}} m d\xi = \frac{1}{2(\alpha+\beta)} \int_{\mathbb{R}} e^{-\frac{|x-\xi|}{\alpha}} m d\xi, \\ u_2 &= \frac{\alpha}{2(\alpha^2-\beta^2)} \int_{\mathbb{R}} e^{-\frac{|x-\xi|}{\alpha}} m d\xi. \end{aligned} \tag{4.14}$$

According to $E_1(t) = \int_{\mathbb{R}} e^{\frac{\xi}{\alpha}} m(\xi,t)d\xi, F_1(t) = \int_{\mathbb{R}} e^{-\frac{\xi}{\alpha}} m(\xi,t)d\xi$, then

$$u_1(x,t) = \frac{1}{2(\alpha+\beta)} e^{-\frac{x}{\alpha}} E_1(t), \qquad u_2(x,t) = \frac{\alpha}{2(\alpha^2-\beta^2)} e^{-\frac{x}{\alpha}} E_1(t), \quad \text{as } x > q(c,t),$$
$$u_1(x,t) = \frac{1}{2(\alpha+\beta)} e^{\frac{x}{\alpha}} F_1(t), \qquad u_2(x,t) = \frac{\alpha}{2(\alpha^2-\beta^2)} e^{\frac{x}{\alpha}} F_1(t), \quad \text{as } x < q(a,t).$$

According to (4.1) and (4.14), we obtain

$$u_2(x,t) - u(x,t) = \frac{\beta}{2(\alpha^2-\beta^2)} \int_{\mathbb{R}} e^{-\frac{|x-\xi|}{\beta}} m d\xi,$$
$$u(x,t) - u_1(x,t) = \frac{\beta}{2(\alpha^2-\beta^2)} \int_{\mathbb{R}} \left(e^{-\frac{|x-\xi|}{\alpha}} - e^{-\frac{|x-\xi|}{\beta}} \right) m d\xi.$$

Then, we obtain

$$\begin{cases} u_1(x,t) \le u(x,t) \le u_2(x,t), & m_0 \ge 0, \\ u_2(x,t) \le u(x,t) \le u_1(x,t), & m_0 \le 0. \end{cases}$$

If $m_0 \ge 0$,

$$\begin{cases} \frac{1}{2(\alpha+\beta)} e^{-\frac{x}{\alpha}} E_1(t) \le u(x,t) \le \frac{\alpha}{2(\alpha^2-\beta^2)} e^{-\frac{x}{\alpha}} E_1(t), & \text{as } x > q(c,t), \\ \frac{1}{2(\alpha+\beta)} e^{\frac{x}{\alpha}} F_1(t) \le u(x,t) \le \frac{\alpha}{2(\alpha^2-\beta^2)} e^{\frac{x}{\alpha}} F_1(t), & \text{as } x < q(a,t). \end{cases} \tag{4.15}$$

If $m_0 \le 0$,

$$\begin{cases} \frac{\alpha}{2(\alpha^2-\beta^2)} e^{-\frac{x}{\alpha}} E_1(t) \le u(x,t) \le \frac{1}{2(\alpha+\beta)} e^{-\frac{x}{\alpha}} E_1(t), & \text{as } x > q(c,t), \\ \frac{\alpha}{2(\alpha^2-\beta^2)} e^{\frac{x}{\alpha}} F_1(t) \le u(x,t) \le \frac{1}{2(\alpha+\beta)} e^{\frac{x}{\alpha}} F_1(t), & \text{as } x < q(a,t). \end{cases} \tag{4.16}$$

The proof of Theorem 4.2 is finished. □

5. LONG TIME BEHAVIOR FOR THE SUPPORT OF MOMENTUM DENSITY

After the global existence of solution is established, we will discuss the long time behavior for the support of momentum density of the FOCH model. Now, we give the lemma and main theorem as follows:

Lemma 5.1. *Let $\alpha > \beta > 0$, Assume the initial value $u_0 \not\equiv 0$ has a compact supported set $[a,c]$.*

(1). *If $m_0(x) \ge 0 (\not\equiv 0)$, $x \in [a,c]$, then we have*

$$\lim_{t\to+\infty} F_1(t) = 0.$$

(2). *If $m_0(x) \le 0 (\not\equiv 0)$, $x \in [a,c]$, then we have*

$$\lim_{t\to+\infty} E_1(t) = 0.$$

Remark 5.1. *By the same argument, we can get a similar conclusion for $\beta > \alpha > 0$. If $m_0(x) \ge 0 (\not\equiv 0)$, $x \in [a,c]$, then we have*

$$\lim_{t\to+\infty} F_2(t) = 0.$$

If $m_0(x) \le 0 (\not\equiv 0)$, $x \in [a,c]$, then we have

$$\lim_{t\to+\infty} E_2(t) = 0.$$

Proof. **(1)** For $m_0(x) > 0$, from (3.2), we have $E_1(t) > 0, F_1(t) > 0, E_2(t) > 0, F_2(t) > 0$, for all $t \ge 0$. As $F_1(t) > 0$, we claim that

$$\lim_{t\to+\infty} F_1(t) = 0.$$

Otherwise, there is a constant $\epsilon_0 > 0$, for any $T > 0$, there will exist a $t > T$, such that $F_1(t) \ge \epsilon_0$.

For any $d < a$, from (4.15) we have

$$\frac{d}{dt} q(d,t) = u(q(d,t),t) \ge \frac{1}{2(\alpha+\beta)} e^{\frac{q(d,t)}{\alpha}} F_1(t)$$
$$\ge \frac{1}{2(\alpha+\beta)} e^{\frac{q(d,t)}{\alpha}} \epsilon_0.$$

It follows that

$$e^{-\frac{q(d,t)}{\alpha}} \leq -\frac{\epsilon_0}{2\alpha(\alpha+\beta)}t + \frac{e^{-\frac{d}{\alpha}}}{\alpha}.$$

Taking $T = \frac{2(\alpha+\beta)}{\epsilon_0}e^{-\frac{d}{\alpha}}$, however, when $t = T+1$,

$$-\frac{\epsilon_0}{2\alpha(\alpha+\beta)}t + \frac{e^{-\frac{d}{\alpha}}}{\alpha} < 0,$$

This is the contradiction. So our claim is right.

(2). For $m_0(x) < 0$, from (3.2), we have $E_1(t) < 0, F_1(t) < 0, E_2(t) < 0, F_2(t) < 0$, for all $t \geq 0$. As $F_1(t) > 0$, we claim that

$$\lim_{t\to+\infty} E_1(t) = 0.$$

Otherwise, there is a constant $\epsilon_0 > 0$, for any $T > 0$, for any $T > 0$, there will exist a $t > T$, such that $E_1(t) \leq -\epsilon_0$.

For any $h > c$, from (4.16) we have

$$\begin{aligned}\frac{d}{dt}q(h,t) = u(q(h,t),t) &\leq \frac{1}{2(\alpha+\beta)}e^{-\frac{q(h,t)}{\alpha}}E_1(t)\\ &\leq -\frac{\epsilon_0}{2(\alpha+\beta)}e^{-\frac{q(h,t)}{\alpha}}.\end{aligned}$$

It follows that

$$e^{\frac{q(h,t)}{\alpha}} \leq -\frac{\epsilon_0}{2\alpha(\alpha+\beta)}t + \frac{e^{-\frac{d}{\alpha}}}{\alpha}.$$

Taking $T = \frac{2(\alpha+\beta)}{\epsilon_0}e^{-\frac{d}{\alpha}}$, however, when $t = T+1$,

$$-\frac{\epsilon_0}{2\alpha(\alpha+\beta)}t + \frac{e^{-\frac{d}{\alpha}}}{\alpha} < 0,$$

This is the contradiction. So our claim is right. □

Theorem 5.2. *If $b > 1$, $\alpha > \beta > 0$, and suppose that $m_0(x) \in L_{\frac{1}{b}}$ and $u_0(x)$ has a compact supported set $[a, c]$.*

(1). *If $m_0(x) \geq 0 (\not\equiv 0), x \in [a, c]$, then we have*

$$e^{\frac{q(c,t)}{\alpha(b-1)}} - e^{\frac{q(a,t)}{\alpha(b-1)}} \longrightarrow +\infty, \quad \text{as } t \longrightarrow +\infty. \tag{5.1}$$

(2). *If $m_0(x) \leq 0 (\not\equiv 0), x \in [a, c]$, then we have*

$$e^{-\frac{q(a,t)}{\alpha(b-1)}} - e^{-\frac{q(c,t)}{\alpha(b-1)}} \longrightarrow +\infty, \quad \text{as } t \longrightarrow +\infty. \tag{5.2}$$

Remark 5.2. *For the case $\beta > \alpha > 0$, by using the properties of E_2 and F_2 in Remark 5.1, one can replace α with β in (5.1) and (5.2).*

Proof. **(1)** By (3.2) and direct calculation, we have

$$\begin{aligned}\left(\int_a^c (m_0)^{\frac{1}{b}}dx\right)^b &= \left(\int_a^c (m(q,t)q_x^b)^{\frac{1}{b}}dx\right)^b = \left(\int_a^c (m(q,t))^{\frac{1}{b}}q_x dx\right)^b\\ &= \left(\int_{q(a,t)}^{q(c,t)} (m(\xi,t))^{\frac{1}{b}}d\xi\right)^b\\ &\leq \left(\int_{q(a,t)}^{q(c,t)} m(\xi,t)e^{-\frac{\xi}{\alpha}}d\xi\right)\left[\int_{q(a,t)}^{q(c,t)} e^{\frac{\xi}{\alpha(b-1)}}d\xi\right]^{b-1}\\ &= F_1(t)\left[\alpha(b-1)\left(e^{\frac{q(c,t)}{\alpha(b-1)}} - e^{\frac{q(a,t)}{\alpha(b-1)}}\right)\right]^{(b-1)}.\end{aligned}$$

It follows

$$\left[\alpha(b-1)\left(e^{\frac{q(c,t)}{\alpha(b-1)}}-e^{\frac{q(a,t)(b-1)}{\alpha}}\right)\right]^{(b-1)}\geq\frac{\left(\int_a^c(m_0)^{\frac{1}{b}}dx\right)^b}{F_1(t)}.$$

Using the limit

$$\lim_{t\to+\infty}F_1=0,$$

we can get

$$e^{\frac{q(c,t)}{\alpha(b-1)}}-e^{\frac{q(a,t)}{\alpha(b-1)}}\longrightarrow+\infty,\quad as\ t\longrightarrow+\infty.$$

(2). Direct calculation, we have

$$\begin{aligned}\left(\int_a^c(-m_0)^{\frac{1}{b}}dx\right)^b&=\left(\int_a^c(-m_0)^{\frac{1}{b}}dx\right)^b=\left(\int_a^c(-m(q,t)q_x^b)^{\frac{1}{b}}dx\right)^b\\&=\left(\int_a^c(-m(q,t))^{\frac{1}{b}}q_xdx\right)^b=\left(\int_{q(a,t)}^{q(c,t)}(-m(\xi,t))^{\frac{1}{b}}d\xi\right)^b\\&\leq\left(\int_{q(a,t)}^{q(c,t)}\left(-m(\xi,t)e^{\frac{\xi}{\alpha}}\right)d\xi\right)\left[\int_{q(a,t)}^{q(c,t)}e^{-\frac{\xi}{\alpha(b-1)}}d\xi\right]^{b-1}\\&=-E_1(t)\left[\alpha(b-1)(e^{-\frac{q(a,t)}{\alpha(b-1)}}-e^{-\frac{q(c,t)}{\alpha(b-1)}})\right]^{b-1}.\end{aligned}$$

It follows

$$\left[\alpha(b-1)\left(e^{-\frac{q(a,t)}{\alpha(b-1)}}-e^{-\frac{q(c,t)}{\alpha(b-1)}}\right)\right]^{b-1}\geq\frac{\left(\int_a^c(-m_0)^{\frac{1}{b}}dx\right)^b}{-E_1(t)}.$$

Using the limit

$$\lim_{t\to+\infty}E_1=0,$$

we can obtain

$$e^{-\frac{q(a,t)}{\alpha(b-1)}}-e^{-\frac{q(c,t)}{\alpha(b-1)}}\longrightarrow+\infty,\quad as\ t\longrightarrow+\infty.$$

□

Theorem 5.3. *If $b=1$, suppose that $m_0(x)\in L_1$ and $u_0(x)$ has a compact supported set $[a,c]$.*

(1). *If $m_0(x)\geq0(\not\equiv0), x\in[a,c]$, then we have*

$$q(c,t)\longrightarrow+\infty,\quad as\ t\longrightarrow+\infty.$$

(2). *If $m_0(x)\leq0(\not\equiv0), x\in[a,c]$, then we have*

$$q(a,t)\longrightarrow-\infty,\quad as\ t\longrightarrow+\infty.$$

Proof. We only present the proof for $\alpha>\beta>0$. The case $\beta>\alpha>0$ can be proved by the same argument. **(1)** As $m_0(x)\geq0$, for any $t\geq0$, we have $F_1(t)>0$. According to Lemma 5.1, we know

$$\lim_{t\to+\infty}F_1(t)=0.$$

Direct calculation, we have

$$\int_a^c m_0dx=\int_a^c m(q,t)q_xdx\leq e^{\frac{q(c,t)}{\alpha}}\int_{q(a,t)}^{q(c,t)}m(\xi,t)e^{-\frac{\xi}{\alpha}}d\xi=e^{\frac{q(c,t)}{\alpha}}F_1(t).$$

It follows

$$e^{\frac{q(c,t)}{\alpha}}\geq\frac{\int_a^c m_0dx}{F_1(t)}\longrightarrow+\infty,\quad as\ t\longrightarrow+\infty,$$

then we can get

$$q(c,t) \longrightarrow +\infty, \quad as\ t \longrightarrow +\infty.$$

(2). As $m_0(x) \le 0$, for any $t \ge 0$, we have $E_1(t) < 0$. According to Lemma 5.1, we know

$$\lim_{t\to+\infty} E_1(t) = 0.$$

Direct calculation, we have

$$\begin{aligned}\int_a^c (-m_0)dx &= \int_a^c (-m(q,t)q_x)dx \\ &\le e^{-\frac{q(a,t)}{\alpha}} \int_{q(a,t)}^{q(c,t)} (-m(\xi,t))e^{\frac{\xi}{\alpha}} d\xi = -e^{-\frac{q(a,t)}{\alpha}} E_1(t).\end{aligned}$$

It follows

$$e^{-\frac{q(a,t)}{\alpha}} \ge \frac{\int_a^c(-m_0)dx}{-E_1(t)} \longrightarrow +\infty, \quad as\ t \longrightarrow +\infty,$$

then we can get

$$q(a,t) \longrightarrow -\infty, \quad as\ t \longrightarrow +\infty.$$

□

Theorem 5.4. *If $0 < b < 1$, $\alpha > \beta > 0$, $m_0(x) \in L_{\frac{1}{b}}$ or $b = 0$, $m_0 \in L_\infty$. Suppose that $u_0(x)$ has a compact supported set $[a,c]$.*

***(1).** If $m_0(x) \ge 0 (\not\equiv 0)$ for $x \in [a,c]$, then we have*

$$e^{\frac{q(c,t)}{\alpha}} - e^{\frac{q(a,t)}{\alpha}} \longrightarrow +\infty, \quad as\ t \longrightarrow +\infty. \tag{5.3}$$

***(2).** If $m_0(x) \le 0 (\not\equiv 0)$ for $x \in [a,c]$, then we have*

$$e^{-\frac{q(a,t)}{\alpha}} - e^{-\frac{q(c,t)}{\alpha}} \longrightarrow +\infty, \quad as\ t \longrightarrow +\infty. \tag{5.4}$$

Remark 5.3. *For the case $\beta > \alpha > 0$, by using the properties of E_2 and F_2 in Remark 5.1, one can replace α with β in (5.3) and (5.4).*

Proof. **(1).** For $m_0(x) \ge 0$, we have $F_1(t) > 0$ for all $t \ge 0$. From Lemma 5.1, we know

$$\lim_{t\to+\infty} F_1(t) = 0.$$

According to the conservation law

$$\int_{\mathbb{R}} m dx = \int_{\mathbb{R}} m_0 dx, \qquad \int_{\mathbb{R}} m^{\frac{1}{b}} dx = \int_{\mathbb{R}} m_0^{\frac{1}{b}} dx.$$

$$If\ \ 0 < b < 1\ \ and \quad \begin{cases} \gamma + \frac{\eta}{b} = 1, \\ 2\gamma + \eta = 1, \\ 0 < \gamma, \eta < 1. \end{cases} \implies \begin{cases} 0 < \eta = \frac{2}{2-b} - 1 < 1, \\ 0 < \gamma = 1 + \frac{1}{b-2} < 1. \end{cases}$$

By direct calculation, we obtain

$$\begin{aligned}\int_{\mathbb{R}} m_0 dx &= \int_{\mathbb{R}} m dx = \int_{\mathbb{R}} m(q,t)q_x dx \\ &= \left[\int_a^c \left(me^{-\frac{q}{\alpha}}q_x\right)^\gamma \left(m^{\frac{1}{b}}q_x\right)^\eta \left(e^{\frac{q}{\alpha}}q_x\right)^\gamma dx\right] \\ &\le \left(\int_a^c me^{-\frac{q}{\alpha}}q_x dx\right)^\gamma \left(\int_a^c m^{\frac{1}{b}}q_x dx\right)^\eta \left(\int_a^c e^{\frac{q}{\alpha}}q_x dx\right)^\gamma \\ &= \left(\int_{q(a,t)}^{q(c,t)} me^{-\frac{\xi}{\alpha}} d\xi\right)^\gamma \left(\int_{\mathbb{R}} m^{\frac{1}{b}} d\xi\right)^\eta \left(\int_{q(a,t)}^{q(c,t)} e^{\frac{\xi}{\alpha}} d\xi\right)^\gamma \\ &= (F_1(t))^\gamma \left(\int_{\mathbb{R}} m^{\frac{1}{b}} d\xi\right)^\eta \left(\int_{q(a,t)}^{q(c,t)} e^{\frac{\xi}{\alpha}} d\xi\right)^\gamma \\ &= (F_1(t))^\gamma \left(\int_{\mathbb{R}} m_0^{\frac{1}{b}} d\xi\right)^\eta \left(\alpha e^{\frac{q(c,t)}{\alpha}} - \alpha e^{\frac{q(a,t)}{\alpha}}\right)^\gamma.\end{aligned}$$

It follows

$$\left(\alpha e^{\frac{q(c,t)}{\alpha}}-\alpha e^{\frac{q(a,t)}{\alpha}}\right)^{\gamma}\geq\frac{\int_{\mathbb{R}}m_0dx}{(F_1(t))^{\gamma}\left(\int_{\mathbb{R}}m_0^{\frac{1}{b}}d\xi\right)^{\eta}}\longrightarrow+\infty,$$

then we can obtain

$$e^{\frac{q(c,t)}{\alpha}}-e^{\frac{q(a,t)}{\alpha}}\longrightarrow+\infty,\quad as\ t\longrightarrow+\infty.$$

If $b=0$, we can obtain

$$\begin{aligned}\int_{\mathbb{R}}m_0dx&=\lim_{b\to0}\int_{\mathbb{R}}mdx=\int_{\mathbb{R}}m(q,t)q_xdx\\&=\lim_{b\to0}\left[\int_a^c\left(me^{-\frac{q}{\alpha}}q_x\right)^{\gamma}\left(m^{\frac{1}{b}}q_x\right)^{\eta}\left(e^{\frac{q}{\alpha}}q_x\right)^{\gamma}dx\right]\\&\leq\lim_{b\to0}\left(\int_a^c me^{-\frac{q}{\alpha}}q_xdx\right)^{\gamma}\left(\int_a^c m^{\frac{1}{b}}q_xdx\right)^{\eta}\left(\int_a^c e^{\frac{q}{\alpha}}q_xdx\right)^{\gamma}\\&=\lim_{b\to0}\left(\int_{q(a,t)}^{q(c,t)}me^{-\frac{\xi}{\alpha}}d\xi\right)^{\gamma}\left(\int_{\mathbb{R}}m^{\frac{1}{b}}d\xi\right)^{\eta}\left(\int_{q(a,t)}^{q(c,t)}e^{\frac{\xi}{\alpha}}d\xi\right)^{\gamma}\\&=\lim_{b\to0}(F_1(t))^{\gamma}\left(\int_{\mathbb{R}}m^{\frac{1}{b}}d\xi\right)^{\eta}\left(\int_{q(a,t)}^{q(c,t)}e^{\frac{\xi}{\alpha}}d\xi\right)^{\gamma}\\&=\lim_{b\to0}(F_1(t))^{\gamma}\left(\int_{\mathbb{R}}m_0^{\frac{1}{b}}d\xi\right)^{\eta}\left(\alpha e^{\frac{q(c,t)}{\alpha}}-\alpha e^{\frac{q(a,t)}{\alpha}}\right)^{\gamma}.\end{aligned}$$

It follows

$$\left(\alpha e^{\frac{q(c,t)}{\alpha}}-\alpha e^{\frac{q(a,t)}{\alpha}}\right)^{\gamma}\geq\frac{\int_{\mathbb{R}}m_0dx}{(F_1(t))^{\gamma}\left(\lim_{b\to0}\int_{\mathbb{R}}m_0^{\frac{1}{b}}d\xi\right)^{\eta}}\longrightarrow+\infty,$$

then we can obtain

$$e^{\frac{q(c,t)}{\alpha}}-e^{\frac{q(a,t)}{\alpha}}\longrightarrow+\infty,\quad as\ t\longrightarrow+\infty.$$

(2). For $m_0(x)\leq0$, we have $E_1(t)<0$ for all $t\geq0$. From Lemma 5.1, we know

$$\lim_{t\to+\infty}E_1(t)=0.$$

Similarly, according to the conservation law

$$\int_{\mathbb{R}}mdx=\int_{\mathbb{R}}m_0dx,\qquad\int_{\mathbb{R}}m^{\frac{1}{b}}dx=\int_{\mathbb{R}}m_0^{\frac{1}{b}}dx.$$

$$\text{If } 0<b<1 \text{ and }\begin{cases}\gamma+\frac{\eta}{b}=1,\\2\gamma+\eta=1,\\0<\gamma,\eta<1.\end{cases}\Longrightarrow\begin{cases}0<\eta=\frac{2}{2-b}-1<1,\\0<\gamma=1+\frac{1}{b-2}<1.\end{cases}$$

By direct calculation, we obtain

$$\begin{aligned}-\int_{\mathbb{R}}m_0dx&=-\int_{\mathbb{R}}mdx=-\int_{\mathbb{R}}m(q,t)q_xdx\\&=\left[\int_a^c\left(-me^{\frac{q}{\alpha}}q_x\right)^{\gamma}\left((-m)^{\frac{1}{b}}q_x\right)^{\eta}\left(e^{-\frac{q}{\alpha}}q_x\right)^{\gamma}dx\right]\\&\leq\left(\int_a^c-me^{\frac{q}{\alpha}}q_xdx\right)^{\gamma}\left(\int_a^c(-m)^{\frac{1}{b}}q_xdx\right)^{\eta}\left(\int_a^c e^{-\frac{q}{\alpha}}q_xdx\right)^{\gamma}\\&=\left(\int_{q(a,t)}^{q(c,t)}-me^{\frac{\xi}{\alpha}}d\xi\right)^{\gamma}\left(\int_{\mathbb{R}}(-m)^{\frac{1}{b}}d\xi\right)^{\eta}\left(\int_{q(a,t)}^{q(c,t)}e^{-\frac{\xi}{\alpha}}d\xi\right)^{\gamma}\end{aligned}$$

$$
\begin{aligned}
&= (-E_1(t))^{\gamma} \left(\int_{\mathbb{R}} m^{\frac{1}{b}} d\xi \right)^{\eta} \left(\int_{q(a,t)}^{q(c,t)} e^{-\frac{\xi}{\alpha}} d\xi \right)^{\gamma} \\
&= (-E_1(t))^{\gamma} \left(\int_{\mathbb{R}} m_0^{\frac{1}{b}} d\xi \right)^{\eta} \left(\alpha e^{-\frac{q(a,t)}{\alpha}} - \alpha e^{-\frac{q(c,t)}{\alpha}} \right)^{\gamma}.
\end{aligned}
$$

It follows

$$
\left(\alpha e^{-\frac{q(a,t)}{\alpha}} - \alpha e^{-\frac{q(c,t)}{\alpha}} \right)^{\gamma} \geq \frac{-\int_{\mathbb{R}} m_0 dx}{(-E_1(t))^{\gamma} \left(\int_{\mathbb{R}} (-m_0)^{\frac{1}{b}} d\xi \right)^{\eta}} \longrightarrow +\infty,
$$

then we can obtain

$$
e^{-\frac{q(a,t)}{\alpha}} - e^{-\frac{q(c,t)}{\alpha}} \longrightarrow +\infty, \quad as\ t \longrightarrow +\infty.
$$

If $b = 0$, we can obtain

$$
\begin{aligned}
-\int_{\mathbb{R}} m_0 dx &= -\lim_{b\to 0} \int_{\mathbb{R}} m dx = -\int_{\mathbb{R}} m(q,t) q_x dx \\
&= \lim_{b\to 0} \left[\int_a^c \left(-m e^{\frac{q}{\alpha}} q_x \right)^{\gamma} \left((-m)^{\frac{1}{b}} q_x \right)^{\eta} \left(e^{-\frac{q}{\alpha}} q_x \right)^{\gamma} dx \right] \\
&\leq \lim_{b\to 0} \left(\int_a^c -m e^{\frac{q}{\alpha}} q_x dx \right)^{\gamma} \left(\int_a^c (-m)^{\frac{1}{b}} q_x dx \right)^{\eta} \left(\int_a^c e^{-\frac{q}{\alpha}} q_x dx \right)^{\gamma} \\
&= \lim_{b\to 0} \left(\int_{q(a,t)}^{q(c,t)} -m e^{\frac{\xi}{\alpha}} d\xi \right)^{\gamma} \left(\int_{\mathbb{R}} (-m)^{\frac{1}{b}} d\xi \right)^{\eta} \left(\int_{q(a,t)}^{q(c,t)} e^{-\frac{\xi}{\alpha}} d\xi \right)^{\gamma} \\
&= \lim_{b\to 0} (-E_1(t))^{\gamma} \left(\int_{\mathbb{R}} m^{\frac{1}{b}} d\xi \right)^{\eta} \left(\int_{q(a,t)}^{q(c,t)} e^{-\frac{\xi}{\alpha}} d\xi \right)^{\gamma} \\
&= \lim_{b\to 0} (-E_1(t))^{\gamma} \left(\int_{\mathbb{R}} m_0^{\frac{1}{b}} d\xi \right)^{\eta} \left(\alpha e^{-\frac{q(a,t)}{\alpha}} - \alpha e^{-\frac{q(c,t)}{\alpha}} \right)^{\gamma}.
\end{aligned}
$$

It follows

$$
\left(\alpha e^{-\frac{q(a,t)}{\alpha}} - \alpha e^{-\frac{q(c,t)}{\alpha}} \right)^{\gamma} \geq \frac{-\int_{\mathbb{R}} m_0 dx}{(-E_1(t))^{\gamma} \left(\lim_{b\to 0} \int_{\mathbb{R}} (-m_0)^{\frac{1}{b}} d\xi \right)^{\eta}} \longrightarrow +\infty,
$$

then we can obtain

$$
e^{-\frac{q(a,t)}{\alpha}} - e^{-\frac{q(c,t)}{\alpha}} \longrightarrow +\infty, \quad as\ t \longrightarrow +\infty. \qquad \square
$$

6. CONCLUSION

We have considered the FOCH model $\alpha \neq \beta, \alpha > 0, \beta > 0$. When α, β is negative, one can get the same results by taking absolute value $|\alpha|$ and $|\beta|$. This model is highly related to the classical Camassa-Holm equation, the Degasperis-Procesi equation and the Holm-Staley b-family equation. We have studied some mathematical property, such as global existence, infinite propagation speed and long time behavior of the support of momentum density. Another highly related equation is (1.1) with $\alpha = \beta$. Due to (1.1) with $\alpha = \beta$ doesn't have the structure in Lemma 2.2 and (2.1), some results in this manuscript may can't been realized for $\alpha = \beta$.

CONFLICTS OF INTEREST

The authors declare they have no conflicts of interest.

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